Mathematics on a Casio 9860/CG20/CG50

Volume 1 Supplement

Activities for Years 9 and 10

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Contents

The activities *Probably Finding* π, *Reaction Times and Statistics* and *Statistics from Birthdays* are associated with Chapter 8, *Probability and Statistics 1*, of Volume 1 of *Mathematics on a Casio 9860/CG20/CG50*. The remaining activities are associated with Chapter 3, *Coordinate Geometry*, of Volume 1.

Calculator versions

The Casio graphics calculator models CG20 AU and CG50 AU are basically the same as the 9860 used here (except for a higher-resolution colour screen). This is probably true of all Casio graphics calculators one level below the ClassPad. There may be minor differences in how the screen looks and in the menus but they all do the same calculations. There are some extras on the CG50 (e.g. colour) but these are not used here.

Calculations, screenshots and figures were (mostly) done on a Casio fx-9860G AU PLUS. The calculator programs were also written on this calculator, and converted to run on the other calculators. The programs are available at *www.canberramaths.org.au* under *Resources*.

1 A Classic Problem — The Hare and Tortoise

Year 10, Level 1; Strand: Algebra; Sub-strand: Sketching Other Graphs.

The graphs of distances covered versus time in this classic race are used to answer various questions about the race, such as who won and by how much. A fun exercise in putting questions into maths and solving equations graphically.

A hare and tortoise compete in a one-kilometre race. The distance each competitor has travelled from the starting point is given by a formula. In time t **minutes**, the distance in **metres** travelled by the hare is given by $H(t) = \frac{500}{3}(2\sqrt{t} + \sqrt[3]{t})$, while the distance in **metres** travelled by the tortoise is given by $T(t) = 100t + 250\sqrt{t}$.

Press MENU 5 (GRAPH) and enter the formulas for H and T in Y1 and Y2 respectively. You have to use X (X,θ,T) as the independent variable. The cube root is SHIFT (.

Set your V-Window (SHIFT F3) so that the two graphs go from the bottom left to the top right of the screen. *Hints*: The race takes about 5 minutes. How far is the race?

If you select Simul Graph On in the SET UP menu of your calculator before graphing, you will get a real-time view of the race.

Answer the following questions, writing down the steps you took.

Trace (F1) and, in the G-Solv menu, ISCT (intersection), Y·CAL (find Y given X) and X·CAL (find X given Y) in the G-Solv menu will be helpful. Press the right arrow after an ISCT operation to find further values.

You may need to increase Ymax or Zoom IN before using ISCT or X·CAL in Questions 2 and 3 below so that the function formulas do not obscure the point you are interested in.

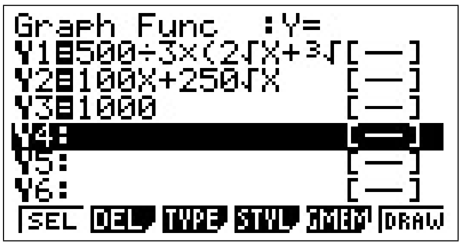
1. Who gets to the halfway point first? How long does it takes them? Verify your answer algebraically.

2. What is the time and distance at which the two runners are neck and neck?

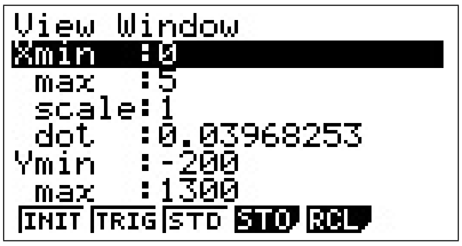
3. Who wins the race, by what time margin and by what distance margin?

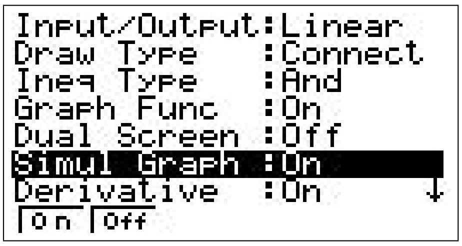
Notes for Teachers

The questions in this version have been written in general terms deliberately for a good class. For a less-advanced class, students may need to be led a little through each question: *What equation do we need to solve to answer this question? What does this mean about the graphs of each side of the equation? How do we solve this equation on the calculator?* and so on.



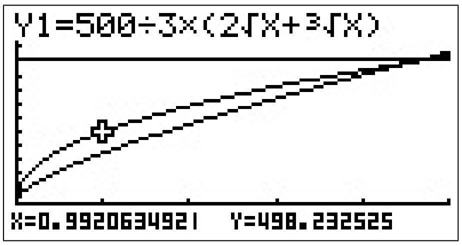
Press MENU 5 (GRAPH) and put the equation for the hare in Y1 and that for the tortoise in Y2. *Watch brackets here.* You might like to discuss with the class how to write the formulas in a suitable form for the calculator. Time t becomes X on the calculator.



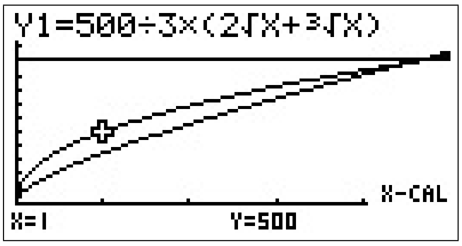


Then set the V-Window. Discuss first with the class what each axis represents and suitable scales.
The Y axis is distance in metres, so $0 < \text{Y} < 1000$.
To avoid function labels and coordinates hiding relevant points on the graphs, we increase Ymax to 1300 and decrease Ymin to -200. The winner is the competitor whose graph first reaches $\text{Y} = 1000$ (providing Simul Graph On is set in SET UP), shown by the horizontal line. *Yscl* is 100 here.

The time (X) scale has to be guessed. The race takes a little less than 5 minutes, so $0 < \text{X} < 5$ gives a good view.
Press EXIT to return to the *Graph Func* screen.



1. From the graph (use Trace and the up/down arrows to see which graph is which), the hare clearly reaches the halfway point (500 m) first.

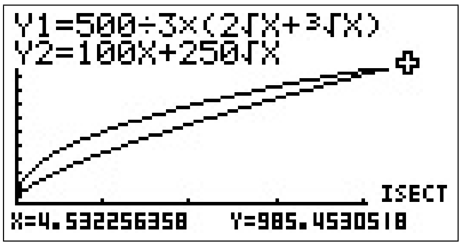


To find how long the hare took, solve $H(t) = 500$ for t using X-CAL in the G-Solv menu.

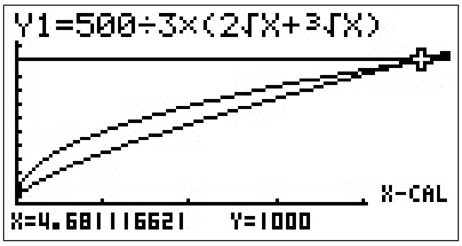
The value for t is 1 minute, a value we can confirm algebraically to be exact by substituting $t = 1$ into the equation for the hare. Note that it is easy to **verify** that $t = 1$ is a solution, but tricky to **solve** $H(t) = 500$ algebraically (it turns into a cubic equation).

The hare reaches the halfway point first in a time of 1 minute.

2. To find when they are neck and neck, we have to solve $H(t) = T(t)$, that is find the intersection of Y1 and Y2 (algebraically, this turns into a quartic equation). We obtain, using ISCT, $t = 4.53$ minutes and distance equal to 985 m (both to 3 significant digits). It might be useful to increase Ymax or Zoom IN (Zoom F3) on this part of the graph to see the two curves more clearly.

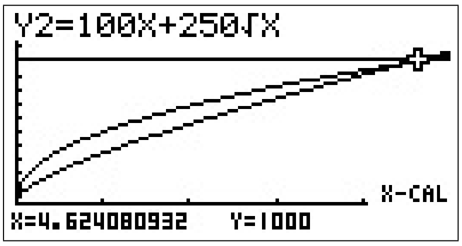


The hare and tortoise are neck and neck after about 4.53 minutes or about 4 minutes 32 seconds, at a distance of about 985 metres from the start.



3. To find the winner, we have to determine the time at which each competitor reaches the finish (1000 m).

Using X-CAL in the G-Solv menu, we find the hare finishes at $t = 4.681$ minutes and the tortoise finishes at $t = 4.624$ minutes.



To find the distance margin, calculate $H(4.624)$, the position of the hare when the tortoise finishes: $H(4.624) \equiv \text{Y1}\,(4.624) = 994.45$ m, rounded to 5 significant digits.

The tortoise wins the race by a margin of 0.057 minutes or 3.42 seconds. The distance margin is 5.55 m.

2 Alien Attack

Years 9 & 10, Levels 1 & 2; Strand: Algebra; Sub-strand: Coordinate Geometry.

Authors: Vanessa Moore and Sherry Morton, in Graphing Calculators in Mathematics Grades 7–12, Center of Excellence for Science and Mathematics Education, University of Tennessee. Modified by Peter McIntyre.

Uses one of Newton's equations of motion to explore properties of quadratic equations both numerically and graphically.

When an object is propelled straight upwards, gravity slows it down and eventually pulls it back to Earth. The graph of height vs time is a parabolic curve, even though the path of the object is straight up and down.

The height of the object as a function of time is given by one of Newton's equations of motion,

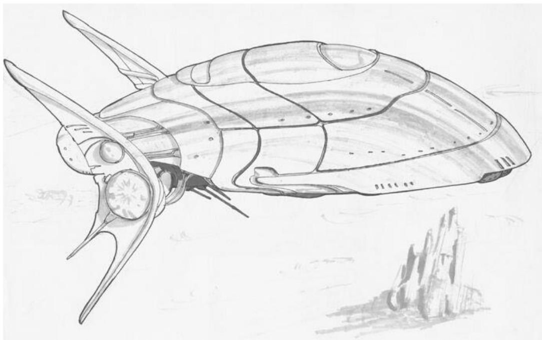
$$h(t) = h_i + v_i t - 0.5gt^2,$$

where h is the height above the ground at time t, h_i is the initial or starting height, v_i is the initial velocity and g is the acceleration due to gravity (a constant).

If we use SI units of height in metres and time in seconds, $g \approx 9.8\,\mathrm{m/s^2}$.

The scene

An alien spaceship is hovering above the city at a height of 100 m. A giant slime blob is housed in a missile-like container. The alien ship launches the container straight up into the air at a velocity of 500 m/s and vanishes immediately into hyperspace. What happens?



www.sheriftariq.org/markers/elegant_alienship.jpg

To answer this question, you need to put the equation for height as a function of time into your calculator.

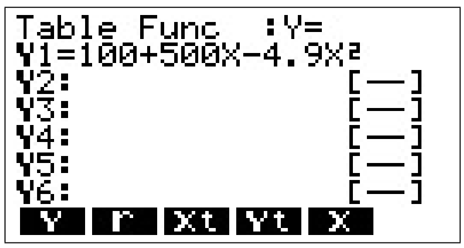
(a) First write the equation for h in the space below, using the given values for h_i, v_i and g. You can probably also work out 0.5×9.8 to shorten the equation.

(b) The calculator uses X for the (independent variable) time and Y for the (dependent variable) height. Write the equation for h in terms of Y and X in the space below.

(c) Now press MENU 7 or select the TABLE icon from the main menu and press EXE.

Enter your equation into Y1 just as you have written it above. The X key is the one labelled X,θ,T in the fourth row. Press EXE to store the equation.

Make sure you use the minus sign $-$, not the change-sign $(-)$.

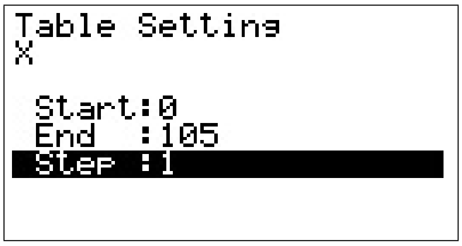


Now you will generate a table of values of the height function you have just entered.

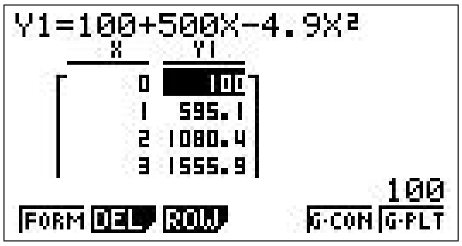
(d) To tell the calculator which X values you want in the table, press [F5] (SET).

Set Start to 0, End to 105 and Step to 1, so that the table of values will start at time X = 0 s and increment in steps of 1 s, up to a maximum of 105 s.

Press [EXIT].



(e) Press [F6] (TABL).



Now answer each of the following questions in the space below the question.

1. What is the height of the container after 25 seconds?

2. Is the container going up or down after 25 seconds? How do you know?

3. What is the maximum height the container reaches?

4. After how many seconds does the container reach its maximum height?

5. How accurate is your value in Question 4? Look at the table and decide between which two X values you are sure the exact answer lies.

The exact time to maximum height is greater than ______, but less than ______.

6. To obtain a more accurate answer to Question 4, set Step in SET to 0.1. You will need to change Start and End too. Scroll in the Y1 column of the table to see the highlighted Y value with more digits at the bottom of the screen.

Find the time to maximum height, *accurate to one decimal place*, and the corresponding maximum height. *Hint*: You may need to change Pitch more than once.

7. When the container hits the ground, the slime blob will envelop the city. How long does our local superhero[1] ____________________ have to come to the rescue?
Hint: Use the table to answer the question. Tenths of seconds are vital here.

[1]Supply an appropriate name.

Next we will look at graphs of height vs time to answer some more questions.

You already have the right function to graph because you used it for the table. You need to set a suitable V-Window so the graph appears on your screen. Press SHIFT F3 (V-Window).

8. What quantity does X represent here? Based on your explorations using TABLE, what are suitable values for Xmin and Xmax? What does Y represent here? What are suitable values for Ymin and Ymax?

Enter these values into your calculator, pressing EXE after each. Set Xscale (the distance between tick marks on the X axis) to 10 and Yscale to 5000. Press EXIT to return to the Table Func screen.

We could press F6 to recalculate the table and F5 (G·CON) to draw a connected graph. However, it is probably easier at this stage to change to GRAPH mode by pressing MENU 5. Press F5 (DRAW) to draw the graph.

Change your V-Window values if necessary and re-graph so that the graph fills the screen, but is not obscured by the formula at the top.

9. Sketch your graph below, being sure to label the axes with what they actually represent and giving some idea of scale (one or two values on each axis).

10. What is the name of the point on the parabola that corresponds to maximum height? What are the approximate coordinates of this point? (Press SHIFT F1 (Trace) and use the left- and right-arrow keys to determine this.) Does this agree with what you found using the table?

On your graph, you will probably find that the cursor coordinates cover the X axis. To fix this, change Ymin in V-Window to -3000 (which minus key?).

11. Find the approximate time that the container hits the ground using the cursor in Trace mode. How accurate is your answer? Does it agree with the answer you found using a table?

The exact time to ground is greater than __________ *, but less than* __________ .

12. A point at which a graph crosses the x axis (has y value 0) is called an x *intercept*, *zero* or *root*.

The time taken for the container to return to the ground is an x intercept of the height graph. In Question 11, you found an approximate value for this.

You will now use the ROOT operation to find the time to ground more accurately.

Press SHIFT F5 (G-Solv) and select F1 (ROOT). The calculator 'thinks' for a while (black rectangle in top right corner), then moves the cursor to the root.

When does the container hit the ground? Give your answer to a *sensible* number of decimal places.

13. Does our superhero save the city? Finish the story.

Notes for Teachers

The equation for height as a function of time is

$$h(t) = 100 + 500t - 4.9t^2,$$

with the corresponding calculator equation and table of values shown in the figures.

1. What is the height of the container after 25 seconds?

Scroll down the table to find that $h(25) = 9537.5$, *so the container is at a height of 9537.5 m after 25 s.*

2. Is the container going up or down after 25 seconds? How do you know?

The container is still going up after 25 s, because the height is increasing with time then.

3. What is the maximum height the container reaches?

Scroll down the table to find that the maximum height the container reaches is apparently 12855 m.

4. After how many seconds does the container reach its maximum height?

According to the table, the container reaches maximum height after 51 s.

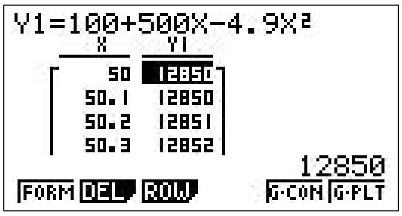
5. How accurate is your value in Question 4? Look at the table and decide between which two X values you are sure the exact answer lies.

The exact time to maximum height is greater than 50 s, but less than 52 s.

The table values are for integer numbers of seconds. We can only say for sure that the maximum time lies between 50 s and 52 s, with a best estimate of 51 s. You could draw a head-up parabola and discuss where the three points in the table near the maximum might lie, in particular that the highest point may not lie right at the vertex.

6. Obtaining more accurate answers.

By changing Step in SET to 0.1, we can see the height every 0.1 s. It's a good idea to set Start to 50 and End to 52 so you don't have to scroll through too many values.



Find the time to maximum height, *accurate to one decimal place*, and the corresponding maximum height.

After changing Step to 0.1, we find the time to maximum height lies between 50.9 s and 51.1 s, so we can only say the answer is 51 s, accurate to 0 decimal places. Changing Step to 0.01 (and Start to 50.9, End to 51.1 say), we find the time to maximum height lies between 51.01 s and 51.03 s, both 51.0 s, accurate to 1 decimal place.

The corresponding maximum height is 12855 m, rounded to the nearest metre.

7. When the container hits the ground, the slime blob will envelop the city. How long does our local superhero ________________________ have to come to the rescue?
Hint: Use the table to answer the question. Tenths of seconds are vital here.

Use the same method to find when $h = 0$ *as you did to find the maximum value of* h*: with Step = 1 and End = 105, scroll down the table until Y1 changes sign; use Step = 0.1 to find the time accurate to one decimal place.*

The time at which the container hits the ground lies between 102.2 s and 102.3 s, with a best estimate of 102.2 s, accurate to the nearest tenth of a second.

Next we will look at graphs of height vs time to answer some more questions.

Here it would be a good idea to run the SLIME program on the overhead projector or simulator, or for students to run it on their calculator (see page 14 for details), and discuss what you see.

You already have the right function to graph because you used it for the table. You need to set a suitable V-Window so the graph appears on your screen. Press SHIFT F3.

8. What quantity does X represent here? Based on your explorations using TABLE, what are suitable values for Xmin and Xmax? What does Y represent here? What are suitable values for Ymin and Ymax?

X represents time, so that Xmin = 0 is the starting time. From our results using the table, we know that the container hits the ground after about 102 s, so set Xmax = 105.

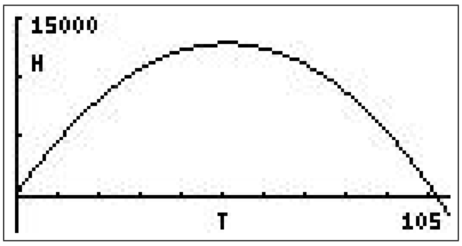
Y represents height above the ground, so that Ymin = 0 is ground level. Again from our table results, we know maximum height is about 12855, so set Ymax = 15000 say.

Enter these into your calculator. Set Xscale (the distance between tick marks on the X axis) to 10 and Yscale to 5000. Press EXIT to return to the Table Func screen.

We could press F6 to recalculate the table and F5 (G·CON) to draw a connected graph. However, it is probably easier at this stage to change to GRAPH mode by pressing MENU 5. Press F5 (DRAW) to draw the graph.

Change your V-Window values if necessary and re-graph so that the graph fills the screen, but is not obscured by the formula at the top.

9. Sketch your graph below, being sure to label the axes with what they actually represent and giving some idea of scale (one or two values on each axis).[2]



10. What is the name of the point on the parabola that corresponds to maximum height? What are the approximate coordinates of this point? (Use Trace and the left- and right-arrow keys to determine this.) Does this agree with what you found before?

The point on the parabola that corresponds to maximum height is called the vertex. Its coordinates are approximately $(51, 12855)$*, roughly the values we found from the table for maximum height.*

On your graph, you will notice that the cursor coordinates cover the X axis. To fix this, change Ymin in V-Window to −3000.

For better students, insert *Finding the equation of the parabola* (page 11) here.

[2]Note that Ymin has been changed to −3000 here to allow the T label to be drawn.

11. Find the approximate time that the container hits the ground using the cursor in Trace mode. How accurate is your answer?

The time to ground is greater than about 101.6 s, but less than about 102.8 s.

These numbers, the X coordinates of adjacent pixels, will vary a little, depending on what values you put in View Window. The best estimate is 102 s.

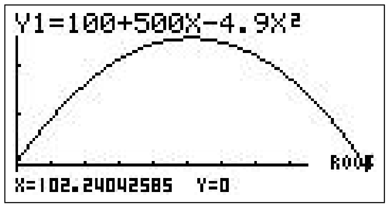
12. A point at which a graph crosses the x axis (has y value 0) is called an x *intercept*, *zero* or *root*.

The time taken for the container to return to the ground is an x intercept of the height graph. In Question 11, you found an approximate value for this.

You will now use the ROOT operation to find the time to ground more accurately.

Press SHIFT F5 (G-Solv) and select F1 (ROOT). The calculator 'thinks' for a while (black rectangle in top right corner), then moves the cursor to the root.

When does the container hit the ground? Give your answer to a sensible number of decimal places.



The container hits the ground at 102.24 s, rounded to 2 decimal places.

Further questions

What happens if the initial velocity is changed to

(a) 1500 m/s? *Answer*: max at (153.06 s, 114896 m); reaches the ground at 306.19 s.

(b) 3000 m/s? *Answer*: max at (306.12 s, 459284 m); reaches the ground at 612.28 s.

(c) 5000 m/s? *Answer*: max at (510.20 s, 1275610 m); reaches the ground at 1020.43 s.

Finding the equation of the parabola

This follows on after Question 10. Suitable for better students.

If we drew a vertical line through this point, we would have an axis of __________ of the parabola.

What is the equation of this line?

What is happening to the slime-blob container on the left-hand side of this line? on the right-hand side?

Write the equation of the parabola in the form $y = a(x-b)^2 + c$.

Check your answer first by entering your equation into Y2, graphing both functions and using Trace.

Are the two functions the same? (The arrow keys will help here. What does the up/down-arrow key do?)

PTO

Second, expand out the brackets in your function here and confirm you obtain the original function. (The coefficients you get doing this may only be approximately correct, because we truncate the numbers we take from either the table or the graph.)

Solutions

The equation of the vertical line is $x \approx 51$. On the left-hand side of the line the container is going up, on the right-hand side going down.

The axis of symmetry of the parabola is $x = b \approx 51$, so that $y \approx a(x-51)^2 + c$.

When $x = 51$, $y = c \approx 12855$, so that $c \approx 12855$. Therefore, $y \approx a(x-51)^2 + 12855$.

The coefficient of x^2 here is a, whereas in the original equation it was -4.9. Therefore, $a = -4.9$ and the equation of the parabola is $y \approx -4.9(x-51)^2 + 12855$. You could also find a by using the fact that $y(0) = 100$.

The two graphs should be the same. Check by toggling between the two curves using the up- or down-arrow key. Look at the Y values at the bottom of the screen. Try this at several points along the curves (use the left- or right-arrow key to do this).

Expanding out the brackets,

$$\begin{aligned} y &= -4.9(x-51)^2 + 12855 \\ &= -4.9(x^2 - 102x + 2601) + 12855 \\ &= -4.9x^2 + 4.9 \times 102x - 4.9 \times 2601 + 12855 \\ &= -4.9x^2 + 499.8x + 110.1 \\ &\approx -4.9x^2 + 500x + 100 \quad \text{the original equation.} \end{aligned}$$

If we used a better approximation for x, say $x \approx 51.02$ rather than 51, we obtain

$$y \approx -4.9x^2 + 499.996x + 100.102,$$

very close to the original equation.

If you go back to just before Question 11 now, turn off Y2 by selecting it with the cursor and pressing [F1] or delete it with [F2]. Press [F5] (DRAW) to regraph Y1.

The SLIME program

This could be done with the whole class just before starting to plot the graph, that is in the middle of page 6.

The SLIME program shows the action in real time.

- Press MENU B, select SLIME with the cursor and press F1 to run the program. The program plots two graphs simultaneously.
- Once the graphs are finished, you can re-live the action in slow motion by pressing F1 (Trace). Move forward in time with the right arrow, backward in time with the left arrow and between graphs with the up/down arrows. Try this to see corresponding points on the graphs. The cursor coordinates are shown at the bottom of the screen, the relevant ones being time T and height Y.

  Why are the two graphs different? What does each one represent?

  The left-hand graph is a plot of height versus time. The right-hand graph is the actual trajectory.
- To rerun the program, just press EXE again.

The SLIME program is available for download at *canberramaths.org.au* under *Resources*.

3 Best Shape for a Can

Year 10, Level 1; Strand: Algebra/Measurement; Sub-strand: Sketching Other Graphs/Volume.

From: Integrating the Graphics Calculator into Years 9 and 10 of the Victorian Mathematics CSF, Teachers Teaching with Technology (T^3), 1998. Modified by Peter McIntyre.

Minimising the surface area of a cylinder (can) for a fixed volume. Numerical and graphical techniques, rather than Calculus, are used to find the minimum. Aspects of mathematical modelling are introduced.

When manufacturers are designing their packaging, they must keep in mind the amount of product that has to fit inside and the amount of material it will take to make the package. Consider the humble soft-drink can. The standard volume is 375 mL or 375 cm^3. Any number of cans can be designed that will hold this volume of liquid, but they will vary in shape and therefore in the amount of material needed to make the can (and therefore cost).

The formula for the volume V of a cylinder (measured here in cm^3) in terms of radius r and height h (both measured in cm), is $V = \pi r^2 h$.

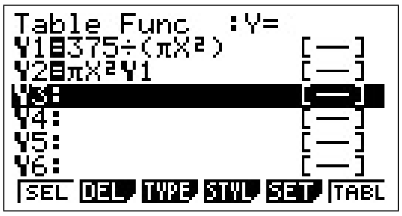
Rearrange the volume formula to make h the subject; let the volume be 375 cm^3: $h =$

Press [MENU] [7] (TABLE) and enter the formula for h in Y1, with X to represent the radius r.

As a check, enter the volume formula: $\text{Y2} = \pi \text{X}^2 \text{Y1}$.
Y1 is [VARS] [F4] [F1] [1].

Then press [EXIT] twice. Press [F5] (TABL).

You may get ERROR for Y1 and Y2 if X = 0 is in your table. *Why?*



Let's specify which X values we want.

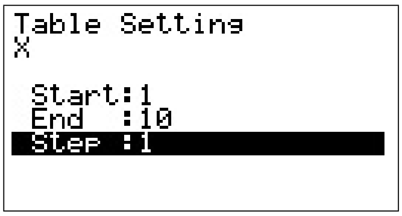
Press [F1] (FORM) to go back to the formula.

Press [F5] (SET).

Set Start = 1, End = 10 and Step = 1.

Press [EXIT], then [F6] again.

Do you get the correct volume in Y2?



Write down the formula for the surface area of a cylinder, including the ends.
The surface area determines the amount of material needed to make the can. *Why?*
Press [F1] and enter the formula for surface area in Y2 in terms of X (radius) and Y1 (height)

SA =

Y2 =

View the table of values again. *What do you notice about the values of the surface area?*

Values of the surface area ______

____

Press [F1], then [F5]; set a new Start and End, and a smaller Step to find the minimum surface area and corresponding radius (radius accurate to 1 decimal place).

minimum surface area =

radius =

Now graph the surface area as a function of radius.

Press MENU 5 (GRAPH).

Set a V-Window (SHIFT F3) of $[0, 12.6, 2] \times [-150, 1000, 100]$ and press EXIT.

Turn off Y1 by pressing F1.

Press F6 (DRAW) to plot the graph.

Draw your graph here with scales on the axes.

Use Trace (F1) and the cursor to find an approximate value for the minimum.

Write down your values for the radius, height, ratio of height to radius, surface area and circumference of the can when the surface area is a minimum.

How do these compare with a soft-drink can? Why might there be differences?
How does the theory fit with the shapes of other cans?

You might like to read the article *The Best Shape for a Tin Can* by PL Roe, either in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993). Search for it online.

Notes for Teachers

There are similar maximum/minimum activities to suit all levels of Years 9 and 10. The activities all follow the same outline as this particular one. This way of doing the problem allows for numeric, graphic and sometimes algebraic approaches. Algebraic approaches without Calculus rely on the function to be minimised/maximised being a quadratic, which is not the case here.[3]

Before you start this activity, you might like to discuss with the class the different shapes of cans one finds on a supermarket shelf (bring in a few examples). Are they just scaled versions of each other? To quantify the shape, measure the ratio of height to radius for different cans. This should be the same if they are scaled versions. *What range of values of h/r do you find?*

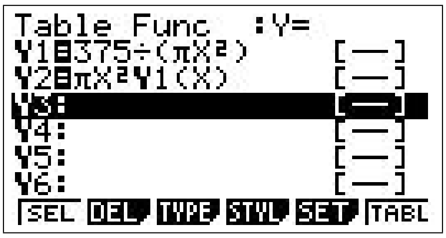
The question then arises: *What is the reason for the manufacturer choosing a particular shape?* This activity explores one possible explanation, that of minimising the amount of metal used to make a can.

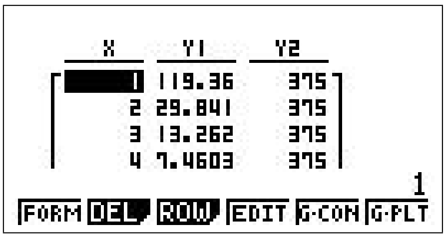
The total volume of the metal, assuming the walls are of uniform thickness, is just the surface area times the thickness. Minimum surface area therefore means minimum volume of metal.

The height of the cylinder is given by $\mathrm{Y1} = 375 \div (\pi \mathrm{X}^2)$, where X is the radius r.

As a check, enter the volume formula $\mathrm{Y2} = \pi \mathrm{X}^2 \mathrm{Y1}$.

Press F6 (TABL). Note that $\mathrm{Y2} = 375$ (except at $\mathrm{X} = 0$) as expected.





If your table starts at $\mathrm{X} = 0$, you will see no value for Y1 at $\mathrm{X} = 0$. This is because we are trying to divide by 0. This causes no value to be shown for Y2 as well, because Y2 is written in terms of Y1. This will only happen of course if the table starts at $\mathrm{X} = 0$. We correct it by starting the table at $\mathrm{X} = 1$ (in SET).

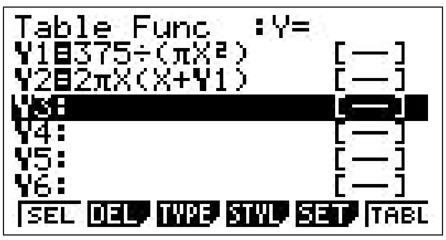
The surface area of a cylinder, including the ends, is given by

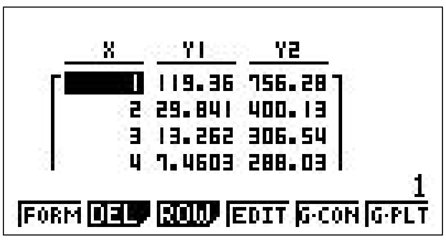
$$\mathrm{SA} = 2\pi r^2 + 2\pi r h = 2\pi r(r + h).$$

If you substitute for h, you find that

$$\mathrm{SA} = 2\pi r^2 + \frac{750}{r},$$

so the function is not a quadratic. For the purposes of graphing the function though, it is easier to leave h in the formula, so that we set $\mathrm{Y2} = 2\pi \mathrm{X}\big(\mathrm{X}+\mathrm{Y1}\big)$.



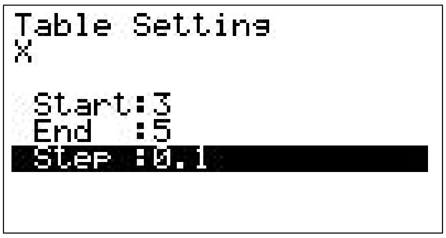


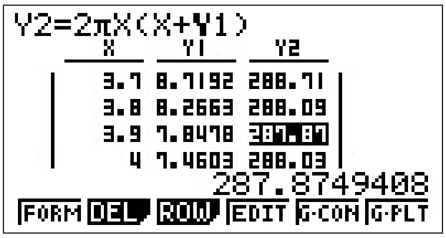
[3] I've included the exact results from Calculus below so this activity can be incorporated into a Calculus class too.

Look at the table of values again. What do you notice about the values of the surface area?

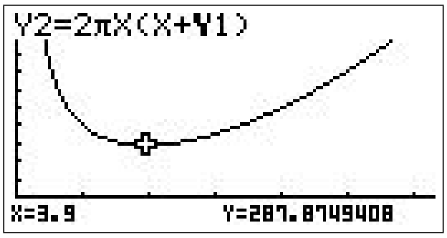
The surface area decreases, then increases as the radius increases. There is a (local) minimum.

With Start = 3, End = 5 and Step = 0.1 (below left), we find, on scrolling down, a radius of 3.9 cm for a minimum surface area of 287.9 cm^2, both values rounded to 1 decimal place (below right).





Now graph the surface area as a function of radius (below) using a V-Window of $[0, 12.6, 2] \times [-150, 1000, 100]$. Use Trace and the cursor to find the minimum surface area.



Note that, because of the choice of V-Window values, the cursor increments in X steps of 0.1, so that eventually you will reach X = 3.9, and it will be clear that the answer is 3.9, not 3.8 or 4.0. With a different choice for Xmax, the X values in Trace will not be nice numbers.

Again, we obtain a value of $r = 3.9$ cm for the radius, giving a minimum surface area of 287.9 cm^2, both values rounded (and accurate) to 1 decimal place.

If your students have sufficiently developed Calculus skills, they could prove algebraically that the global minimum lies at $r = \sqrt[3]{375/2\pi} \approx 3.9$, with a corresponding height $h = \sqrt[3]{1500/\pi} \approx 7.8$. More generally, for a given volume V, it is not too hard to show that $h = 2r$ (height = diameter) for minimum surface area.

Collecting our results so far and calculating several more, we have (to 1 decimal place):

radius $r = 3.9$ cm height $h = 7.8$ cm ratio $h/r = 2$

surface area $= 287.9\,\text{cm}^2$ circumference $= 24.6$ cm

Does it matter if the radius is not the exact minimum value? (follow up)

If the radius varies by say 5%, 10%, etc from the minimum value, by what percentage does the surface area and therefore cost change? Alternatively, by how much does the radius need to change from its minimum value to make the surface area change by 5%? by 10%? Use the calculator graph and Trace. Discuss the difference between a flat minimum and a sharply pointed one.

How do these compare with a soft-drink can? Why might there be differences? How does the theory fit with the shapes of other cans?

Standard soft-drink cans have a radius of 3.25 cm and a height of 13 cm, so that $h/r = 4$. The surface area is about 332 cm^2 and circumference 20.4 cm. Clearly, considerations other than minimum surface area are involved. These might be what circumference is comfortable for the average human hand, the wastage of material when cutting the ends and the cost of making the joins.

The article *The Best Shape for a Tin Can* by P.L. Roe, in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993) goes into why there might be differences between the theory here and the actual values. A good example of mathematical modelling.

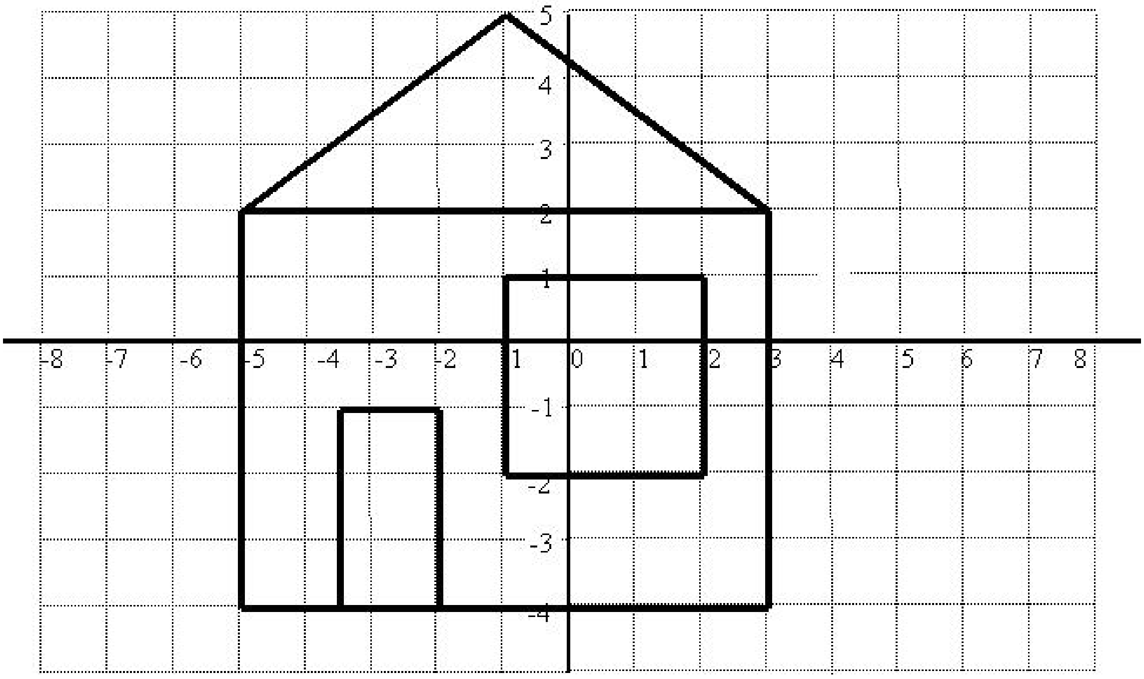
4 Coordinate Geometry Art

Year 9, Levels 1 & 2; Strand: Algebra; Sub-strand: Coordinate Geometry.

Author: Rex Boggs in The Sub-AToMIC Project, V Geiger et al, QAMT, 1999.
Modified by Peter McIntyre.

A simple picture consisting of straight-line segments is 'coded' using the coordinates of its vertexes. These are used 'transmit' the picture to someone else. A graphics calculators is used 'decode' and check the 'transmitted' picture.

Consider the beautiful work of art below.



Instructions for drawing this picture

This picture can be drawn by joining the points below in the given order. No line or part of a line is drawn twice. The instruction *lift pencil* means do not join the previous point to the next point.

$(-2,-4)$	$(-2,-1)$	$(2,1)$
$(-5,-4)$	$(-3.5,-1)$	$(2,-2)$
$(-5,2)$	$(-3.5,-4)$	$(-1,-2)$
$(-1,5)$	*lift pencil*	$(-1,1)$
$(3,2)$	$(-5,2)$	$(2,1)$
$(3,-4)$	$(3,2)$	
$(-2,-4)$	*lift pencil*	

1. Create your own drawing on grid paper (next page). The rules for this are:

- all line segments must be straight;
- use no more than two *lift pencil* instructions. Sometimes a clever choice of starting point can reduce the number of *lift pencil* instructions.

2. Give a set of instructions as a table of points like that one above so someone else can reproduce your drawing. No line segments are to be drawn twice.

3. Enter your coordinates into a graphics calculator and draw the picture on the screen. Your teacher will show you how. This is a good way to check that your instructions are correct.

Notes for Teachers

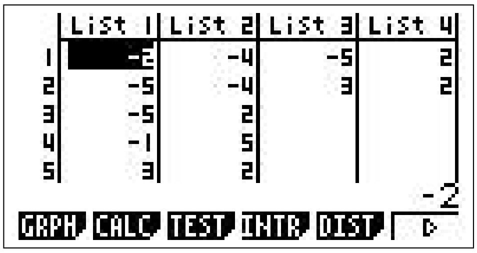
Grid paper is also provided on page 22. The artwork is displayed as follows.

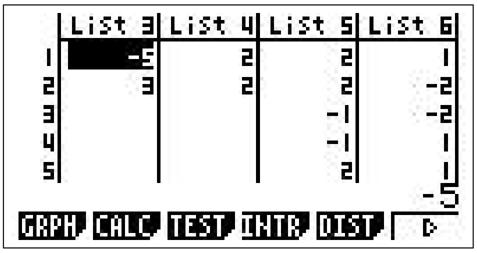
Press MENU 2 (STAT) to enter the Statistics area.

Clear all the lists: F6, then F4 for each list.

Press SHIFT MENU to get to SET UP. Make *Stat Wind* Manual. This allows you to set the viewing window. Press EXIT.

Enter the points, pressing EXE after each value. Put the x coordinates of the first group of coordinates (up to the first *lift pencil*) in List 1 and the y coordinates in List 2; put the second group of coordinates in List 3 and List 4, the third group in List 5 and List 6. The calculator allows a maximum of three plots at one time, hence no more than two *lift pencil* instructions can be used.

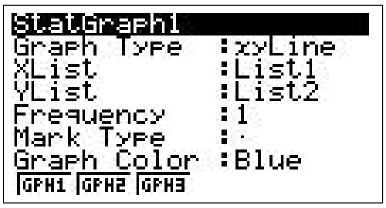


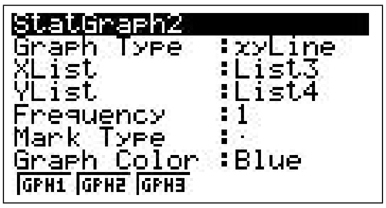


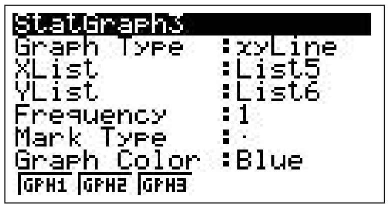
Note that not all the values in List 1 and List 2 are shown in the screen above left.

Set a suitable V-Window (guided by the house figure) with SHIFT F3, pressing EXE after each value. Press EXIT.

Press F1 (GRPH) and choose SET. Set up each of the three graphs as shown below, using the cursor to move down the lines. Press EXIT.







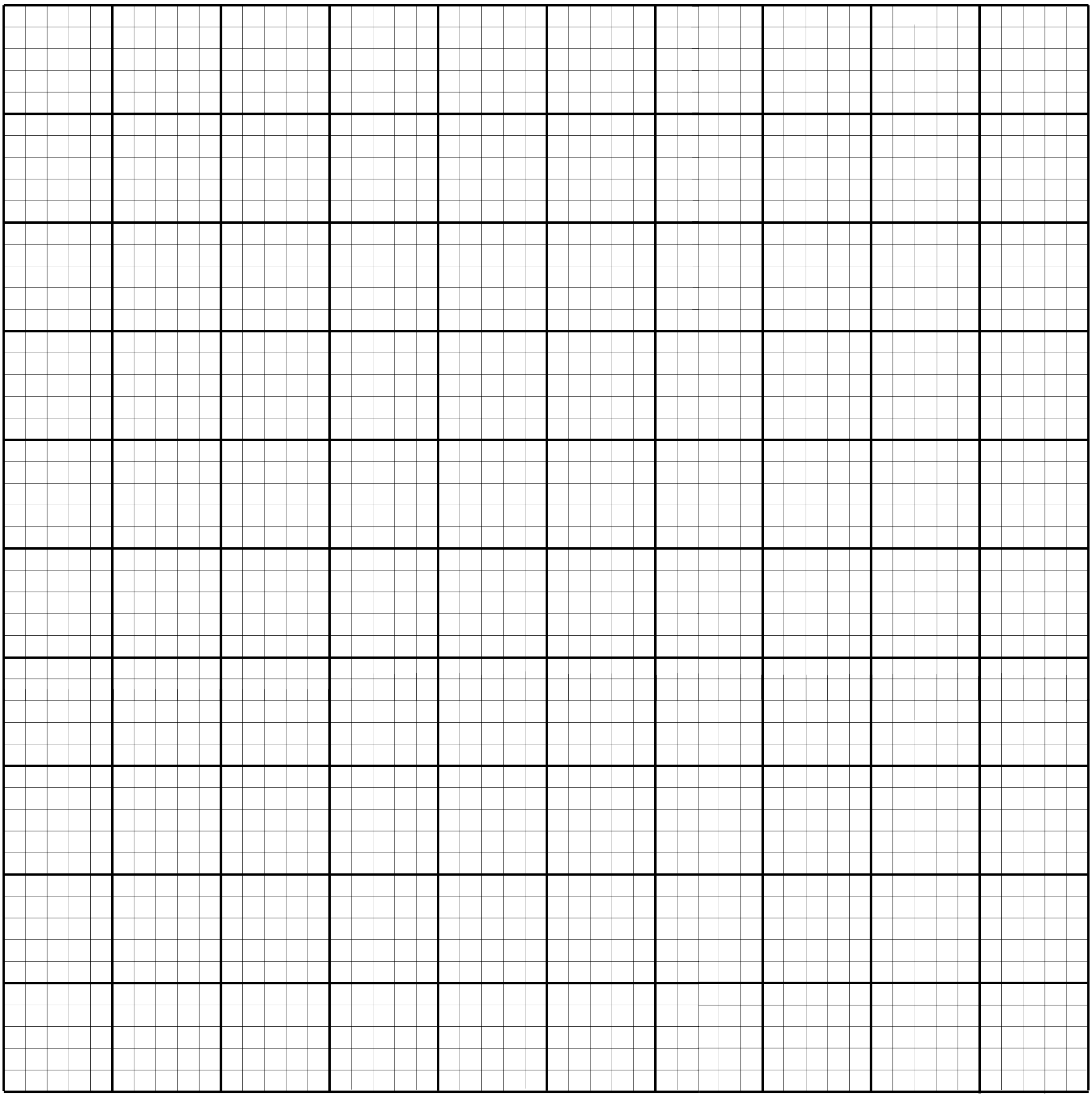
Press SEL and turn on all StatGraphs. Press DRAW.

If you want to turn off the axes, press SHIFT MENU to get to SET UP, scroll down and turn off the axes, followed by EXE. Press GRPH, SEL, DRAW to redisplay the graph.

Press SHIFT F1 (Trace) and use the left/right arrow keys to move between the points; the up/down arrow keys move between the three plots. This will enable you to find out which points, if any, were incorrect. Try F2 (Zoom) F6 (MORE) F2 (SQR) to set equal scales on both axes.

Don't forget to clear the lists and turn the axes back on when you have finished.

This is a great time to introduce a little bit of network theory. Give the students some pictures that appear to require more than two *lift pencil* instructions, and have them figure out where to start so as to reduce the number of *lift pencil* instructions to a maximum of two. Challenge the students to find the rule for determining the minimum number of *lift pencil* instructions needed to draw the graph.



5 Graphing Straight Lines

Years 9, 10, Levels 1, 2; Strand: Algebra; Sub-strand: Coordinate Geometry – Straight Lines.

Author: Margie Smith.

Using the graphics calculator to explore the $y = mx + b$ form of a straight line.

Graphing Straight Lines Worksheet 1

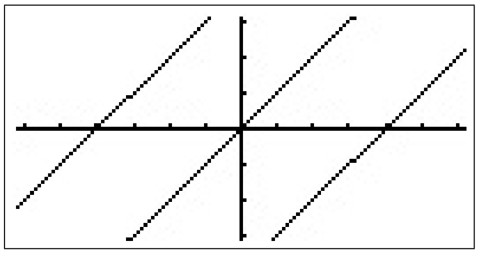
1. Press [MENU] [5] for the GRAPH menu.
Clear any current graphs with [F2] [F1].

2. Press [F2] (Zoom) [F6] (MORE) [F2] (SQR) to set equal scales on both axes.

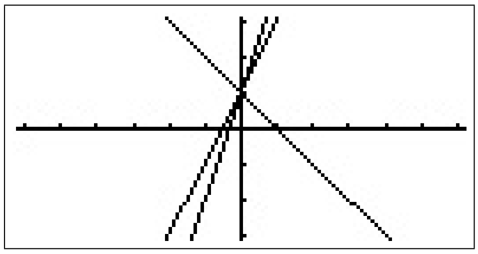
The graphs should appear like this.

↓

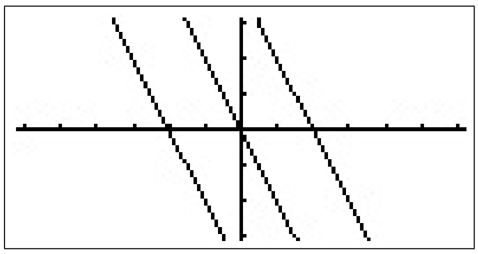
3. Put $y = x$ in Y1: [X,θ,T] gives X.
Put $y = x + 4$ in Y2.
Put $y = x - 4$ in Y3.
Press [F6] (DRAW).
What can you say about these lines?



4. Clear the three graphs in 3 and draw the graphs of:
$y = 2x + 1$
$y = 3x + 1$
$y = -x + 1$
What can you say about these lines?



5. Clear the three graphs in 4 and draw the graphs of:
$y = -2x$
$y = -2x + 4$
$y = -2x - 4$
What can you say about these lines?



PTO

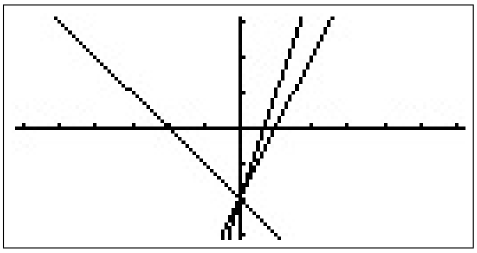
6. Clear the three graphs in 5 and draw the graphs of:

$y = 2x - 2$

$y = 3x - 2$

$y = -x - 2$

What can you say about these lines?

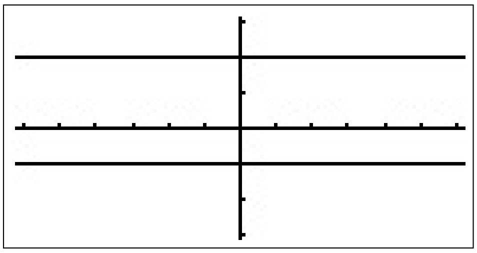


7. Clear the three graphs in 6 and draw the graphs of:

$y = 2$

$y = -1$

What can you say about these lines?



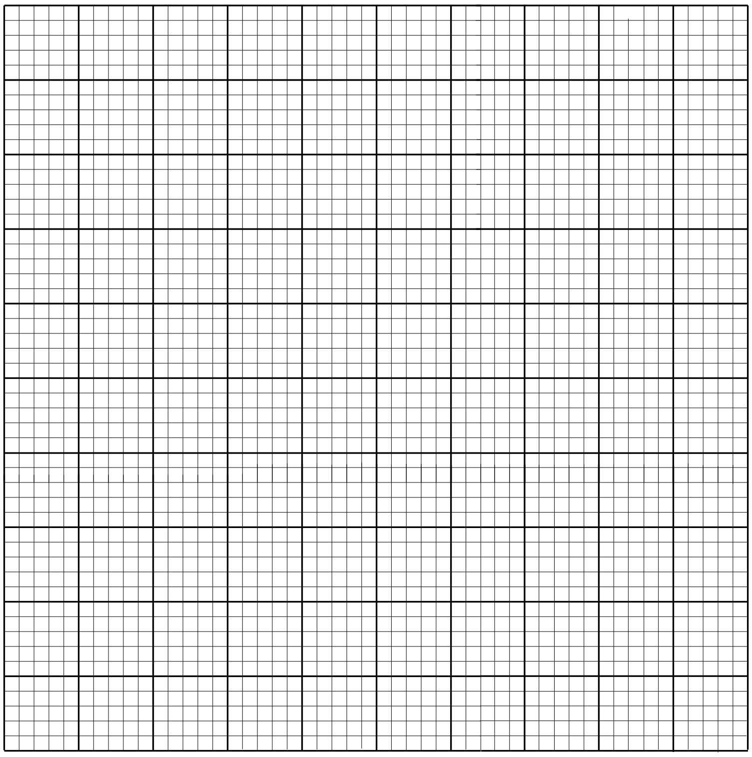
Graphing Straight Lines   Worksheet 2

Clear all current graphs from your calculator.

Press SHIFT F3 (V-Window) F1 (INIT) for 'decimal' axes.

1. Graph the function $y=x$ in Y1.

Draw a sketch on the number plane below. The thick lines are 1 unit apart.



2. Graph the function $y=x+1$ in Y2.

(i) What is the y intercept of $y=x+1$?

(ii) Draw a sketch of $y=x+1$ on the number plane above.

3. Graph the function $y=x+2$ in Y3.

(i) What is the y intercept of $y=x+2$?

(ii) Draw a sketch of $y=x+2$ on the number plane above.

4. Graph the function $y=x-1$ in Y4.

(i) What is the y intercept of $y=x-1$?

(ii) Draw a sketch of $y=x-1$ on the number plane above.

5. What are the y intercepts for the following curves?

$y=x+7$	$y=x+2.7$	**Check these answers**
$y=x-2$	$y=x-3.5$	**using your calculator**

6. Try to generalise your results, i.e. what is the y intercept of $y=x+b$, where b is any number? Test your conjecture with some more graphs.

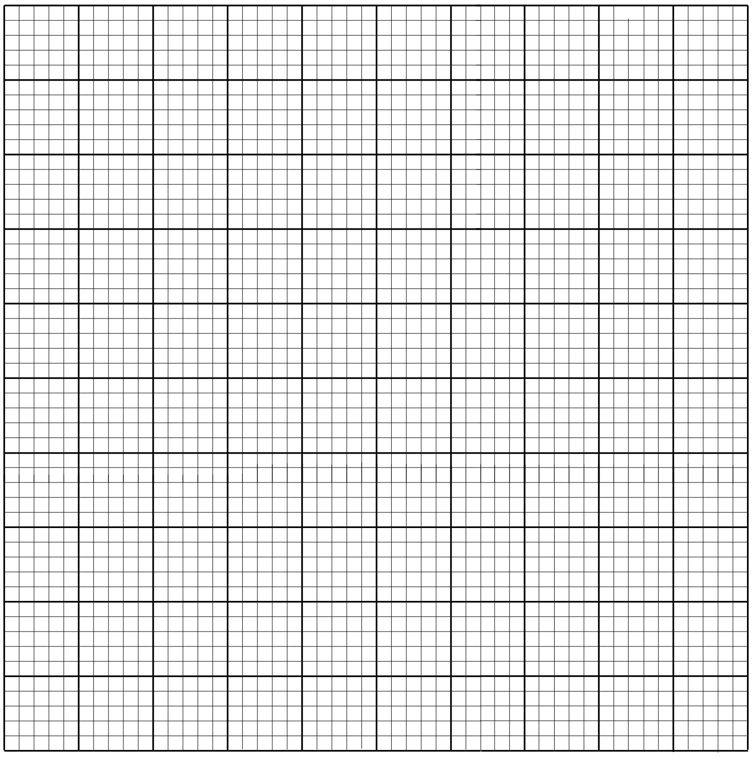
Graphing Straight Lines   Worksheet 3

Clear all current graphs from your calculator.

Change the V-Window of your calculator so that $-5 < x < 5$.

1. Graph the function $y = 2x/3$ in Y1.

Draw a sketch on the number plane below.



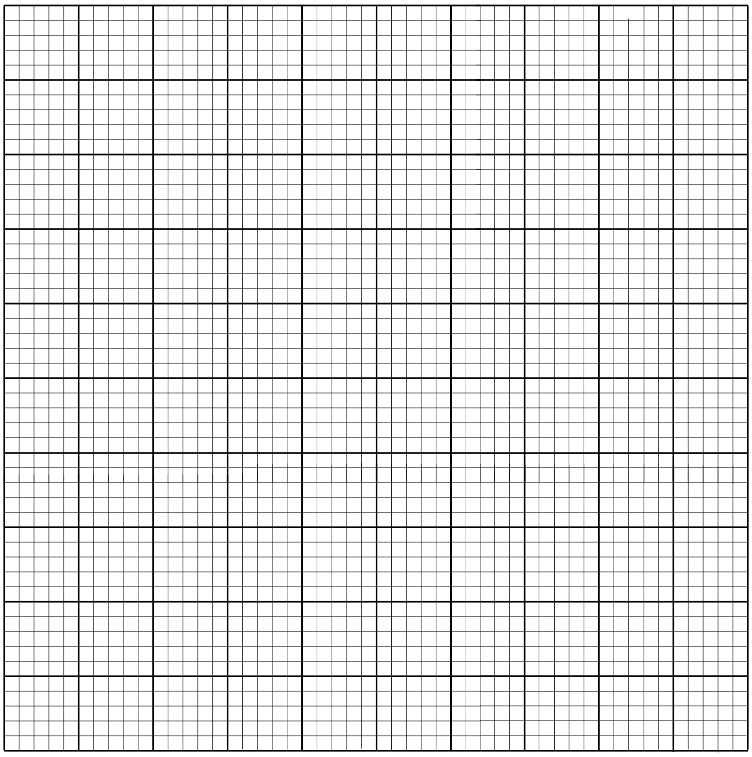
2. Imagine you are a trace dot. Starting at the origin, move in a positive horizontal (x) direction for 3 units ('run'), then in a positive vertical (y) direction ('rise') for 2 units. Where do you end up?   Coordinates: $x =$___ $y =$___

3. Draw the triangle on the grid that you have traced by this move.
What sort of triangle is it?   ____________

PTO

4. Graph the function $y = x/4$ in Y2.

Draw a sketch on the number plane below.



5. Move from the origin in a positive x direction for 4 units, then in a positive y direction for 1 unit. Is the triangle produced the same type as in Question 3? _________

6. Do you notice a pattern between the x coefficient in the equations and the 'rise' and 'run' of your triangles?

____

PTO

7. Graph the following functions on your calculator and see if the results are consistent with your previous findings: $y = 0.4x$; $y = x/3$; $y = 2x$; $y = 3x$.

8. Complete the tables below for each function in 7, putting them in order, left to right, from the smallest x coefficient to the largest.

Choose three x values and calculate the corresponding y values for each equation. Use these to draw each graph on the number plane below.

$y =$

$\boldsymbol{x}$			
$\boldsymbol{y}$			

$y =$

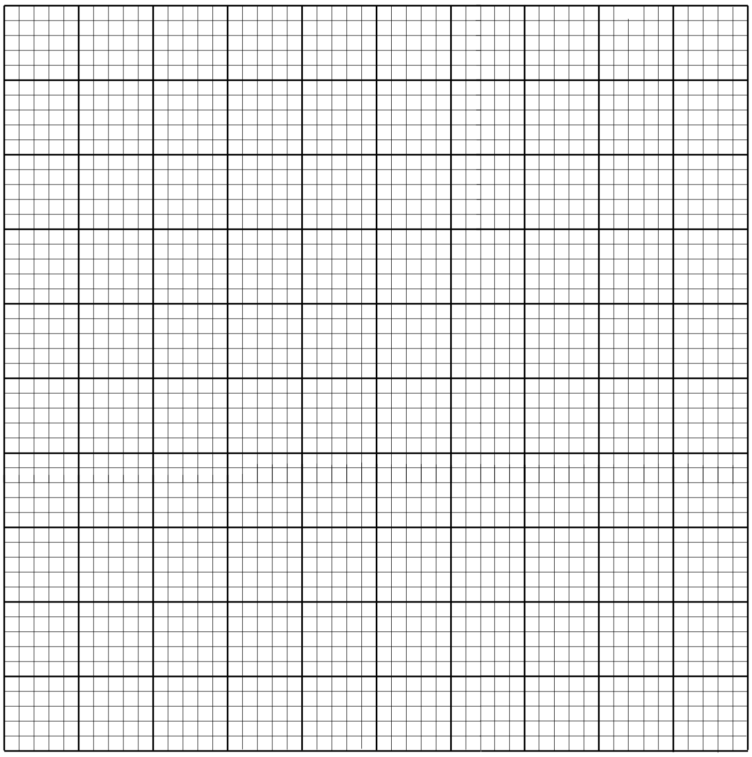
$\boldsymbol{x}$			
$\boldsymbol{y}$			

$y =$

$\boldsymbol{x}$			
$\boldsymbol{y}$			

$y =$

$\boldsymbol{x}$			
$\boldsymbol{y}$			



9. Is there a relationship between the size of the coefficient and the steepness (or slope) of the line?

____

____

10. Having completed Worksheets 1, 2 and now 3, can you generalise your results, i.e. say what happens to the graph of $y = mx + b$ when m and b are changed?

____

____

____

____

6 Guess the Line

Years 9, 10, Levels 1, 2; Strand: Algebra; Sub-strand: Coordinate Geometry – Straight Lines.
Author: Margie Smith.
Guess the equations of straight lines generated by the calculator. The calculator keeps score.

GUESSLIN program
The program generates random values for M and B in the straight line Y = MX + B, and graphs the line. B is a non-zero integer between −4 and 4. M is a non-zero integer between −3 and 3 divided by B. From the graph, you have to guess values for M and B.

1. Select PRGM to bring up the program list (Fig. 1).

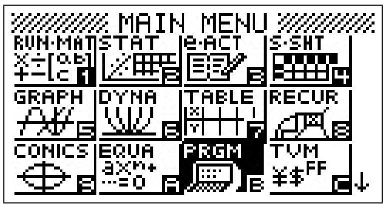


Figure 1

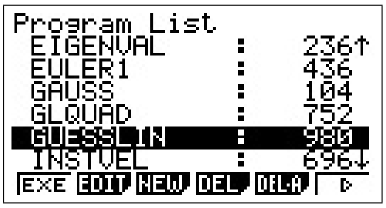


Figure 2

2. Scroll down to highlight GUESSLIN (Fig. 2), then press EXE.
3. Use EXE to continue through the introduction (Figs. 3 and 4) until you come to the graph of a straight line generated by the calculator (Fig. 5).

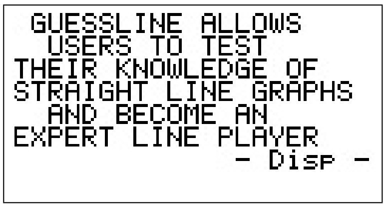


Figure 3

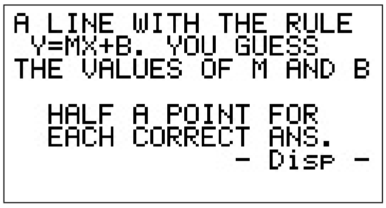


Figure 4

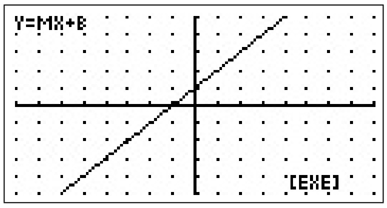


Figure 5

4. At this point, you should work out the gradient of the line M and the Y intercept B.
YOU CANNOT ACCESS THE GRAPH AGAIN ONCE YOU PROCEED!
5. Press EXE: the program prompts you to input a value for M (EXE) and for B (Fig. 6).

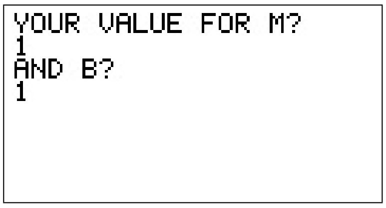


Figure 6

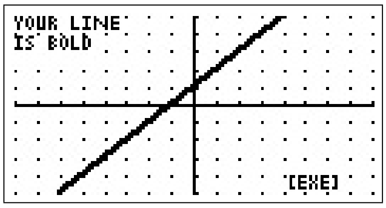


Figure 7

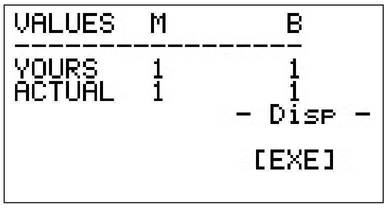


Figure 8

6. Press EXE again to redraw the line.; a bold line is then drawn using the values you inputted for M and B — hopefully the original line will turn bold (Fig. 7).
7. Press EXE to display your values for M and B, and the actual values (Fig. 8).
8. Press EXE to play again or quit (Fig. 9). If you quit, your final score is shown (Fig. 10).

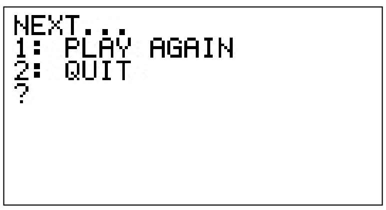


Figure 9

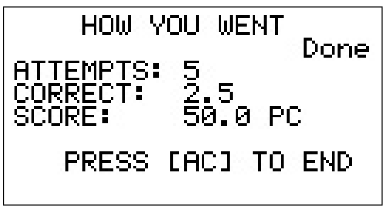


Figure 10

7 Let's Be Rational

Year 10, Level 1; Strand: Algebra; Sub-strand: Sketching Other Graphs, Polynomials.

Author: Brenda Batten, Ridgeland, South Carolina, USA. Modified by Peter McIntyre.

Understanding the local and global behaviour of rational functions.

The first five letters in *rational* spell *ratio.*

- A rational **number** is the ratio of two **integers**.
- A rational **function** is the ratio of two **polynomials**.
  The simplest rational function is $y = 1/x$.
  It is the ratio of the polynomials $f(x) = 1$ and $g(x) = x$.

There are two types of behaviour we need to investigate in rational functions:

1. when x values are very small in absolute value (*up-close and personal*);

2. when x values are large in absolute value (*like an astronaut*).

Understanding these two features, the **local and global behaviour** of rational functions, will allow you to appreciate the properties of these functions and to sketch graphs of rational functions . . . without the aid of a graphics calculator.

To reach that point, however, the capabilities of a graphics calculator are very useful.

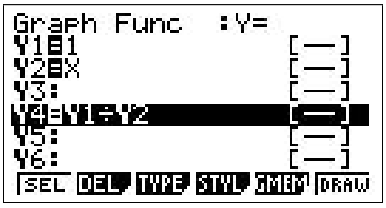
PTO to begin

Local behaviour: Up-close and personal

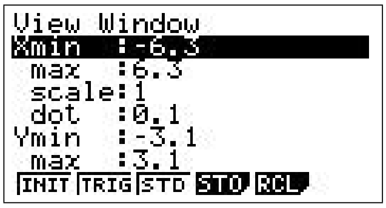
Graphically: In MENU 5 (GRAPH), set up your calculator to graph Y1 = 1, Y2 = X and Y4 = Y1÷Y2 (figure below left). We'll come back to Y3 shortly.

Y is in the VARS GRPH menu; you can't just use the letter Y.

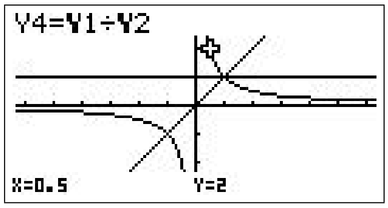
In V-Window, press INIT (below centre), then EXIT DRAW and Trace (below right).



Define functions



'Friendly' window



Trace along graph

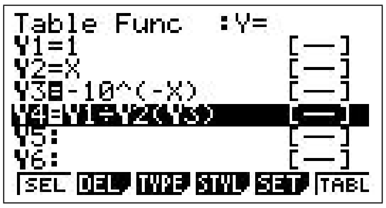
What View Window does INIT produce? ________________________________

Numerically: Here, we'd like to calculate values of Y1/Y2 at values of X closer and closer to 0. The standard TABLE makes this a slow process, so we look to speed it up.

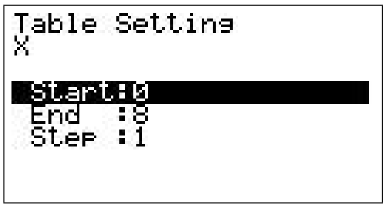
Press MENU 7 (TABLE) and set up your calculator to produce a table of values of Y3 = 10^(−X) and Y4 = Y1÷Y2 (Y3) (figure below left).

Y3 is our new independent variable: as X goes 0, 1, 2, …, Y3 goes $-10^0 = -1$, $-10^{-1} = -0.1$, $-10^{-2} = -0.01$, …, i.e. approaches 0 from below much faster than X.

Set the table range in SET (below centre), then press EXIT TABL for the table (below right).



Turn off Y1 and Y2



SET

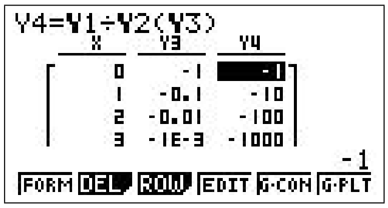


Table values

Remember that Y3 is the X value that produces the Y4 value; ignore the X column here.

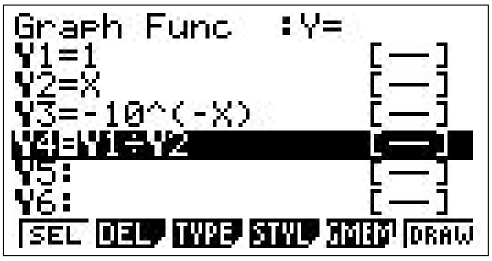
Use these two calculator modes to answer Questions 1, 2 and 3 below.

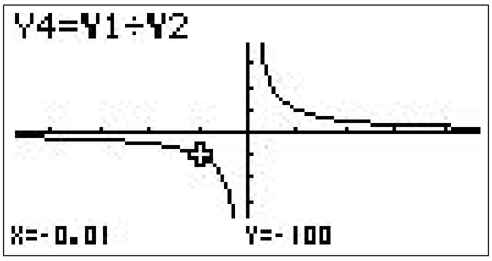
1. As the x values approach 0 from the left, what happens numerically to the y values?

2. What happens graphically?

Look at Y4 on the graph on the previous page, then turn off Y3, set Y4 = Y1÷Y2 again and graph it with a V-Window of $[-0.047, 0.047, 0.01] \times [-500, 500, 100]$.

Use Trace to investigate.



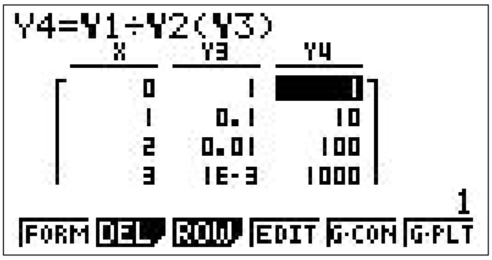


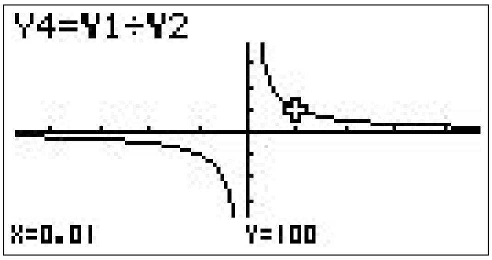
Bigger window

In words: As x tends to 0 from the left, $f(x)$ tends to ______________

In symbols: As $x \to 0^-$, $f(x) \to$ ______________

3. As the x values approach zero from the right, what happens to the y values?





Remenber that Y3 is the X value that produces the Y4 value; ignore the X column here.

In words: As x tends to 0 from the right, $f(x)$ tends to ______________

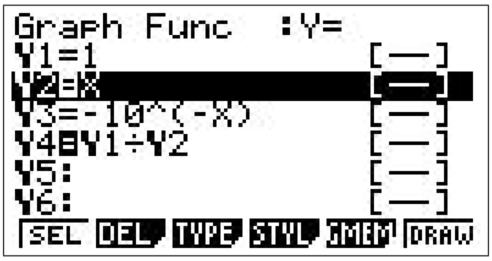
In symbols: As $x \to 0^+$, $f(x) \to$ ______________

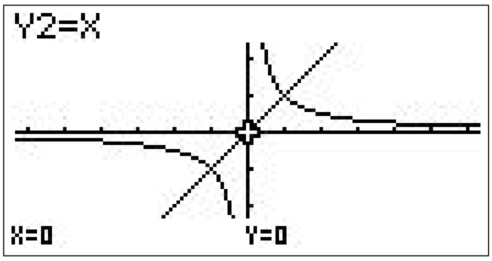
The rational function $f(x) = 1/x$ is undefined at $x = 0$, as the denominator is zero at that point. The value of the numerator is not zero. When both of these conditions are true, there is a ________________________ on the graph.

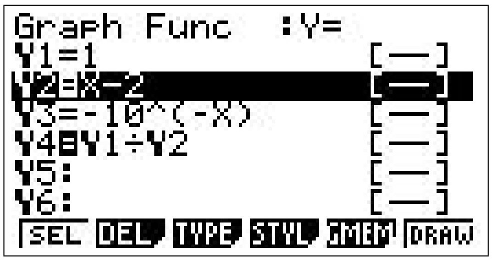
The role of the denominator

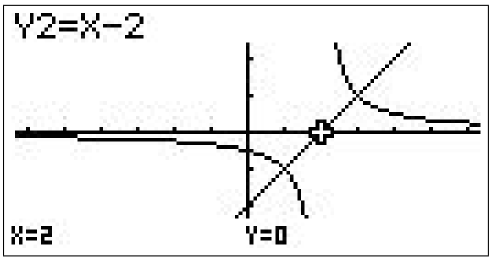
Now look at the functions $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x-2}$.

Use a V-Window INIT window.





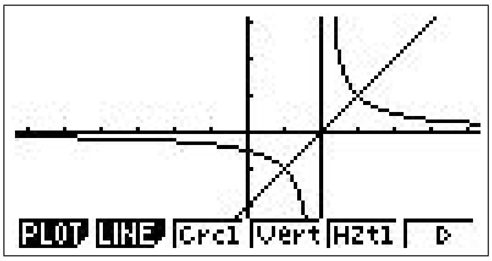


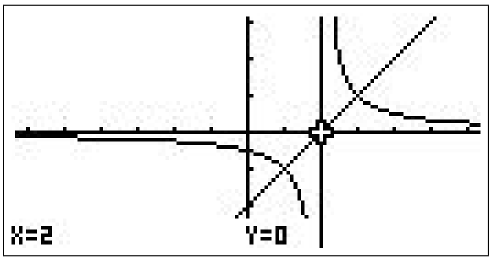


1. For what values of x is the denominator of the rational function in each case equal to zero?

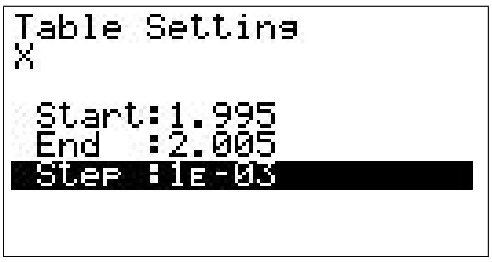
2. What is the equation of the vertical asymptote in each case?

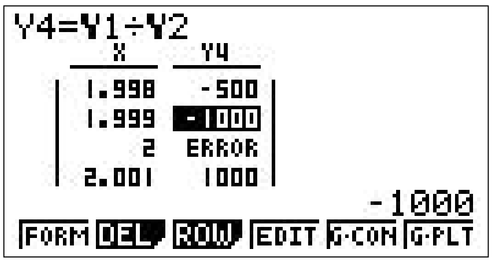
To plot the vertical asymptote, use Vert in the Sketch menu: use the arrow keys to move the line to the correct position.





Now let's investigate numerically values of the second function, $g(x) = \dfrac{1}{x-2}$, when x is close to 2.





3. For what values of x is the rational function Y4 undefined?

4. As the x values approach that value from the left, what happens numerically to the y values?

5. As the x values approach that value from the right, what happens numerically to the y values?

As $x \to 2^-$, $g(x) \to$ _____ and, as $x \to 2^+$, $g(x) \to$ _____

The equation of the vertical asymptote is ___________

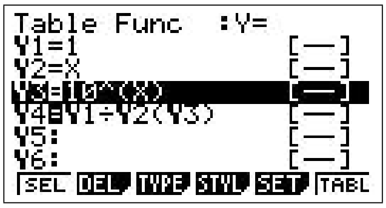
Concept check: What would the equation of the vertical asymptote be if Y2 were changed to $x+3$?

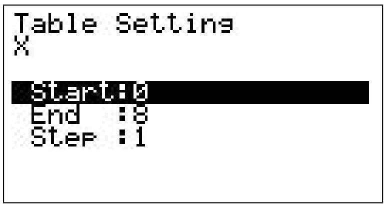
Global behaviour: Like an astronaut

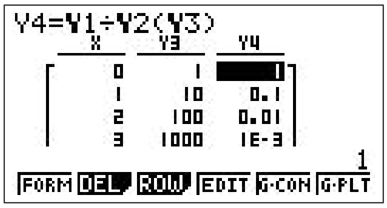
What happens at the extreme 'edges' of the graph of a rational function?

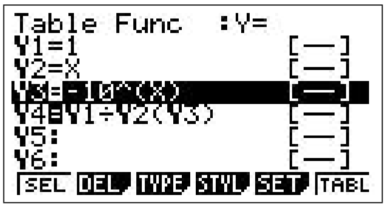
We have investigated the behaviour when values of x get close to the zeros of the denominator. Now we turn our attention the behaviour of the function when the values of x get very large, either positive or negative, that is when the absolute values of x get very large or 'tend to infinity'.

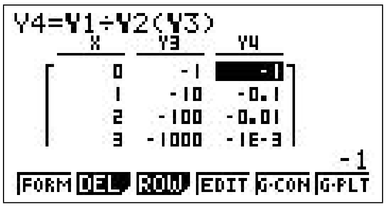
The calculator can do the arithmetic for us.











Remenber that Y3 is the X value that produces the Y4 value; ignore the X column here.

As x values get very large and positive, we say x approaches positive infinity.

Here the function values (Y4) then approach ____________

We write: As $x \to \infty$, $f(x) \to$ ____________

As x values get very large and negative, we say x approaches negative infinity.

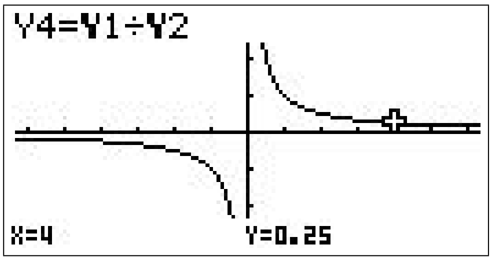
Here the function values then approach ____________

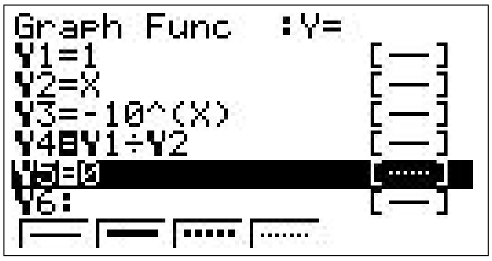
We write: As $x \to -\infty$, $f(x) \to$ ____________

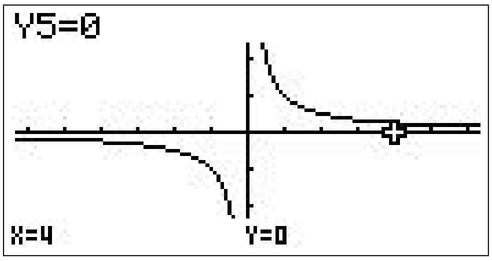
The Rule of Four

We have investigated global behaviour **numerically** and written it **algebraically**. Now investigate global behaviour **graphically**.

Set Y2 = X again, then turn it off so we are just plotting Y4. Use a V-Window INIT window. Press Trace.

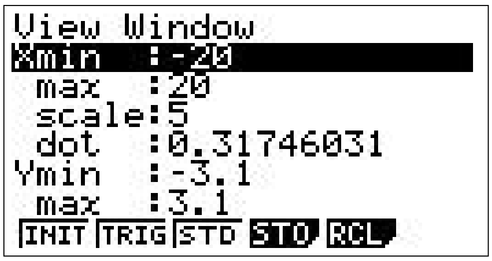


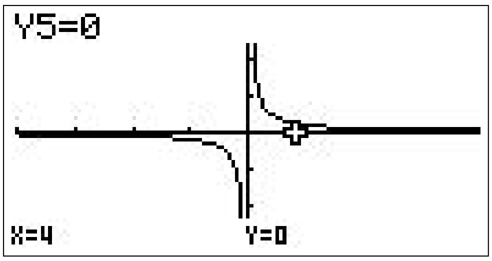




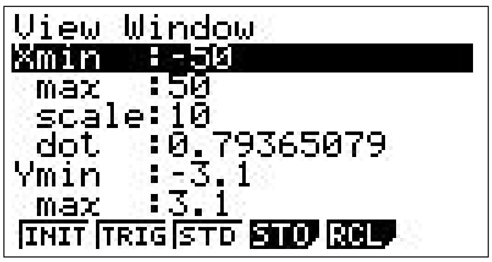
Define Y5

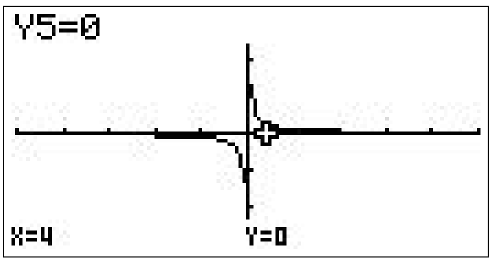
Plot Y4 and Y5





Broaden X window



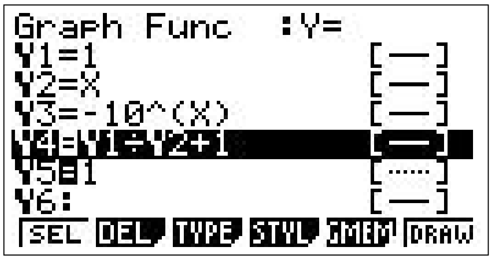


Broaden X window

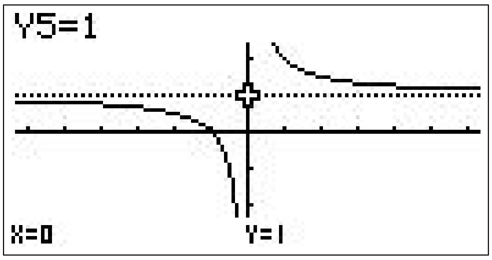
Graphically: (describe what happens as we broaden the window)

There is a horizontal asymptote at ____________

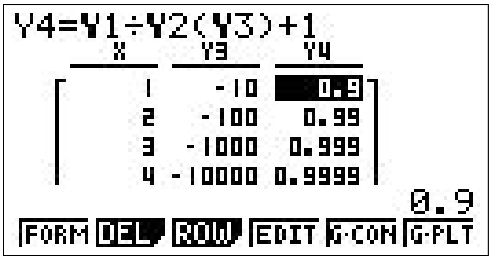
For a different horizontal asymptote, add a constant to the rational function. This will increase the value of every y coordinate by that amount. It shifts the graph vertically. Here the constant is 1.

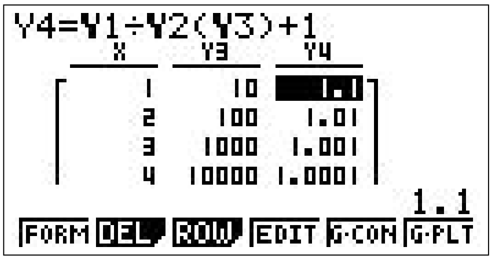


Y5 = 1 is an asymptote



Graphically





Numerically

Remenber that Y3 is the X value that produces the Y4 value; ignore the X column here.

Verbally: As x tends to $\pm\infty$, $f(x)$ tends to ____________

Exercise: Predict the behaviour of the graph of $f(x) = \dfrac{1}{x-2} + 1$.

1. Which part of the function determines the position of the vertical asymptote?
2. What is the equation of the vertical asymptote? Sketch the vertical asymptote **without** the aid of a graphics calculator.
3. Which part of the function determines the position of the horizontal asymptote?
4. What is the equation of the horizontal asymptote? Sketch the horizontal asymptote **without** the aid of a graphics calculator.
5. Now sketch the graph of the function **without** the aid of a graphics calculator.
6. Confirm your conjecture in 5 using a graphics calculator.

Homework

For each of the following rational functions

1. $y = \dfrac{1}{x+5} - 3$ **2**. $y = \dfrac{1}{x-1} + 1$ **3**. $y = \dfrac{1}{x} - 2$

(a) write the equation of the vertical asymptote.

(b) write the equation of the horizontal asymptote.

(c) sketch the graphs of (a) and (b) **without** the aid of a graphics calculator.

(d) sketch the graph that you predict for the rational function **without** the aid of a graphics calculator.

(e) confirm your conjecture in (d) using a graphics calculator.

(f) combine the terms of each function so that it is written as the ratio of two polynomials.

Notes for Teachers

Local behaviour: Up-close and personal

What window does [V-Window] [INIT] produce?

$-6.3 < x < 6.3$, $-3.1 < y < 3.1$ or $[-6.3, 6.3, 1] \times [-3.1, 3.1, 1]$

1. As the x values approach 0 from the left, what happens numerically to the y values?

As the x values approach 0 from the left, the y values become more and more negative.

2. What does it mean graphically?

As the x values approach 0 from the left, the graph of the function drops down further and further, and comes closer and closer to the negative y axis.

In words: As x tends to 0 from below, $f(x)$ tends to negative infinity.

In symbols: As $x \to 0^-$, $f(x) \to -\infty$.

3. As the x values approach zero from the right, what happens to the y values?

As the x values approach 0 from the left, the y values become larger and larger.

In words: As x tends to 0 from above, $f(x)$ tends to positive infinity.

In symbols: As $x \to 0^+$, $f(x) \to \infty$.

The rational function $f(x) = 1/x$ is undefined at $x = 0$, since the denominator is zero at that point. The value of the numerator is not zero. When both of these conditions are true, there is a vertical asymptote on the graph.

The role of the denominator

1. For what values of x is the denominator of the rational function in each case equal to zero?
$f(x)=0$ when $x=0$. $g(x)=0$ when $x=2$.

2. What is the equation of the vertical asymptote in each case?
The vertical asymptote for f is (the vertical line) $x=0$.
The vertical asymptote for g is $x=2$.

Now investigate numerically the values of the second function, $g(x)=\frac{1}{x-2}$, when x is close to 2.

3. For what values of x is $g(x)$ undefined?
$g(x)$ is undefined at $x=2$

4. As the x values approach that value from the left, what happens numerically to the y values?
As the x values approach 2 from the left, the y values tend to negative infinity.

5. As the x values approach that value from the right, what happens numerically to the y values?
As the x values approach 2 from the right, the y values tend to positive infinity.

As $x \to 2^-$, $g(x) \to \underline{-\infty}$ and, as $x \to 2^+$, $g(x) \to \underline{+\infty}$.

The equation of the vertical asymptote is $\underline{x=2}$.

Concept check: What would the equation of the vertical asymptote be if Y2 were changed to $x+3$?

The rational function would now be $h(x) = 1/(x+3)$: the denominator has a zero at $x=-3$, the numerator is non-zero there (and everywhere else), so that the vertical asymptote is $x=-3$.

Global behaviour: Like an astronaut

What happens at the extreme 'edges' of the graph of a rational function?

We have investigated the behaviour when values of x get close to the zeros of the denominator. Now we turn our attention the behaviour of the function when the values of x get very large, either positive or negative, that is when the absolute values of x get very large or 'tend to infinity'.

As x values get very large and positive, we say x tends to positive infinity.

Here the function values (Y3) then approach zero from above.

We write: As $x \to \infty$, $f(x) \to \underline{0^+}$.

As x values get very large and negative, we say x tends to negative infinity.

Here the function values then tend to zero from below.

We write: As $x \to -\infty$, $f(x) \to \underline{0^-}$.

The Rule of Four

We have investigated global behaviour **numerically** and written it **algebraically**. Now investigate global behaviour **graphically** and state it **verbally**.

Graphically: As the x window gets broader and broader, the graph of Y3 becomes indistinguishable from that of the asymptote, illustrating the global behaviour of the function.

For a different horizontal asymptote, add a constant to the rational function. This will increase the value of every y coordinate by that amount. It shifts the graph vertically. Here the constant is 1.

There is a horizontal asymptote at $\underline{y=0}$.

Verbally: As x tends to positive or negative infinity, $f(x)$ tends to $\underline{1}$.

Exercise: Predict the behaviour of the graph of $f(x) = \dfrac{1}{x-2} + 1$.

1. Which part of the function determines the position of the vertical asymptote?

   The denominator (or more specifically, the zero of the denominator) determines the position of the vertical asymptote.

2. What is the equation of the vertical asymptote? Sketch the vertical asymptote **without** the aid of a graphics calculator.

   The equation of the vertical asymptote is $x = 2$.

3. Which part of the function determines the position of the horizontal asymptote?

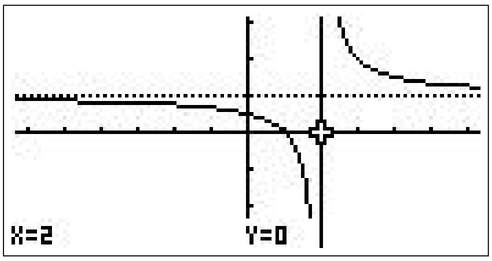
   The number that is added to the rational function determines the position of the horizontal asymptote.

4. What is the equation of the horizontal asymptote? Sketch the horizontal asymptote **without** the aid of a graphics calculator.

   The equation of the horizontal asymptote is $y = 1$.

6. Confirm your conjecture in Exercise 5 using a graphics calculator.

   V-Window INIT



Homework

For each of the following rational functions

1. $y = \dfrac{1}{x+5} - 3$ **2.** $y = \dfrac{1}{x-1} + 1$ **3.** $y = \dfrac{1}{x} - 2$

(a) write the equation of the vertical asymptote.

(b) write the equation of the horizontal asymptote.

(c) sketch the graphs of (a) and (b) **without** the aid of a graphics calculator.

(d) sketch the graph that you predict for the rational function **without** the aid of a graphics calculator.

(e) confirm your conjecture in (d) using a graphics calculator.

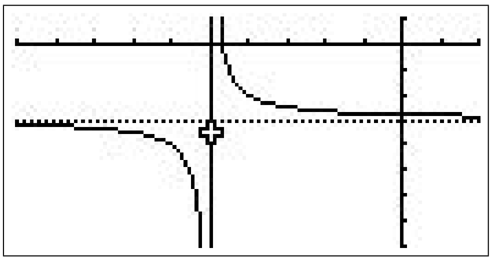
(f) combine the terms of each function so that it is written as the ratio of two polynomials.

Answers

1. **(a)** The equation of the vertical asymptote is $x = -5$.

(b) The equation of the horizontal asymptote is $y = -3$.

(e)



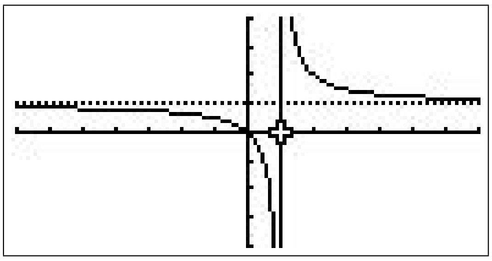
V-Window $[-10, 2, 1] \times [-8, 1, 1]$

(f) $y = \dfrac{1}{x+5} - 3 = \dfrac{1 - 3(x+5)}{x+5} = \dfrac{-14 - 3x}{x+5}.$

2. **(a)** The equation of the vertical asymptote is $x = 1$.

(b) The equation of the horizontal asymptote is $y = 1$.

(e)



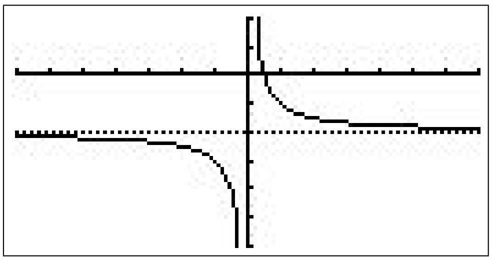
View Window $[-7, 7, 1] \times [-4, 4, 1]$

(f) $y = \dfrac{1}{x-1} + 1 = \dfrac{1 + (x-1)}{x-1} = \dfrac{x}{x-1}.$

3. **(a)** The equation of the vertical asymptote is $x = 0$.

(b) The equation of the horizontal asymptote is $y = -2$.

(e)



V-Window $[-7, 7, 1] \times [-6, 2, 1]$

(f) $y = \frac{1}{x} - 2 = \frac{1 - 2(x)}{x} = \frac{1-2x}{x}$.

8 Parabolic Aerobics

Year 9, Levels 1 & 2; Strand: Algebra; Sub-strand: Coordinate Geometry.

Based on Parabola Guessing Game, an activity from Activities Integrating the TI-83+ into Algebra by Vicki Fortson Shirley. Modified by Peter McIntyre.

The first activity investigates the effect of changing the numbers A, B and C on the graphs of the family of parabolas Y=A(X−B)^2+C. In the second activity, you have to guess the numbers A, B and C for the graph of a mystery parabola generated by the calculator. The calculator checks your answers and keeps score.

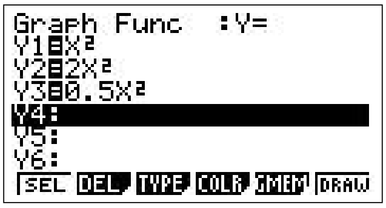
Warm-ups

Press [MENU] [5] or use the cursor and [EXE] to select GRAPH mode from the Main Menu.

Set a V-Window for plotting the graphs by pressing [SHIFT] [F3] (V-Window), then [F1] (INIT).

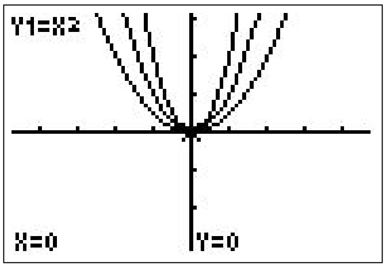
Press [EXIT] to return to the Graph Func screen.

1. Stretches and reflections



Set $Y1 = X^2$, $Y2 = 2X^2$ and $Y3 = 0.5X^2$.

X is produced with the [X,θ,T] key.



Press [F6] (DRAW), then [F1] (Trace).

Use the arrow keys to see which graph is which.

Let A stand for the number multiplying X^2. You have just plotted graphs for $A = 1$, $A = 2$ and $A = 0.5$.

Use your graphs to decide what happens when you multiply X^2 by different positive numbers. Test your ideas with some other values of A, that is by graphing $Y = AX^2$ for different values of A. Press [F6] to go back to the Graph Func screen. Write down your conclusions in the space below.

What if A is a negative number? Again, test your ideas by plotting suitable graphs. Write down your conclusions in the space below.

2. Shifts or translations

Set $Y1 = X^2$, $Y2 = (X-2)^2$ and $Y3 = (X+3)^2$. What happens when you vary the number B in the family of graphs $Y = (X-B)^2$? Test your ideas by trying some more values of B. Write down your conclusions in the space below.

Set $Y1 = X^2$, $Y2 = X^2 + 1$ and $Y3 = X^2 - 2$. What happens when you vary the number C in the family of graphs $Y = X^2 + C$? Test your ideas by trying some more values of C. Write down your conclusions in the space below.

3. Summary

In the space below, summarise the effects of changing the numbers A, B and C in the family of graphs $Y = A(X-B)^2 + C$.

What parabola is that?

Press MENU. With the cursor and EXE, select the PRGM icon.

Run the PARABOLA program by moving the cursor to PARABOLA and pressing F1 (EXE).

This program plots the graph of a mystery parabola $Y = A(X-B)^2 + C$ as a solid line, where A, B and C are generated randomly. The program also plots the basic parabola $Y = X^2$ as a dotted curve to use for comparison.

Just to make it a bit easier, A, B and C can take only a restricted number of values.

A: ±0.5, ±1 or ±2.

B, C: 0, ±1 or ±2.

Your job is to decide what values of A, B and C the calculator has chosen. When you have decided, press EXE and input your values. The calculator will then plot the mystery parabola again as a normal line and the parabola with your values of A, B and C as an bold line.
Did you get the right values?

Press EXE again to see the your values and the calculator values for A, B and C, and EXE once more for the NEXT ... menu.

Here you can generate another mystery parabola by pressing 1 followed by EXE or quit by pressing 2. When you quit, you will see your final score.

The PARABOLA program is available at *canberramaths.org.au* under *Resources.*

9 Probably Finding π

Year 10, Levels 1 & 2; Strand: Chance and Data; Sub-strand: Probability.

Author: Michael McNally, Lower Canada College, Montreal, Canada.
Modified by Peter McIntyre.

An experimental-probability method for finding π.

1. **Turn off the axes**

   Press MENU 1 for RUN mode, SHIFT MENU for SET UP, scroll down and set *Axes Off*. Press EXIT.

2. **Set a View Window**

   Press SHIFT F3 for V-Window and set the values shown in the figure, pressing EXE after each value. Press EXIT.

   Press MENU 5 (GRAPH) and make sure all the functions are unselected. Press EXIT.

3. **Draw and store a unit circle**

   Press SHIFT F4 for Sketch.

   Press F1 EXE to clear any existing graphics.

   Press F6 and then F3 for Crcl, which puts *Circle* on the screen.
   Complete the command *Circle 0, 0, 1* and press EXE to draw the circle.

   Press OPTN, F1 (PICT), F1 (STO) and F1 again to store the circle picture in Pic1.

   To have the circle as a background to all plots, press MENU 2 (STAT), SHIFT MENU for SET UP, scroll down and set *Background* to Pict1.

   While you are here, check that *Stat Wind* (the first item) is set to Manual.
   Press EXIT.

4. **Store the coordinates of 50 random points**

   Press MENU 1 to return to RUN mode. Press OPTN F1 to select the List menu. Enter the command below and press EXE.

   seq $(-4 + 8\text{Ran\#}, \text{X}, 1, 50, 1) \rightarrow$ List 1

   Seq: F5.

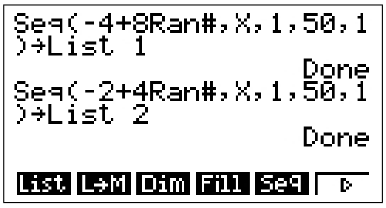
   Ran#: EXIT F6 F3 (PROB) F4.

   List: OPTN F1 (LIST) F1.

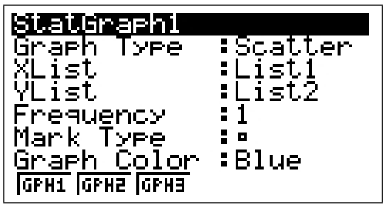
   The arrow represents the STO key.

   This stores the X coordinates of the points in List 1. To store the Y coordinates in List 2, use the left arrow to recall the Seq command you just executed, edit it to give the command below and press EXE.

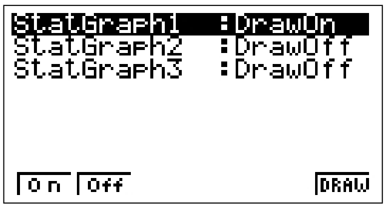
   seq $(-2 + 4\text{Ran\#}, \text{X}, 1, 50, 1) \rightarrow$ List 2



5. Set up the plot screen

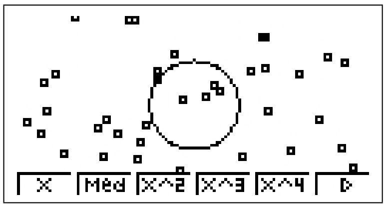


Press MENU 2 (STAT), F1 for GRPH and F6 for SET.
Set up *StatGraph1* as shown in the figure.
Press EXIT.



Press F4 for SEL. Turn on *StatGraph1*.

6. Plot the 50 random points on the screen



Press F6 (DRAW) to plot the points and the circle.

How many points landed inside the circle? If a point lands on the circle, toss a coin to decide if it is in or out.

7. Answer the following questions to find an estimate for π.

(a) What is the area of the View Window?

(b) What is the area of the circle?

(c) What is the theoretical probability of a random point on the screen landing in the circle? *Hint*: Think areas.

(d) What is the experimental probability? *Hint*: How many points are there on the screen? How many of these lie in the circle?

(e) Assuming the experimental probability and the theoretical probability are approximately equal, find an estimate for π.

Pool a series of results to obtain a better estimate for π. To generate another experiment, repeat Step 4. Then press MENU 2 (STAT), F1 for GRPH and F1 for GPH1.

When you have finished, turn off the background (see Step 3 above) and StatGraph1 (Step 5), and clear all the lists: press EXIT until the lists appear. Press F6 and then F4 (DEL·A) for each list.

Notes for Teachers

Calculator operations

The X coordinates of each point generated are stored in List 1 by the first *Seq* command. The *Ran#* command generates a random number between (but not equal to) 0 and 1. Therefore, 8*Ran#* generates a random number between 0 and 8, and $-4+8$*Ran#* a random number between -4 and 4, the range of x values in the View Window.

Similarly, the second *Seq* command, containing $-2+4$*Ran#*, generates random numbers between -2 and 2 for the Y coordinates and stores them in List 2.

The command Circle x, y, r draws a circle, centre (x, y) and radius r, on the screen.

Question 7

(a) The area of the WINDOW is $8\times4=32$.

(b) The area of the circle is $\pi\times1^2=\pi$.

(c) The theoretical probability of a point in the View Window lying in the circle is the ratio of the area of the circle to the area of the View Window, i.e. $\pi/32$.

(d) The experimental probability is the ratio of the number of points (N say) in the circle to the total number of points in the View Window, i.e. $N/50$.

(e) If we assume that the experimental probability and the theoretical probability are approximately equal, we have

$$\frac{\pi}{32} \approx \frac{N}{50},$$

so that

$$\pi \approx \frac{32N}{50} = 0.64N.$$

Counting N then gives us an estimate for π.

It's a good idea to pool the data from all the students to obtain (hopefully) a better estimate for π than individual students will obtain. You could discuss why more data should give a better estimate (experimental probability $\to$ theoretical probability as the number of data points $\to\infty$).

If you pool the data, find the mean number of points that land in the circle and multiply by 0.64 to find the mean estimate for π. It is easier to average integers than to average all the individual estimates for π.

You can gain some idea of the expected accuracy in your estimate for π by using the mean $N \pm$ one standard deviation as values for N in the formula. Does the actual value of π lie in this range?

Extension

How would you get the calculator to decide whether a point (x, y) lies inside the circle?

A point (x, y) lies inside the circle if $\sqrt{x^2+y^2} < 1$ or, equivalently, if $x^2+y^2 < 1$.

Write a calculator program to generate the random points, calculate the number lying inside the circle and hence estimate π automatically.

Further research

Find out about Buffon's needle problem, a 'hands-on' forerunner of this problem.

10 Reaction Times and Statistics

Years 9 &10, Levels 1 & 2; Strand: Chance and Data; Sub-strand: Statistics.

Authors: Jamie Alford and Dan Graeme, Queensland University; Peter McIntyre, UNSW.
Modified by Margie Smith and Peter McIntyre.

Programs are used to measure reaction times in various scenarios, including driving a car. The data are displayed as box-and-whisker plots for subsequent analysis.

Introduction

There are three activities here, based around a common theme of reaction times. In each activity, a calculator program is used to measure and record reaction times under different conditions.

1. Use the REACTHND program to measure reaction times using first one hand, then the other. Analyse the resulting box-and-whisker plots, one for each hand, to draw conclusions from your data.

Based on a presentation by Jamie Alford of the University of Queensland.

2. Use the REACT program to measure reaction times under different conditions:

(a) intense concentration and complete silence;

(b) relaxed in relative quiet;

(c) distracted in a noisy environment.

These conditions pertain to a person's reaction time when driving a car, and therefore to the time to apply the brakes and stopping distances. The data are plotted as box-and-whisker plots and are also available in lists for further analysis.

Real-World Statistics #1

Introduction

You are a statistician employed by Microsoft. You have been commissioned to assist in the design of a new control pad for the XBOX® III. The data you are tasked to compile are the reaction times of the 'better' and 'worse' hands of users.

You are required to measure and analyse the reaction times, and to comment on the differences between the two hands. To help with your research, you are provided with a graphics-calculator program to measure reaction times and a set of steps to work through.

Method

Use the REACTHND program to measure your reaction times using *at least* 15 trials on each hand.

Rest your finger on the [F1] key while waiting for *NOW* to appear.

Write the means in the table below.

Sketch the boxplots in the space below and complete the quartile analysis in the table. Press [EXE] to plot the boxplots, then [SHIFT] [F1] (Trace) and use the arrow keys to read off the values. The data remain in List 1 and List 2.

	Left Hand	**Right Hand**
Mean		
MinX		
Q1		
Median		
Q3		
MaxX		

Analysis

Compare your results with those of others in your group. Detail any differences in the data obtained from different individuals.

From the data, describe the differences between your 'better' hand and your 'worse' hand. How accurate are these terms?

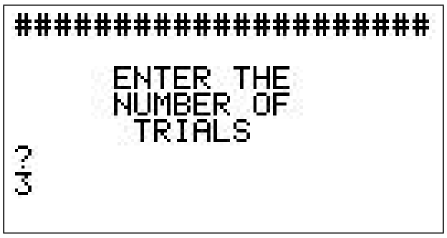
Further Activities

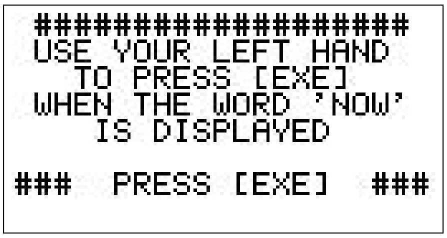
1. What, from your findings, should Microsoft consider when designing the control pad?
2. Present a rough design of the new control pad. This should be light on artistic focus, but high on thinking. Microsoft has paid designers. You are the statistician. Your job is to give a rough outline of the control pad **with a high level of justification**. Your design should be a one-third-page diagram, with the other two-thirds of the page used for justification and conclusion.
3. Reflect on the activity. Explain your work as a statistician, including formal descriptions of the types of tasks you performed and why you believe they were relevant. You should also address what you think the purpose behind statistical work is.

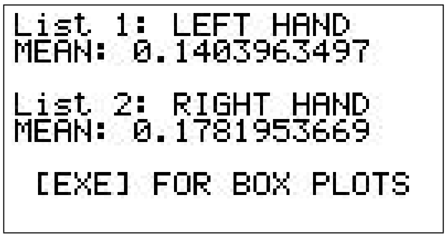
Notes for Teachers

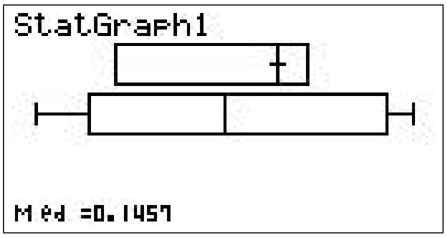
This activity uses a calculator program REACTHND to create box plots using reaction-time data collected from the students.

To run the REACTHND program, select the PRGM icon from the main menu, scroll down to the REACTHND program and press F1 to start it. Below are some of the screens that students will see in collecting the data.









Note: Students should rest their finger on the F1 key instead of having it hover over the EXE key.

At the end of the trials, the means are displayed on the screen (the data remain in List 1 and List 2). Press EXE to see the boxplots, then SHIFT F1 (Trace) and use the arrow keys to read off the values.

- MinX: minimum — smallest value in the data
- Q1: lower quartile — cutoff point for the bottom 25% of the data
- Med: median — cutoff point for the bottom 50% of the data
- Q3: upper quartile — cutoff point for the top 25% of the data
- MaxX: maximum — largest value in the data

Real-World Statistics #2

Introduction

You are a statistician employed by a car company ROLLER. In their current advertising campaign, they claim that their cars have the least road noise of any current car on the market and that their cars are therefore the safest, as drivers are not distracted as they are in other noisier makes of car.

You have been commissioned to compile data to measure the effect of noise on the reaction times of drivers. The data are required to be collected under each of the following conditions.

Scenario 1: Total concentration and complete silence.

Scenario 2: Relaxed in relative quiet.

Scenario 3: Distracted in a noisy environment.

Method

Use the graphics-calculator program REACT to measure reaction times for each of the three scenarios. The program leads you through the steps. Note that the reaction time just pressing a calculator key is likely to be significantly shorter than when you move your foot onto the brake pedal and push it. The simulation will be more realistic if you rest your finger on the [F1] key while waiting for *NOW* to be displayed.

Do *at least* 15 trials for each scenario. After you have done all the trials, the program will tell you the means (write these in the table below) and present the results as boxplots, one for each scenario.

Press [SHIFT] [F1] (Trace) and use the arrow keys to read off the values from the boxplots.

Results

Sketch your boxplots and complete the quartile analysis in the table — use the arrow keys to find all the numbers (except the means) from the boxplots.

	Scenario 1	**Scenario 2**	**Scenario 3**
Mean			
MinX			
Q1			
Median			
Q3			
MaxX			

The data are available in List 1, List 2 and List 3 for further analysis.[4]

Analysis

From your results, describe the differences between your reaction times in the three scenarios. Explain what these mean in plain English — don't just say the means are different, or something similar.

From your findings, should ROLLER continue with their current advertising campaign? Why or why not?

Compare your results and conclusions with those of other 'drivers'. Summarise any similarities and differences.

[4] Select STAT from the main menu to see the data. You can replot the boxplots by pressing F1 (GRPH), F4 (SEL), then F6 (DRAW).

Further analysis

On one calculator, enter the *median* for Scenario 1 from each participant into List 1, the median for Scenario 2 into List 2 and the median for Scenario 3 into List 3.

Press [F1] (GRPH), [F4] (SEL), then [F6] (DRAW) to plot the boxplots of these medians.

Sketch the boxplots of the class medians and fill in the quartile analysis below.

	Scenario 1	**Scenario 2**	**Scenario 3**
MinX			
Q1			
Median			
Q3			
MaxX			
Mean			

Now do the same for the *means*. Sketch the boxplots of the class means and fill in the quartile analysis below.

	Scenario 1	**Scenario 2**	**Scenario 3**
MinX			
Q1			
Median			
Q3			
MaxX			
Mean			

What are the differences between the boxplots of the medians and the boxplots of the means of the group data?

Which of these would ROLLER use to support their ads? Why?

11 Simultaneous Equations

Year 9, Levels 1 & 2; Strand: Algebra; Sub-strand: Solve Simultaneous Linear Equations.

Authors: Margie Smith, Peter McIntyre.

Solving simple simultaneous equations numerically (with a table), graphically and algebraically.

Simultaneous Equations 1

Example: Solve the simultaneous equations

$$x - y = -1 \qquad (1)$$

$$x + y = 3. \qquad (2)$$

This means we have to find pairs of values, x and y, that satisfy both equations simultaneously, i.e. such that $x-y=-1$ and $x+y=3$. Each pair of values, x, y, is called a solution of the simultaneous equations.

To use a table or graph to find the solutions, first re-write the equations with y as subject,

$$y = x + 1 \qquad (3)$$

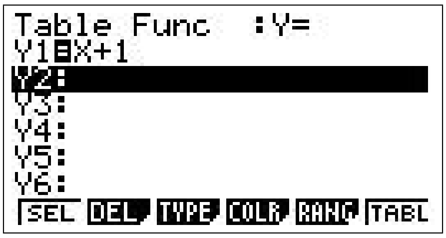
$$y = 3 - x. \qquad (4)$$

Using a table

Here we use the calculator to generate a table of values of x and the corresponding y values from each of equations (3) and (4).

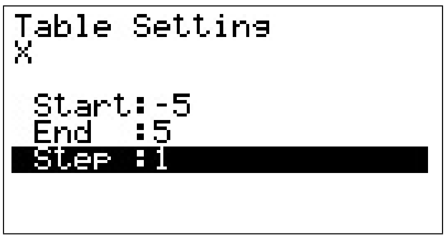
Press [MENU] [7] and set Y1 = X + 1 (Eq. (3)).

For X, press the [X,θ,T] key.



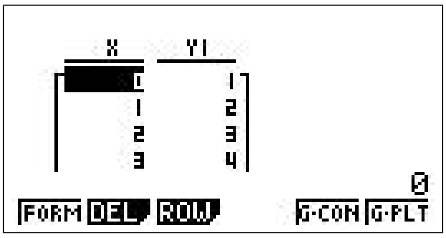
We'd like the table to start at X = −5 and increment by 1.

Press [F5] (SET): set *Start* to −5, *End* to 5 and *Step* to 1.



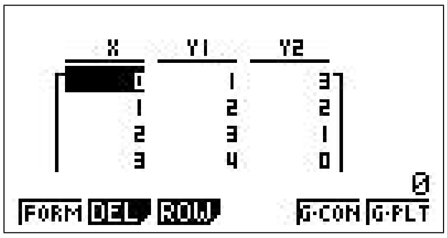
Press [EXIT] to return to the Table Func screen and [F6] (TABL) to generate the table.

For each pair of values X, Y1 on the screen, verify that it satisfies Eq. (1), i.e. verify that X − Y1 = −1, and Eq. (3), i.e. verify that Y1 = X + 1.



Now enter the second equation: press F1 (FORM) and enter Eq. (4) in Y2.

Press F6 (TABL).



Verify that each pair of values of X and Y2 on the screen satisfies both Eq. (2), i.e. verify that $X+Y2=3$, and Eq. (4), i.e. verify that $Y2=3-X$.

Now we wish to find all pairs of values X, Y that satisfy Eqs. (3) **and** (4) (or equivalently Eqs. (1) and (2)) **simultaneously**.

In the table, all pairs (X, Y1) satisfy Eq. (3) and all pairs (X, Y2) satisfy Eq. (4).

If, for some X, we have $Y1=Y2$, then the pair $(X, Y1)=(X, Y2)$ satisfies both equations simultaneously, and is therefore a solution of the simultaneous equations (3) and (4).

Look at your table to find this solution — there's only one solution in this case, but scroll down the table just to make sure. You might like to expand the range of values in SET.

Finally check your solution back in Eqs. (1) and (2).

Exercises

Use a table to find the solutions to the following simultaneous equations. The Table Setting above is appropriate here, but you should be aware that a solution may not lie in the range chosen. Check your answers.

Be aware of the difference between the change-sign key (−) *and the minus key* − .

1. $$\begin{aligned} y &= 2x+3 \\ y &= 3x+1. \end{aligned}$$

2. $$\begin{aligned} 4x-y &= -5 \\ 2x+y &= -7. \end{aligned}$$

3. $$\begin{aligned} 2x-y &= -8 \\ x+y &= -1. \end{aligned}$$

Simultaneous Equations 2

Example: Solve the simultaneous equations (as in *Simultaneous Equations 1*)

$$x - y = -1 \qquad (1)$$

$$x + y = 3 \qquad (2)$$

or, equivalently

$$y = x + 1 \qquad (3)$$

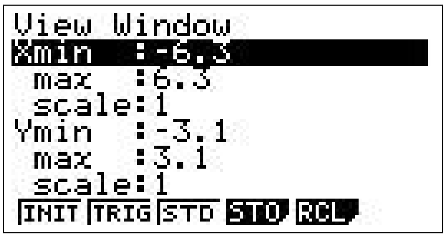
$$y = 3 - x. \qquad (4)$$

Using a graph

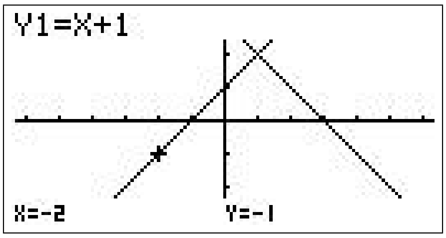
Press MENU 5 (GRAPH) and make sure you still have Eqs. (3) and (4) entered into your calculator.

Press SHIFT F3 (V-Window) F1 (INIT) to set suitable axes.

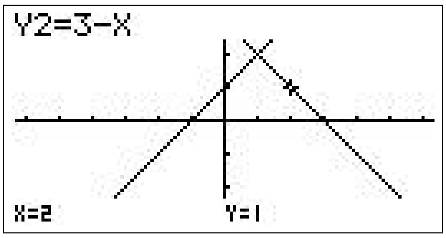
Press EXIT, then F6 (DRAW) to plot the graphs of the functions (3) and (4).



Press F1 (Trace) and move the cursor along the graph of Y1 using the left- and right-arrow keys. The coordinates at the bottom of the screen are points on this graph. Verify, for a couple of points, that they satisfy Eq. (1), i.e. that $X - Y = -1$.



Press the down-arrow key to move the cursor to Y2. Again, verify, for a couple of points, that they satisfy Eq. (2), i.e. that $X + Y = 3$. Move the cursor to $X = 0$ and press the down-arrow key several times to toggle between Y1 and Y2. *What happens to the Y value? Is this an intersection point?*

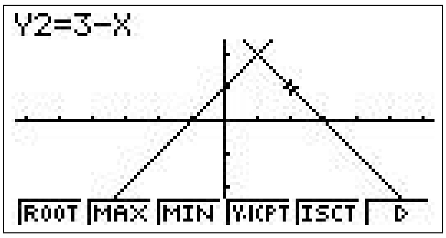


Which point satisfies both equations? Move the cursor there and press the down-arrow key several times to verify that the point does lie on both graphs. This should be the same solution you found in *Simultaneous Equations 1*.

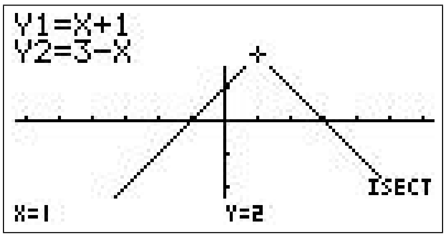
In this case we were lucky. We could move the cursor exactly to the point we wanted. *What if it had turned out that the solution was at $X = 0.25$?* Try to move the cursor there along either curve.

We can use a built-in calculator operation ISCT to find the intersection of any two graphs on the calculator screen. Try it here for practice, so you know how to use it when it is necessary.

Press F5 G-Solv to bring up the G-Solv menu.



Press F5 to select ISCT. The calculator thinks for a while (black box in top right-hand corner), then moves the cursor to the intersection point. *Did it get the right point here? How accurate was its answer in this case?*



Exercises

Find the solutions to the following pairs of simultaneous equations by graphing each pair of lines accurately on graph paper and finding the point of intersection. Label each axis and put on scales. Label the point of intersection.

Check your answer using a calculator graph and ISCT if necessary. You may need to change the View Window to see the point of intersection. Check your solution in the original equations.

1. $$\begin{aligned} y &= 2x+3 \\ y &= 3x+1. \end{aligned}$$

2. $$\begin{aligned} 4x-y &= -5 \\ 2x+y &= -7. \end{aligned}$$

3. $$\begin{aligned} 2x-y &= -8 \\ x+y &= -1. \end{aligned}$$

Simultaneous Equations 3

Example: Solve the simultaneous equations (as in *Simultaneous Equations 1, 2*)

$$x - y = -1 \quad (1)$$

$$x + y = 3 \quad (2)$$

or, equivalently

$$y = x + 1 \quad (3)$$

$$y = 3 - x. \quad (4)$$

Solving algebraically

These simple simultaneous equations are easy to solve algebraically. Between the two equations, we can eliminate one of the variables and solve the resulting equation for the other variable.

If we add Eqs. (1) and (2) (sum of LHS = sum of RHS), we have

$$x - y + x + y = -1 + 3.$$

$$\therefore\ 2x = 2.$$

$$\therefore\ x = 1.$$

Putting $x=1$ back into Eq. (1) $\big($or Eq. (2)$\big)$ gives $y=2$.

Therefore, the solution to the simultaneous equations is $x=1$, $y=2$, as we found previously.

Alternatively, subtracting Eq. (4) from Eq. (3) gives $0=2x-2$, so that again $x=1$. Substituting into Eq. (3) $\big($or Eq. (4)$\big)$ gives $y=2$. Or, we could have added Eqs. (3) and (4) to find y first.

Exercises

Find the solutions to the following simultaneous equations algebraically. Write down what operations you do to achieve this, e.g. Eq. (2) − Eq. (1), Eq. (1) + Eq. (2), etc. Check your solution in the original equations.

1. $$y = 2x + 3$$
$$y = 3x + 1.$$

2. $$4x - y = -5$$
$$2x + y = -7.$$

3. $$2x - y = -8$$
$$x + y = -1.$$

In the following exercises, solve the simultaneous equations first using a table, then a graph and finally algebraically.

4. $$2x - y = -3$$
$$x - y = 1.$$

PTO

5.
$$\begin{aligned} x + y &= -1 \\ x - y &= -5. \end{aligned}$$

6.
$$\begin{aligned} x - y &= 6 \\ 2x + y &= 6. \end{aligned}$$

7.
$$\begin{aligned} 2x - y &= 1 \\ x - y &= -5. \end{aligned}$$

8.
$$\begin{aligned} x + y &= 3 \\ x - y &= 6. \end{aligned}$$

9.
$$\begin{aligned} 2x + y &= 6 \\ y - 4x &= -9. \end{aligned}$$

10.
$$\begin{aligned} 3x + y - 1 &= 0 \\ 10x - y - 25 &= 0. \end{aligned}$$

12 Speeding — A Study in Linear Functions

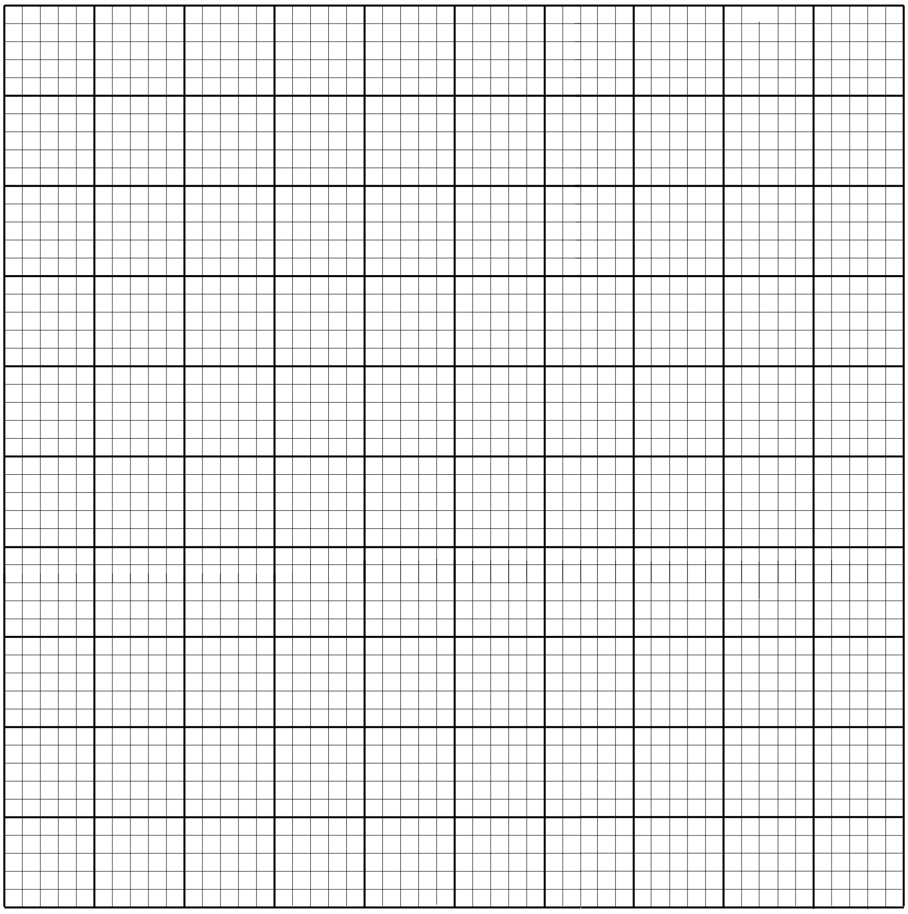
Year 9, Levels 1 & 2; Strand: Algebra; Sub-strand: Coordinate Geometry.

Authors: Bonnie Peterson and Eddie Keel in Graphing Calculators in Mathematics Grades 7–12, Center of Excellence for Science and Mathematics Education (CESME) at The University of Tenessee at Martin, USA. Modified by Peter McIntyre.

Students learn and apply basic knowledge of linear functions to problems involving speeding tickets.

Activity 1: Review

1. Plot the following points: $A\,(2,5)$; $B\,(-1,3)$; $C\,(0,2)$; and $D\,(4,-3)$.

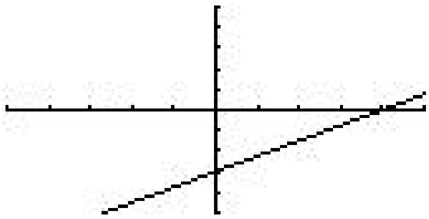


2. From the graph, using a ruler, work out the slope of the line through the points $(2,5)$ and $(-1,3)$.

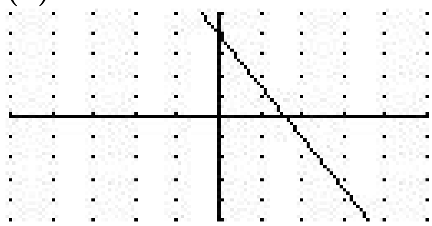
3. Calculate the slope of the line through the points $(-2,5)$ and $(-4,0)$.

4. Calculate the slopes of the two lines below. The tick marks are 1 unit apart.

(a)

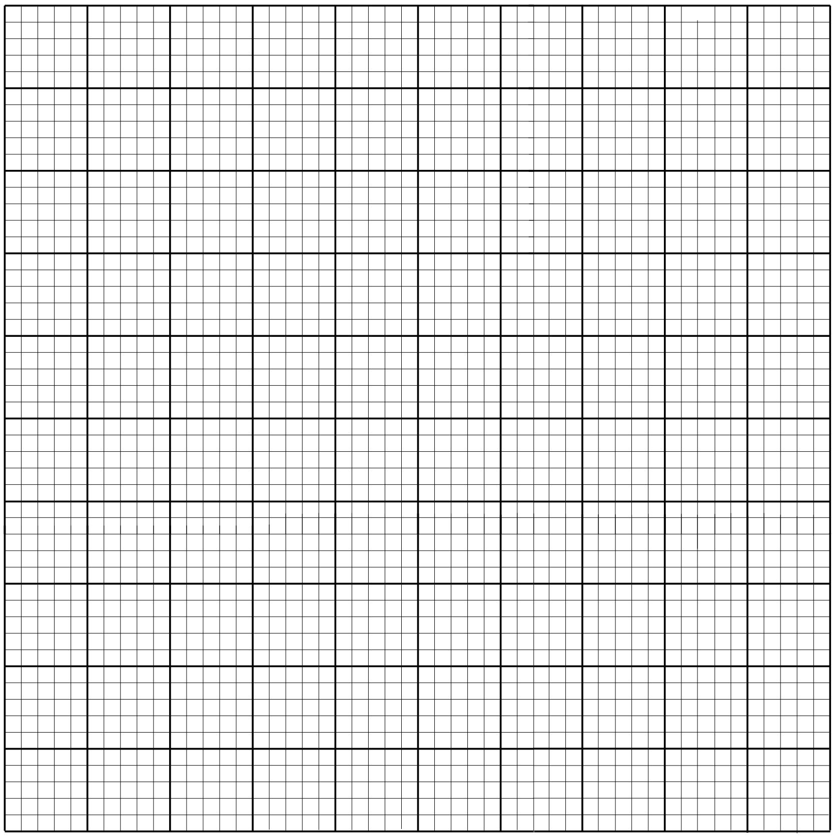


(b)



5. Without using a graphics calculator, graph the following lines:

(a) $y=2x+3$; **(b)** $y=\frac{2}{3}x+1$; **(c)** $2x+3y=6$.



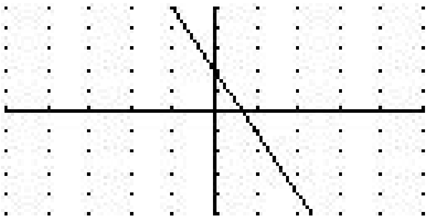
6. Draw the following lines on your graphics calculator using a suitable window:

(a) $y=3x-4$; **(b)** $y=\frac{4}{3}x+2$; **(c)** $2x+3y=6$.

See page 68 for instructions.

7. Find the equation of the straight line through the points $(1,5)$ and $(9,2)$.

8. Find the equation of the line in the following graph using the slope-intercept form of the line $y = mx + b$. The grid points are 1 unit apart.



9. Using linear regression on your calculator, find the equation of the line through the points $(-1, -3)$, $(1, 1)$, $(2, 3)$, $(12, 23)$. Check that each point lies on the line.

See page 69 for instructions.

10. Using your graphics calculator, complete the following table for $f(x) = x^2 + 5x^6$.

See page 70 for instructions.

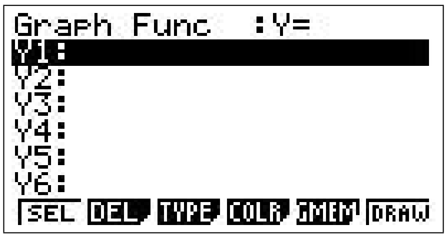
x	$f(x)$
0	
1	
2	
3	
4	
5	
6	
7	
8	

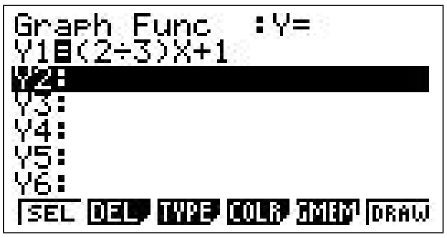
Instructions

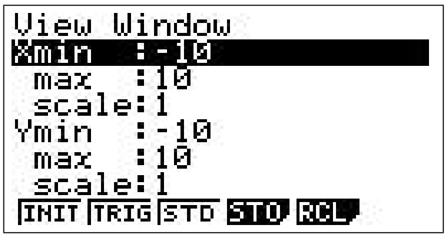
Graphing lines *Problem 6 of Activity 1*

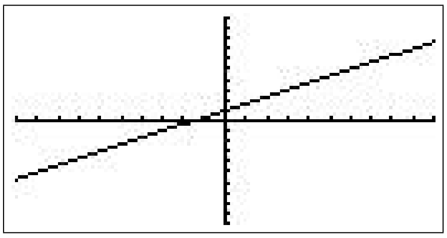
- Press [MENU] [5].
- Type in the equation. Use the [X, θ, T] key for the independent variable X.

  Here the equation is $y = \frac{2}{3}x + 1$. Brackets around $2 \div 3$ are necessary.
- Press [SHIFT] [F3] (V-Window).
- Press [F3] to select STD.
- Press [EXIT], then [F6] to graph.







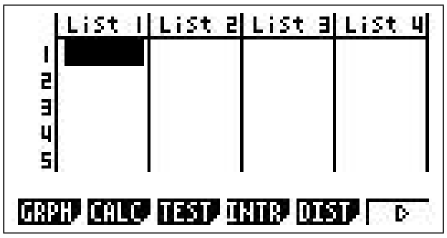


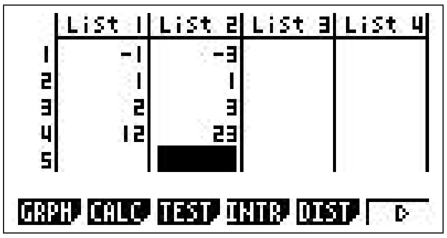
Regression *Problem 9 of Activity 1*

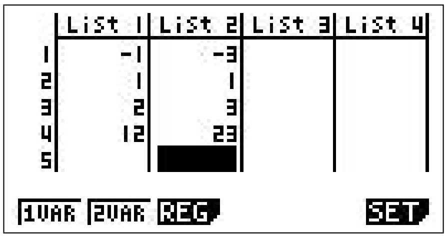
- Press [MENU] [2] (STAT).
- If there are numbers in the lists already, you will need to clear them. Press [F6]. Put the cursor in the list you want to clear, then press [F4] (DEL·A) and [F1] (YES).
  Repeat as necessary for the other lists.
- Enter the x coordinates of the points in List 1: move to the beginning of List 1, enter the first x coordinate and press [EXE]. Repeat for all the x coordinates.
- Next move to the beginning of List 2, enter the first y coordinate and press [EXE]. Repeat for all the y coordinates.

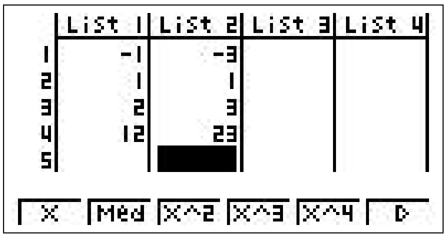
  Check that the two lists are of the same length and that the numbers you entered are correct.
- To run the linear regression, press [F2] (CALC) (if you cleared lists, press [F6] to see the CALC menu).

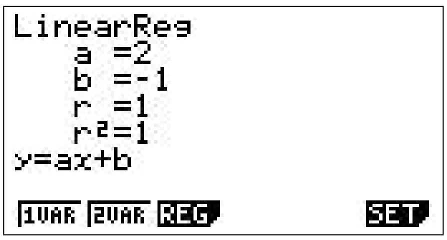
  Then press [F3] (REG) and [F1] (X) for linear regression. The results of the calculation will appear on the screen.











If you want to graph the regression line, press [F1] (GRPH) instead of [F2] (CALC).

Press [F6] (SET) to make sure that the Graph Type is set to Scatter, XList is set to List 1 and YList to *List 2*.

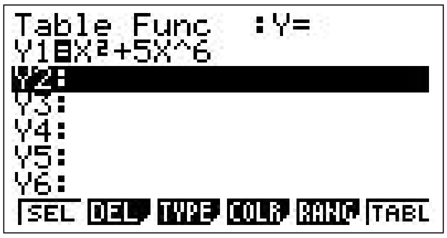
Press [EXIT], then [F1] (GPH1) to draw the scatterplot. Change the V-Window if necessary.

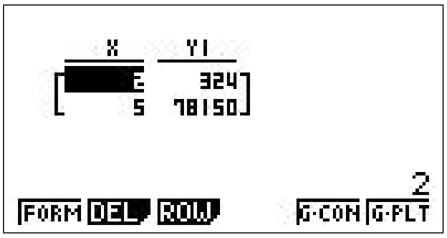
To do the linear regression, press [F1] (X) to see the results and [F6] (DRAW) to draw the regression line over the points.

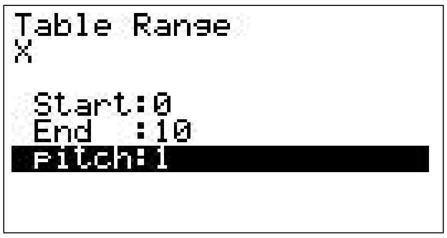
Making a table *Problem 10 of Activity 1*

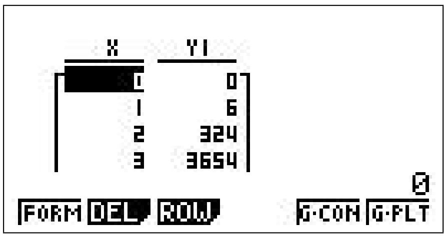
- Press MENU 7 (TABLE).
- Enter the equation you wish to use.
- Press F6 (TABL).
- If the starting X value or the X increment is not what you want, press F1 (FORM), then F6 (RANG). Change *Start*, *End* and *Pitch* (X increment) to what you want.

  Press EXIT, then F6 to return to the table.
- Scroll up or down with the arrow keys. Press F1 (FORM) to return to the equation.









Activity 2: An introduction to speeding tickets

You are driving along the highway, when you are stopped by a policeman. To your disgust, you are given a speeding ticket. The cost of a ticket T (in dollars) on a road with a speed limit of 100 km/h is determined by the following function:[5]

$$T = 120 + 1(S - 110).$$

S is the speed you are travelling at in kilometres per hour.

1. What would be the cost of your speeding ticket if you were travelling at 130 km/h? Explain how you arrived at your answer.

2. Find T if $S = 90$.

3. Explain why you would not get a traffic ticket if you were travelling at 90 km/h, even though your answer in 2 indicates your speeding ticket is a certain amount.

4. For what values of S would you not use this formula? Why?

5. Your teacher received a $135 speeding ticket on the same road. How fast was he or she going? Explain how you arrived at your answer.

[5] The 1 before the brackets is here for a reason: revealed in Activity 3.

6. The local government has hired you to establish a way for the police to determine quickly the cost of a traffic ticket. You decide a table would be the best solution to this problem. Using your graphics calculator, make a speeding-ticket table for speeds between 110 km/h and 135 km/h.

Speed (km/h)	**Cost of ticket ($)**
110	
111	
112	
113	
114	
115	
116	
117	
118	
119	
120	
121	
122	
123	
124	
125	
126	
127	
128	
129	
130	
131	
132	
133	
134	
135	

7. Suburban streets have a speed limit of 50 km/h. The local government has hired you as a consultant to establish a formula that determines the cost of a speeding ticket on these streets. They have given you this stipulation:

The minimum ticket should be $50.

Create a fair and reasonable function using S and T. Explain how you arrived at your function, and why it is fair and reasonable.

Activity 3: When is a ticket the same?

Michelago in the ACT and Bredbo in NSW (both on the Monaro Highway on the way from Canberra to Cooma) are having a special blitz on motorists speeding to and from the snowfields on the highway near each town (speed limit 100 km/h).

The local police use the following functions to determine the cost of a speeding ticket:

Michelago: $T = 130 + 2(S - 110)$

Bredbo: $T = 10 + 10(S - 110)$

1. Use a graphics calculator to complete the table below, listing the costs of speeding tickets near Michelago and near Bredbo for speeds between 120 km/h and 135 km/h.

At what speed is the cost of a speeding ticket the same in both places? Circle this speed on your table.

Michelago

Speed	**Cost of ticket**
120	
121	
122	
123	
124	
125	
126	
127	
128	
129	
130	
131	
132	
133	
134	
135	

Bredbo

Speed	**Cost of ticket**
120	
121	
122	
123	
124	
125	
126	
127	
128	
129	
130	
131	
132	
133	
134	
135	

2. For every additional km/h you go over the speed limit near Michelago, what happens to the cost of a speeding ticket? *Hint*: Use the table in Question 1.

3. Write $T = 130 + 2(S - 110)$ in slope-intercept form ($T = mS + b$). What is the slope of the equation? What does the slope mean? What are its units?

4. For every additional km/h you go over the speed limit near Bredbo, what happens to the cost of a speeding ticket?

5. Write $T = 10 + 10(S - 110)$ in slope-intercept form. What is the slope of the equation? What does the slope mean? What are its units?

6. Explain how slope and the change in the cost of a speeding ticket are related.

7. Sketch the speeding-ticket functions with the help of your graphics calculator. What V-Window did you use and why?

Michelago	**Bredbo**	**Both on the same axes**

8. Describe how the graphs are similar. Why do you think they are similar?

9. Describe how the graphs are different.

10. Find the speed at which the cost of a speeding ticket is the same for both Michelago and Bredbo. There are at least three ways to attack this problem.

(a) **Algebraic method:** Solve the equations for Michelago and Bredbo simultaneously. Show all working.

(b) **Graphical method:** Graph the two functions on your calculator. Where do they intersect? Justify your answer — show the graphs, the point of intersection and the V-Window. Use ISCT in the [G-Solv] menu if you know how.

(c) **Numerical method:** You already did this in Question 1.

Activity 4: Piecewise and other functions

Some places use a schedule similar to the following to work out the cost of a speeding ticket.

Traffic offence	**Cost**
Speeding — Exceed limit	\$50
Speeding — Limit + 10 km/h	\$70
Speeding — Limit + 20 km/h	\$90
Speeding — Limit + 30 km/h	\$120
Speeding — Limit + 40 km/h	\$160

This is a piecewise function.

1. Draw the graph that represents the speeding fines above above for a speed limit of 80 km/h.

2. What would be the cost of a ticket if your speed in a 80 km/h zone is

92 km/h?

110 km/h?

115 km/h?

130 km/h?

Explain how you arrived at these answers.

3. Write down this function in standard functional form.

$$T(S) = \begin{cases} 50 & 80 < S < 90 \\ & \\ & \\ & \end{cases}$$

Another example of speeding fines in an 80 km/h zone is given below.

Speed (km/h)	**Fine**
90–99	$50
100–109	$100
110–119	$120
120–129	$140
130 or greater	$180

This is another piecewise function.

4. Explain the pattern for speeds from 100 km/h to 129 km/h.

5. Why do you think the $50 fine does not fit the pattern?

6. Why does the $180 fine not fit the pattern?

7. Why do you think there is no listing for speeds from 81 km/h to 89 km/h?

8. You have been hired by the government, who wants to deter people from speeding. You have been asked to create a function that is not linear and will particularly punish high speeds.

Create a function and explain why your function would accomplish this goal. Use tables and graphs to support your claim.

Notes for Teachers

Activity 2: An introduction to speeding tickets

You are driving along the highway, when you are stopped by a policeman. To your disgust, you are given a speeding ticket. The cost of a ticket T (in dollars) on a road with a speed limit of 100 km/h is determined by the following function:

$$T = 120 + 1(S-110).$$

S is the speed you are travelling at in kilometres per hour.

1. What would be the cost of your speeding ticket if you were travelling at 130 km/h? Explain how you arrived at your answer.

The cost of the ticket T is $120 + 1(130-110) = \$140$.

2. Find T if $S=90$.

$T=120+1(90-110)=100$.

3. Explain why you would not get a traffic ticket if you were travelling at 90 km/h, even though your answer in 2 indicates your speeding ticket is a certain amount.

The speed 90 km/h is less than the speed limit.

4. For what values of S would you not use this formula? Why?

You would not use this formula for speeds less than or equal to the speed limit, i.e. for $S \leqslant 100$. (The formula appears to indicate that it might not actually be used for speeds less than 110 km/h, perhaps to allow for speedometer error, but this is not certain.)

5. Your teacher received a \$135 speeding ticket on the same road How fast was he or she going? Explain how you arrived at your answer.

We have to find S such that $T = 135$, that is solve

$$120 + 1(S-110) = 135$$

for S. This gives $S=110+135-120=125$.

The teacher was travelling at 125 km/h.

6. The local government has hired you to establish a way for the police to determine quickly the cost of a traffic ticket. You decide a table would be the best solution to this problem. Using your graphics calculator, make a speeding-ticket table for speeds between 110 km/h and 120 km/h, the first part of the full table.

Speed (km/h)	**Cost of ticket ($)**
110	120
111	121
112	122
113	123
114	124
115	125
116	126
117	127
118	128
119	129
120	130
121	131
122	132
123	133
124	134
125	135
126	136
127	137
128	138
129	139
130	140
131	141
132	142
133	143
134	144
135	145

7. Suburban streets have a speed limit of 50 km/h. Suppose the local government has hired you as a consultant to establish a formula that determines the cost of a speeding ticket on these streets. They have given you this stipulation:

The minimum ticket should be $50.

Create a fair and reasonable function using S and T. Explain how you arrived at your function, and why it is fair and reasonable.

Any sensible answer here. Class discussion?

Activity 3: When is a ticket the same?

Michelago in the ACT and Bredbo in NSW (both on the Monaro Highway on the way to Cooma) are having a special blitz on motorists speeding to and from the snowfields on the highway near each town (speed limit 100 km/h).

The local police use the following functions to determine the cost of a speeding ticket:

Michelago: $T = 130 + 2(S - 110)$

Bredbo: $T = 10 + 10(S - 110)$

1. Use a graphics calculator to complete the table below, listing the costs of speeding tickets near Michelago and near Bredbo for speeds between 115 km/h and 130 km/h.

At what speed is the cost of a speeding ticket the same in both places? Circle this speed on your table.

Michelago

Speed	Cost of ticket
115	140
116	142
117	144
118	146
119	148
120	150
121	152
122	154
123	156
124	158
125	160
126	162
127	164
128	166
129	168
130	170

Bredbo

Speed	Cost of ticket
115	60
116	70
117	80
118	90
119	100
120	110
121	120
122	130
123	140
124	150
125	160
126	170
127	180
128	190
129	200
130	210

The cost of a speeding ticket is the same in both places at a speed of 125 km/h.

2. For every additional km/h you go over the speed limit near Michelago, what happens to the cost of a speeding ticket? *Hint*: Use the table in Question 1.

For every additional km/h you go over the speed limit near Michelago, the cost of a speeding ticket goes up by $2.

3. Write $T = 130 + 2(S - 110)$ in slope-intercept form ($T = mS + b$). What is the slope of the equation? What does the slope mean? What are its units?

In slope-intercept form, $T = 2S - 90$. The slope is 2. This is the number of dollars that the cost of a speeding ticket increases for an increase of 1 km/h in the speed. The units are dollars per km/h.

4. For every additional km/h you go over the speed limit near Bredbo, what happens to the cost of a speeding ticket?

For every additional km/h you go over the speed limit near Bredbo, the cost of a speeding ticket goes up by \$10.

5. Write $T = 10 + 10(S - 110)$ in slope-intercept form ($T = mS + b$). What is the slope of the equation? What does the slope mean? What are its units?

In slope-intercept form, $T = 10S - 1090$. The slope is 10. This is the number of dollars that the cost of a speeding ticket increases for an increase of 1 km/h in the speed. The units are dollars per km/h.

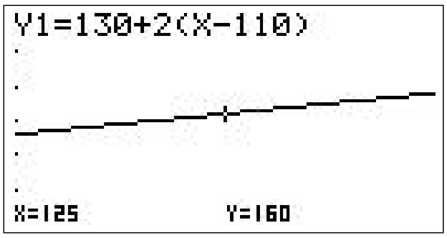
6. Explain how slope and the change in the cost of a speeding ticket are related.

The slope is equal to the change in the cost of a speeding ticket with an increase in speed of 1 km/h.

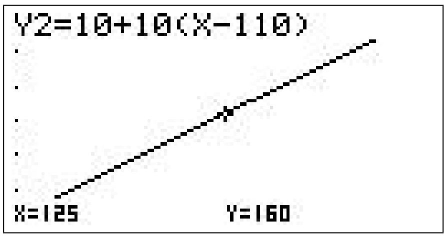
7. Sketch the speeding-ticket functions with the help of your graphics calculator. What V-Window did you use and why?

V-Window of $[110, 140, 10] \times [0, 300, 50]$ to cover the range of speeds (X axis) and the range of fines (Y axis).

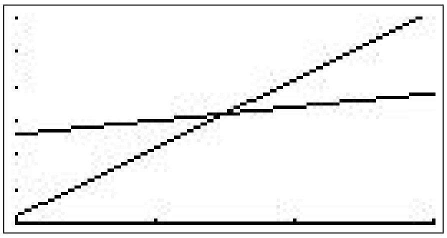
Michelago



Bredbo



Both on the same axes



8. Describe how the graphs are similar. Why do you think they are similar?

The graphs are both straight lines. Both functions are linear functions.

9. Describe how the graphs are different.

The graphs have different slopes.

10. Find the speed at which the cost of a speeding ticket is the same for both Michelago and Bredbo. There are at least three ways to attack this problem.

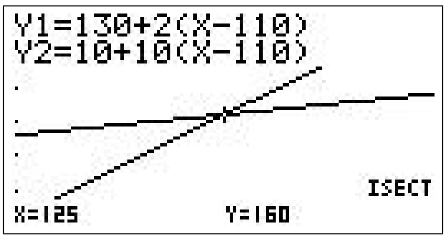
(a) Algebraic method: Solve the equations for Michelago and Bredbo simultaneously. Show all working.

Equating the two expressions for T gives

$$\begin{aligned} 130 + 2(S - 110) &= 10 + 10(S - 110). \\ \text{Therefore, } 8(S - 110) &= 120. \\ \text{Therefore, } S - 110 &= \frac{120}{8} \\ &= 15. \\ \text{Therefore, } S &= 110 + 15 = 125. \end{aligned}$$

The cost of a speeding ticket is the same for both Michelago and Bredbo at a speed of 125 km/h.

(b) Graphical method: Graph the two functions on your calculator. Where do they intersect? Justify your answer — show the graphs, the point of intersection and the V-Window. Use *ISCT* in the G-Solv menu if you know how.



(c) Numerical method: You already did this in Question 1.

Activity 4: Piecewise and other functions

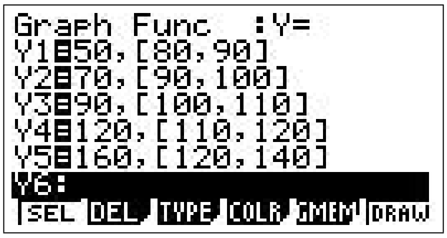
Some places use a schedule similar to the following to work out the cost of a speeding ticket.

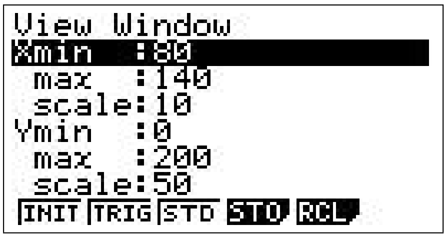
Traffic offence	Cost
Speeding — Exceed limit	$50
Speeding — Limit + 10 km/h	$70
Speeding — Limit + 20 km/h	$90
Speeding — Limit + 30 km/h	$120
Speeding — Limit + 40 km/h	$160

This is a piecewise function.

1. Draw the graph that represents the speeding fines above for a speed limit of 80 km/h.

   Note that students are not expected to use their graphics calculator to do this. The procedure using a graphics calculator is shown below to give the graphs and show teachers how to do it if they wish.





2. What would be the cost of a ticket if your speed in a 80 km/h zone is

   92 km/h? $70

   110 km/h? $120

   115 km/h? $120

   130 km/h? $160

   Explain how you arrived at these answers.

   Work this out by looking at the table above.

3. Write down this function in standard functional form.

$$T(S) = \begin{cases} 50 & 80 < S < 90 \\ 70 & 90 \leq S < 100 \\ 90 & 100 \leq S < 110 \\ 120 & 110 \leq S < 120 \\ 160 & 120 \leq S \end{cases}$$

Another example of speeding fines in an 80 km/h zone is given below.

Speed (km/h)	**Fine**
90–99	$50
100–109	$100
110–119	$120
120–129	$140
130 or greater	$180

This is another piecewise function.

4. Explain the pattern for speeds from 100 km/h to 129 km/h.

The fine increases by $20 each time we reach a speed ending in 0.

5. Why do you think the $50 fine does not fit the pattern?

The fine is relatively smaller for speeds exceeding the speed limit by only a small amount.

6. Why does the $180 fine not fit the pattern?

The fine is relatively larger for speeds exceeding the speed limit by a large amount.

7. Why do you think there is no listing for speeds from 81 km/h to 89 km/h?

This allows for inaccuracy in speedometer readings (supposedly less than 10%).

8. You have been hired by the government, who wants to deter people from speeding. You have been asked to create a function that is not linear and will particularly punish high speeds.

Create a function and explain why your function would accomplish this goal. Use tables and graphs to support your claim.

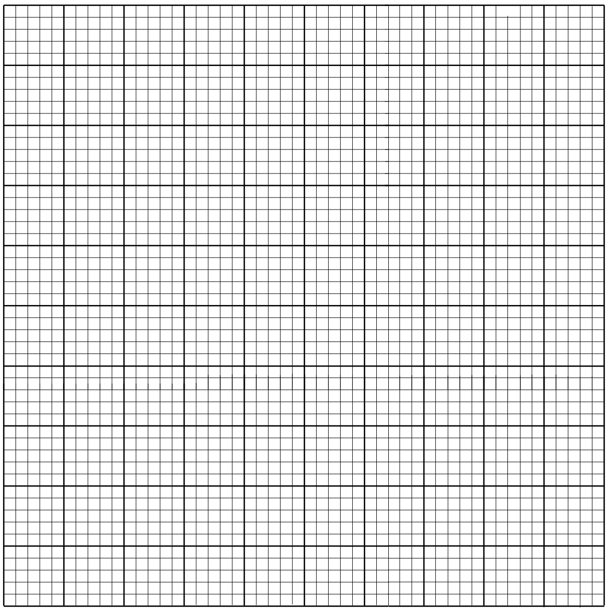
Anything they like here, as long as they can justify it. If they can come up with a formula, graphics calculators can be used. Otherwise you might give them some graph paper on which to draw their function, so they can read off values and make up a table. Working in small groups might be a good idea here.

13 Starburst

Year 9, Levels 1 & 2; Strand: Algebra; Sub-strand: Coordinate Geometry.

A study of straight lines: slope and intercept.

On your graphics calculator, draw the graphs of $y = 2x$ and $y = -2x$ for $-5 < x < 5$ and $-5 < y < 5$. Sketch the two graphs on the grid below. Explain the similarities and differences.



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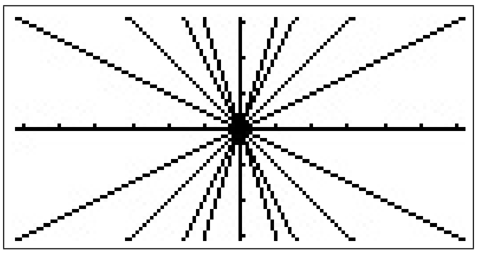
Now draw $y = 3x$ and $y = -3x$ on the grid above.

Once you have drawn those lines, suggest how you could make the picture like the one below. Try your idea out on your graphics calculator and see if you were correct.



How I would make this picture

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How would you describe this picture to your friend over the phone so that they could draw it on their graphics calculator?

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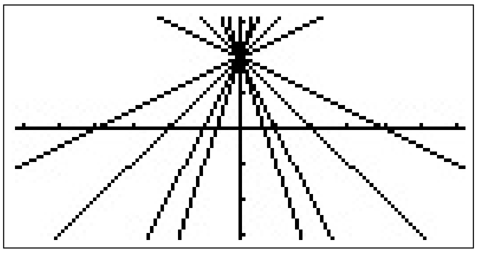
Now try to draw the star pattern below. You may first need to work out how you can shift the straight-line graphs up the y axis.



How I shifted the straight lines up the y axis

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Explain what would happen if we drew the graphs of $y = 3x - 2$ and $y = -3x - 2$.

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The equation $y = mx + c$ gives a straight-line graph. Explain in your own words what changing m does to the straight line. What about changing c?

If you are not sure, look back on what you have already done or try graphing some more straight lines on your calculator to see their orientation and location.

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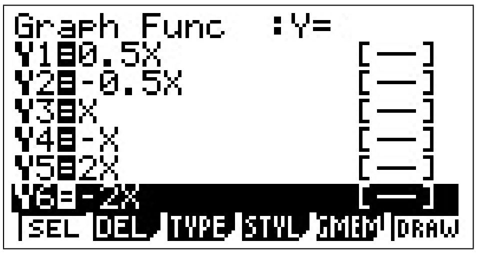
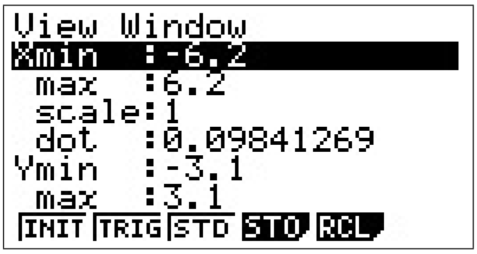
Notes for Teachers

The equations for the first starburst are $y = \pm 0.5x$, $y = \pm x$, $y = \pm 2x$ and $y = \pm 3x$ (left-hand figure below: Y7 = 3X and Y8 = −3X are not shown).





These figures were generated by first graphing the functions, then [SHIFT] [F2] (Zoom) [F6] [F2], the square option, which makes the scales on each axis the same. The right-hand figure above shows the resulting View Window (Yscale is 1).

Depending on the scales you select, your graphs may look a little different.

The equations for the second starburst are just those of the first, but with 2 added to the right-hand side of each equation.

14 Statistics from Birthdays

Year 9, Levels 1 & 2; Strand: Chance and Data; Sub-strand: Statistics.

Author: Stephen Arnold in Integrating Technology in the Middle School, T^3 Publication, 2003. Modified by Peter McIntyre.

Class data on day and month of birth are used to provide an introduction to data presentation on a graphics calculator.

Preparation

To enter the data, press MENU 2 (STAT). This takes you to the list editor.

If there are numbers in List 1 or List 2, move the cursor to the list, press F6, then F4 (DEL-A) and F1 for YES.

Collecting the data

Go round the class and get every student to say the day of the month on which he or she was born. Enter these numbers into List 1 as you go, pressing EXE after each number, including the last.

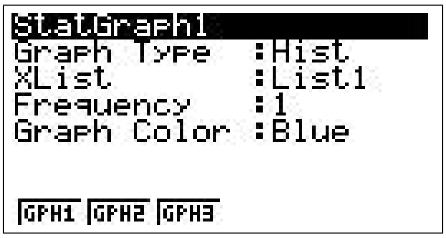
Move the cursor to List 2 and enter, in the same order, the month in which each person was born. Make sure there is the same number of values in each column.

Plotting the data

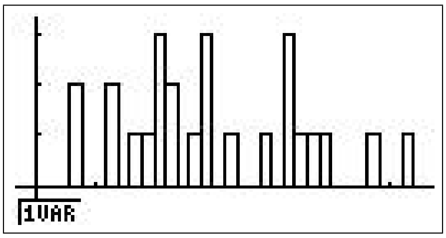
Histogram

What would you expect if you plot a histogram of the birth days, particularly if the class is large? Would you expect more births on some days of the month than on others?[6]

Press F1 (GRPH) (you will need to press F6 first if you cleared a list as above).
Press F6 (SET) and select GPH1 with F1. Move the cursor to *Graph Type*, press F6 and choose *Hist*.
The XList should be List 1, the Frequency 1 and the Graph Color whatever you like.



Next press EXIT, then F4 (SEL) and make sure *StatGraph1* has *DrawOn*, the others *DrawOff*.
Press F6 (DRAW) and set *Start* to 0 and *Step*, the width of each box, to 1, pressing EXE after each.
Press F6 (DRAW) again to generate the histogram. Yours should be similar to, but not the same as, the figure.



Press SHIFT F1 (Trace) and use the left- and right-arrow keys to move around the plot to see the frequency of each day (the number of students who have a birthday on this day of any month).

Press SHIFT F1 again to exit Trace. From the graph, press F1 (1VAR) to give the mean, standard deviation, etc for List 1.

Did you see what you expected? If not, why not?

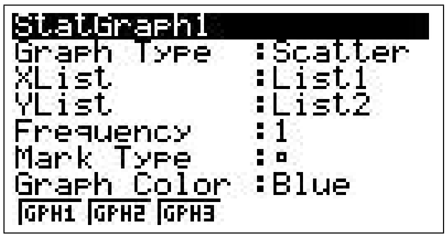
Now plot a histogram of month of birth. Explain your graph.

[6] The answer is actually yes because the days 29, 30 and 31 don't occur in all months.

Scatter plot

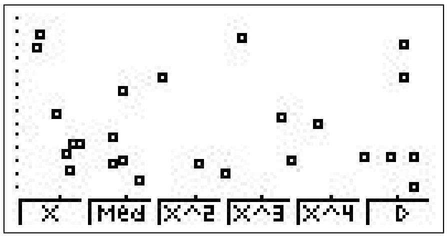
Here we will plot birth day (List 1) along the X axis and birth month (List 2) along the Y axis. *What do you expect to see this time?*

Press (MENU 2) F1 (GRPH) F6 (SET) and set up StatGraph1 as shown.



Press EXIT , change the V-Window if necessary, then press GRPH1 to generate the plot.

Press SHIFT F1 (Trace) and move around with the left- and right-arrow keys to see the coordinates of the points. Your plot will be different to the figure, but the conclusions should be the same.



Did you see what you expected? If not, why not?

The viewing window is set automatically for statistics plots.[7] To set it manually, press SHIFT MENU and set *Stat Wind* to *Man*. Then press EXIT and SHIFT F3 (V-Window) to enter the View Window parameters. Discuss with students what the View Window should be. Press EXIT to return to the list editor.

Sorting the data

We will now sort List 1 and List 2 separately in ascending order. *What do you expect to see this time when we plot a scatter plot of Month versus Day?*

In the list editor, press F6 and select F1 (SRT·A). We only want to sort one list at a time, so type in 1 to the prompt *How Many Lists?* and press EXE . We start with List 1, so type in 1 again to the prompt *Select List* and press EXE . List 1 will then be sorted.

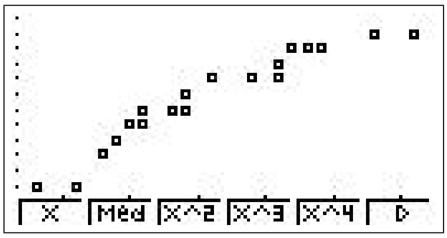
Repeat for List 2.

Press F6 and the usual sequence F1 (GRPH) and F1 (GPH1) to graph.

What do you see?
Hopefully a graph in which the points lie approximately on a straight line.

What's the equation of the line?

What do you predict it to be?



[7]This is why you have to keep setting *Start* and *Step* for the histogram.

Notes for Teachers

When trying out this activity beforehand, you can use the random-number generator to generate some data. To generate birth days for a class of 25, press [MENU] [1] (RUN) and [OPTN]. Enter the following command followed by [EXE].

$$\text{Seq(Int(31Rand\#+1)}, \text{X}, 1, 25, 1) \rightarrow \text{List}\,1,$$

where *Seq* is in the List menu ([F1]), *Int* is in the NUM menu ([EXIT] [F6] [F4]), *Rand#* is in the PROB menu (next to NUM) and *List* is in the List menu ([EXIT] [F6] [F6] [F1]).

To generate birth months: $\text{Seq(Int(12Rand\#+1)}, \text{X}, 1, 25, 1) \rightarrow \text{List}\,2$.

Shortcut: Use the left-arrow key to recall the previous command and edit before pressing [EXE].

Note that you may end up with some impossible pairs here, such as 31/2, but it doesn't matter in the testing phase.

If you want to find an equation of the line of best fit, press [F2] (Med). This calculates the median-median line of best fit. The coefficients of the fit are shown, and if you press [F6] (DRAW), the line will be plotted over the data.

The equation of a line between the first birthday of the year, 1 January $(1, 1)$, and the last, 31 December $(31, 12)$, is

$$y - 1 = 11(x-1)/30 \qquad \text{or} \qquad y \approx 0.367x+0.633.$$

15 Tangrams and Straight Lines

Year 9, Levels 1,2 & 3; Strand: Algebra; Sub-Strand: Coordinate Geometry.

Original article: Tangrams and Technology, by Mark Ellingham, University of Queensland. Modified by Margie Smith.

Students make a picture using Tangram pieces, place the picture onto grid paper and use their knowledge of xy coordinates and straight-line equations to redraw the picture on a graphics calculator.

Using Tangrams to Investigate $y = mx + b$

Teaching Notes

On using this multi-level activity in full, the student will be able to:

- construct a picture using the tangram pieces from a selection of examples given by the teacher;
- record X and Y coordinates on a sheet provided;
- transfer collected data into a graphics calculator;
- find the equation of the lines of the edges of the picture using two points for each line; alternatively, working in pairs, use the method of guess and check to find the equation of a particular line in the teacher's picture;
- enter the equations and make sure that they match the data points previously entered;
- determine where they want the line to begin and end, i.e. find the domain of the line.

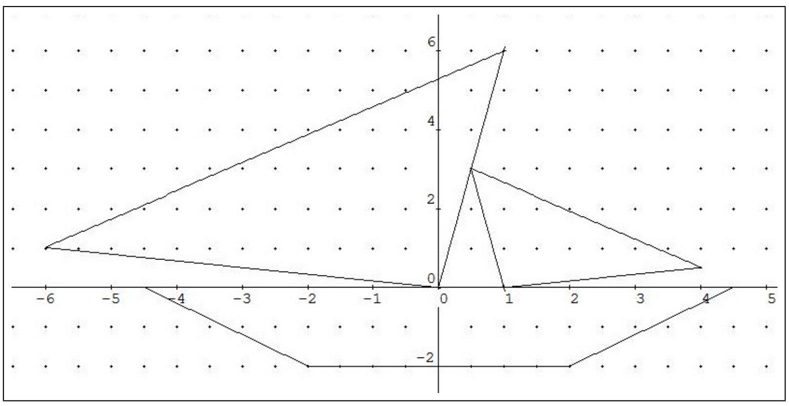
Because the activity is based on a picture created by students, the solutions obtained will vary greatly, depending on the shape they choose to construct and where they place the picture on the grid. We work here with one particular picture and set of data points in order to demonstrate the process.

The following is a worked solution of a selected picture — a boat.

Data Collection

1. Select a tangram picture to construct.
2. Paste picture pieces together on a worksheet and place on an XY grid (see page 98).
3. Find the X and Y coordinates for each corner – round to the nearest half point on the grid.
4. Record these points on a worksheet.

Point	X	Y
A	−6	1
B	1	6
C	0	0
D	0.5	3
E	4	0.5
F	1	0
G	4.5	0
H	2	−2
I	−2	−2
J	−4.5	0



Data Transfer to Graphics Calculator

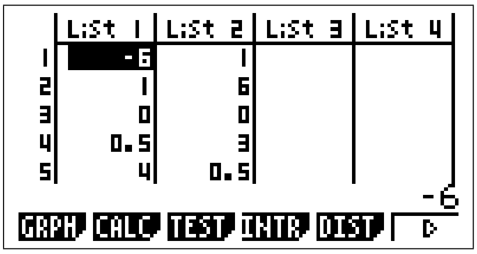
Now we transfer the data from the picture into List 1 for the X coordinates and List 2 for the Y coordinates.

1. Go to the main menu MENU and select STAT.

Press SHIFT SET UP and set Stat Wind to Auto. Press EXIT.

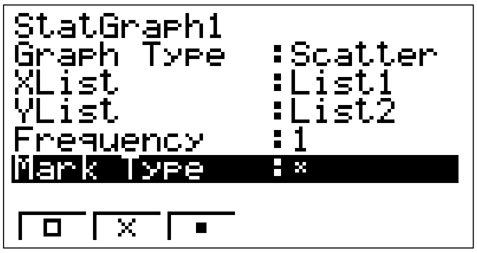
2. Clear any numbers in the lists if necessary: F6 F4.

Enter the X coordinates from the table into List 1 and the Y coordinates into List 2.



3. Select F1 (GRPH) F6 (SET) F1 (GPH1) and set StatGraph1 as follows.

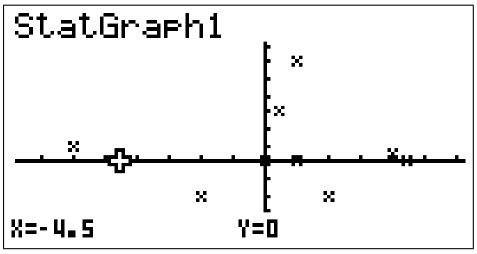
Graph Type: Scatter; XList: List 1; YList: List 2 ; Frequency: 1; Mark Type: $\times$.



4. Press EXIT F1 (GPH1). You should see a scatterplot of the points in List 1 and List 2.

If required, you can change the V-Window by pressing SHIFT F3. The window can be ‘squared up’ by pressing SHIFT F2 (ZOOM) F6 F2 (SQR).

Press F1 (Trace) and use the left/right arrows to move between the points.



5. Store the picture by pressing OPTN F1 (PICT) F1 (STO) 1 (Pic1).

6. Now set this picture as the background: SHIFT SET UP.

Scroll down and change Background to Pic1: F2 (PICT) 1 (Pic1) EXIT.

PTO

Fitting Linear Equations to Data Points

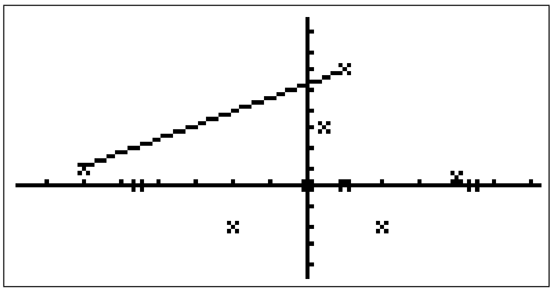
Students are now required to fit linear equations to connect pairs of the plotted points in their calculator. They will have to use their calculator graph to locate the data points they wish to fit for each line. Trace will help here.

The equations of the edges for our example, found using the two-point formula, and the domains are:

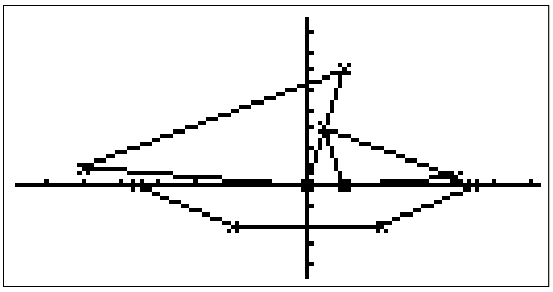
Line	Equation	m		b	Domain	
A to B	$y =$	$0.71x$	$+$	5.29	$x > -6$	$x < 1$
B to C	$y =$	$6.00x$	$+$	0.00	$x > 0$	$x < 1$
A to C	$y =$	$-0.17x$	$+$	0.00	$x > -6$	$x < 0$
D to E	$y =$	$-0.71x$	$+$	3.36	$x > 0.5$	$x < 4$
E to F	$y =$	$0.17x$	$+$	-0.17	$x > 1$	$x < 4$
D to F	$y =$	$-6.00x$	$+$	6.00	$x > 0.5$	$x < 1$
G to H	$y =$	$0.80x$	$+$	-3.60	$x > 2$	$x < 4.5$
H to I	$y =$	$0.00x$	$+$	-2.00	$x > -2$	$x < 2$
I to J	$y =$	$-0.80x$	$+$	-3.60	$x > -4.5$	$x < -2$
J to G	$y =$	$0.00x$	$+$	0.00	$x > -5$	$x < 4.5$

To enter these equations into your calculator and graph them over the data points, follow the instructions below.

1. Go to the main MENU and select GRAPH.
2. Type in the equation of the first line: $Y1 = 0.71X + 5.29$.
3. When you press F6 (DRAW), you will notice that the line passes right across the screen. Press F6 to return to the Graph Func screen and enter the domain in square brackets after the function: $Y1 = 0.71X + 5.29$, $[-6,1]$. Don't forget the commas.
4. Now press F6 and you will see that the line is drawn only over the specified domain.



5. Repeat Steps 1–4 for all the other lines on the tangram picture, using Y2, Y3, Y4, etc, to draw your picture on the graphics calculator.



Some problems you may encounter doing different pictures

- Plotting vertical lines (e.g. $x=2$) cannot be done with the usual graph functions.

  To overcome this hurdle, you need to do the following: in the Graph Func window, press F3 (TYPE) F4 (x=c), then type in the X value and hit EXE.

  To change back, press F3 (TYPE) F1 (Y=) and continue entering equations as normal.

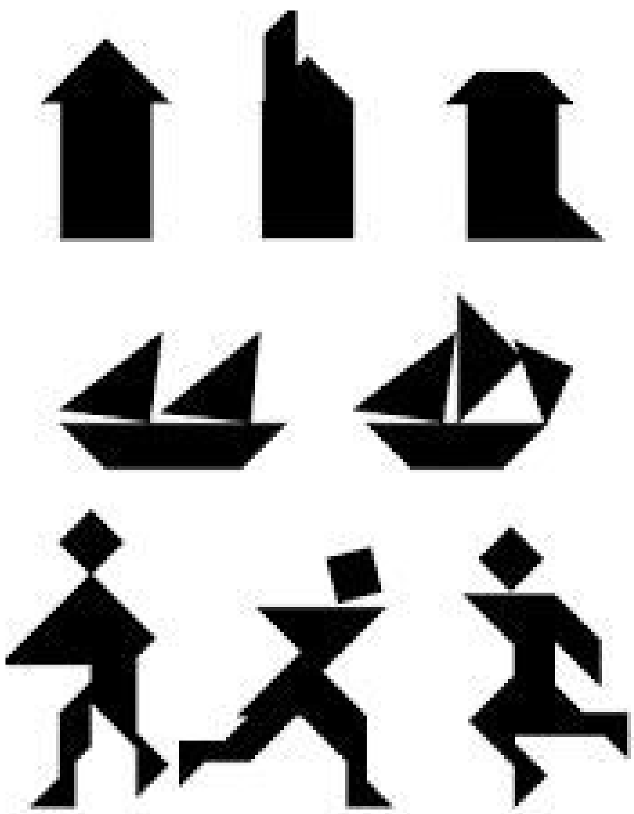
- Any tangram shapes could be used (see below), although we did find some more useful than others because they gave a variety of gradients when fitting lines. Tangram shapes can also be rotated on the grid to achieve a range of linear functions.

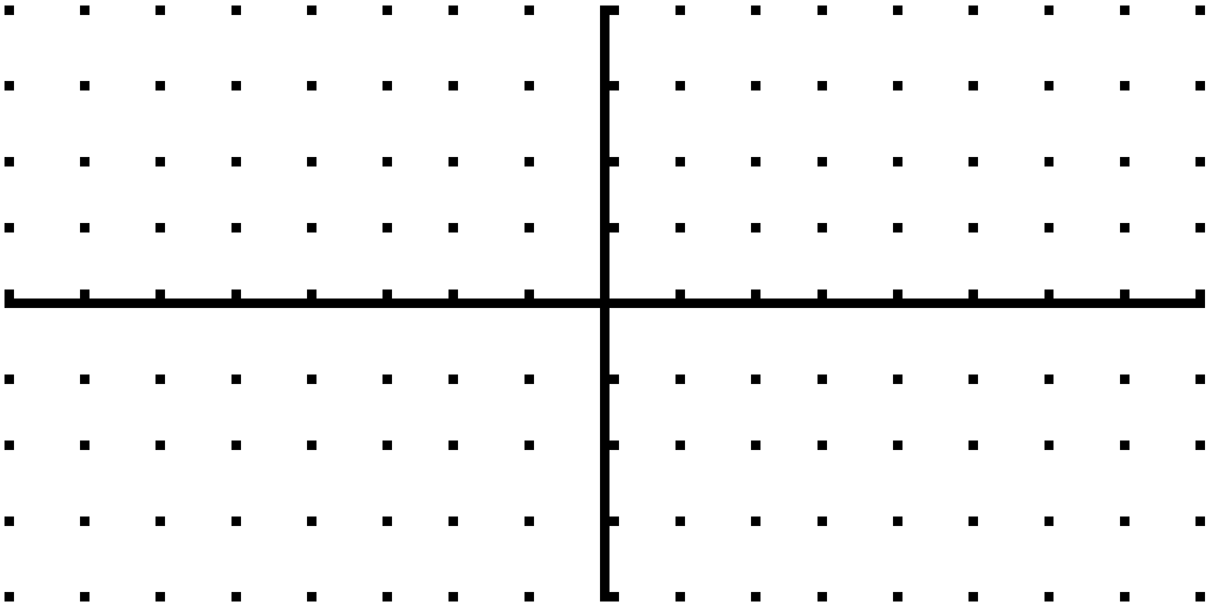
- Students are required to find the domain of the function in this activity, but the range is also easily found from reading values off the calculator; this could also be included in the activity.

- To be more precise when finding the domain for the linear functions, simultaneous equations could be introduced or the intersection function on the calculator could be used as a challenging exercise for a more advanced class.

- Something to think about when choosing the tangram picture: the calculator can only graph 20 functions at a time. Therefore, pictures with more than 20 sides cannot be graphed as a whole on the screen; they will have to be created in parts.

Some sample tangram pictures





$-8 < X < 8; \ -4 < Y < 4$

16 Temperature Conversions

Year 9, Levels 1 & 2; Strand: Algebra; Sub-Strand: Coordinate Geometry.

From 45 Single Lessons, University of Melbourne, 1996. Downloaded from the Casio Australia website. Modified by Peter McIntyre.

An application of linear functions to conversions between degrees Celsius and degrees Fahrenheit.

Celsius and Fahrenheit

John and David have recently been on a trip to the United States, where temperature is measured using the Fahrenheit scale rather than the Celsius or Centigrade scale used in Australia.

They were told, when converting from Celsius to Fahrenheit, to apply the rough rule: *double and add thirty.*

1. On the day they left Canberra, the maximum temperature was 20°C. To prepare for the US, John and David estimated this temperature in degrees Fahrenheit using the rough rule. What was this temperature?

2. They flew in to Los Angeles, where the predicted maximum temperature was 81°F. Using the rough rule, what is this temperature in degrees Celsius?

3. The actual conversion rule is $F = \frac{9}{5}C + 32$, where F is degrees Fahrenheit and C is degrees Celsius. What is the difference between the rough rule and the actual rule of conversion for a temperature of 20°C?

- On your graphics calculator, press [MENU] [7], put the rough rule in Y1 and the actual rule in Y2. What does X stand for? What about Y?

- Press [F5] (SET), set *Start* to 0, *End* to 40 and *Step* to 1. Press [F6] to generate a table of values for Y1 and Y2.

- Next graph degrees Fahrenheit versus degrees Celsius for both rules: press [MENU] [5]; set a sensible V-Window ([F3]); press [EXIT]; and [F6] to graph.

  Use your graph or table to help you answer the remaining questions.

4. On a 'hot' day, the temperature is 30°C or more.

(a) What is a 'hot' day on the Fahrenheit scale?

(b) How does the difference between the rough rule and the actual rule change as the days get hotter?

5. For what temperature in degrees Celsius will the rough rule give the same converted value as the actual rule? Describe how you find this out. Can you think of more than one way to do this?

6. The rough rule is considered to be good enough if it is in error by no more than 5°F. For what range of temperatures (degrees Celsius) would the rough rule be considered good enough? Describe how you find this out. Do this in at least two ways.

7. There is a temperature at which the numerical value is the same in degrees Celsius and in degrees Fahrenheit. What is this temperature? How far out, in degrees Fahrenheit, is the rough rule in this case?

Notes for Teachers

John and David have recently been on a trip to the Unites States, where temperature is measured using the Fahrenheit scale rather than the Celsius or Centigrade scale used in Australia.

They were told, when converting from Celsius to Fahrenheit, to apply the rough rule: *double and add thirty.*

1. On the day they left Canberra, the maximum temperature was 20°C. To prepare for the US, John and David estimated this temperature in degrees Fahrenheit using the rough rule. What was this temperature?

Temperature = $2 \times 20 + 30 = 70$°F.

2. They flew in to Los Angeles, where the predicted maximum temperature was 81°F. Using the rough rule, what is this temperature in degrees Celsius?

Here students should ask the question *What temperature when doubled and added to 30 gives 81?*, and answer it in one of several ways:

- by trial and error, giving the temperature as 25.5°C;
- by using some elementary algebra: if x is the unknown temperature, then $2x+30=81$, which when solved for x gives the temperature as 25.5°C;
- by thinking ‘inversely’: if I double and add 30 to go from Celsius to Fahrenheit, then to go from Fahrenheit to Celsius I must subtract 30 and halve the result.

3. The actual conversion rule is $F = \frac{9}{5}C + 32$, where F is degrees Fahrenheit and C is degrees Celsius. What is the difference between the rough rule and the actual rule of conversion for a temperature of 20°C?

From Question 1, the rough rule gives 70°F.

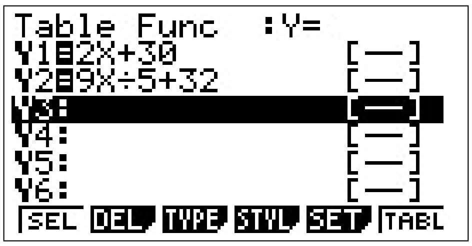
The actual rule gives $9 \times 20 \div 5 + 32 = 68$°F, so the difference is 2°F.

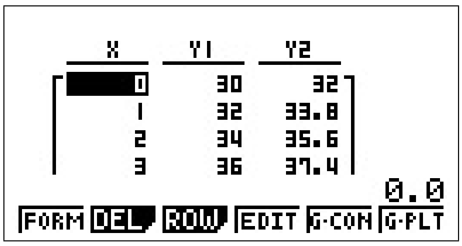
- On your graphics calculator, put the rough rule in Y1 and the actual rule in Y2. What does X stand for? What about Y?

  Y1 = 2X + 30   Y2 = 9X ÷ 5 + 32

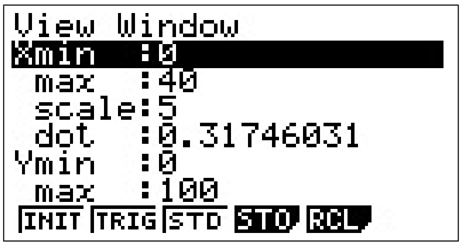
  X stands for degrees Celsius; Y for degrees Fahrenheit.

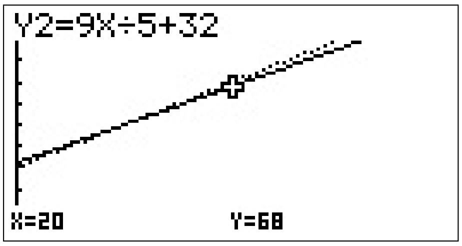
- Generate a table of values for Y1 and Y2.





- Next graph degrees Fahrenheit versus degrees Celsius for both rules. Think about a sensible V-Window. Trace (F1) has been pressed in the right-hand figure.





Use your graph or table to help you answer the remaining questions.

4. On a 'hot' day, the temperature is 30°C or more.

(a) What is a 'hot' day on the Fahrenheit scale?

From the graph, table or simple calculation, a hot day is when the temperature is 86°F or more.

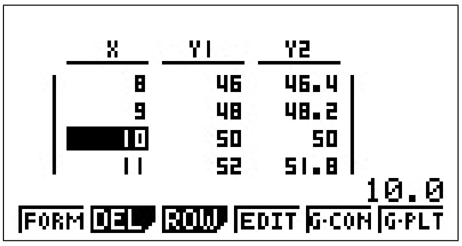
(b) How does the difference between the rough rule and the actual rule change as the days get hotter?

The difference becomes larger as the days get hotter, with the rough rule giving the higher temperature.

5. For what temperature in degrees Celsius will the rough rule give the same converted value as the actual rule? Describe how you find this out. Can you think of more than one way to do this?

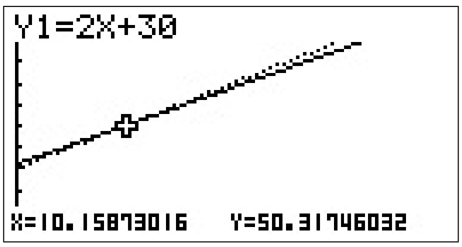
There are at least three ways to do this.

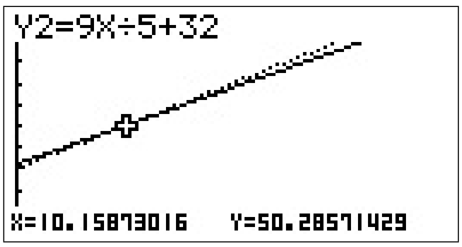
(a) The easiest way is probably from the table. Scroll up or down the table until the value for Y1 (rough rule) is the same as that for Y2 (actual rule). This happens when X (degrees Celsius) is 10, so the two rules give the same answer at 10°C.



X	Y1	Y2
8	46	46.4
9	48	48.2
10	50	50
11	52	51.8

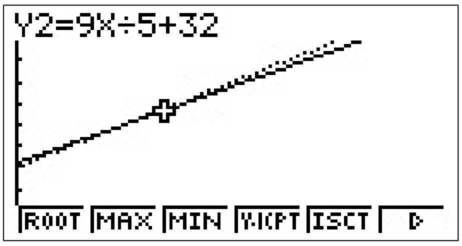
(b) The same result come from the graph — it is the X value of the intersection point of the two lines. Find this approximately using Trace (F1).

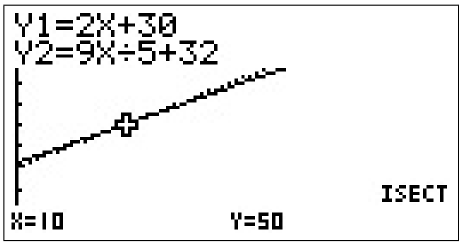




Using ISCT in the G-Solv menu is a faster way, but students may get more out of Trace.

To use ISCT, make sure the intersection point appears on the screen, select ISCT from the G-Solv menu, wait until the cursor moves to the intersection point and read off its coordinates at the bottom of the screen.





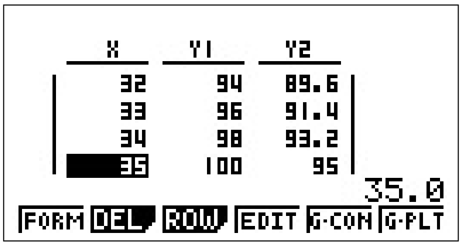
(c) Algebraically, set the two formulas equal and solve for C:

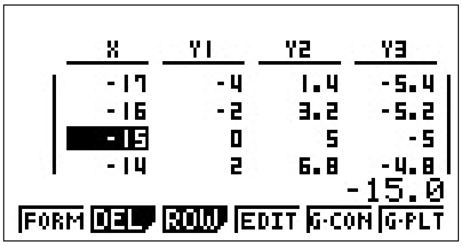
$$2C + 30 = \frac{9}{5}C + 32 \Rightarrow \frac{1}{5}C = 2 \Rightarrow C = 10.$$

6. The rough rule is considered to be good enough if it is in error by no more than 5°F. For what range of temperatures (degrees Celsius) would the rough rule be considered good enough? Describe how you find this out. Do this in at least two ways.

The answer is $-15 \leqslant C \leqslant 35$.

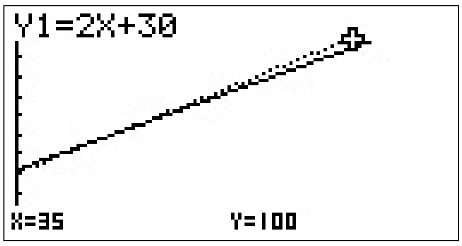
(a) The simplest way to find these values is again to use the table. By scrolling and checking the difference between Y1 and Y2, you can very soon find when the difference is 5 or -5 (left-hand figure below). The value -5 may need discussion. It may also not be apparent that there is a lower value for X, -15.

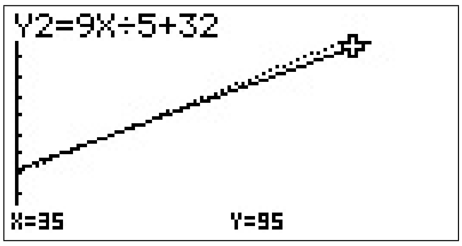


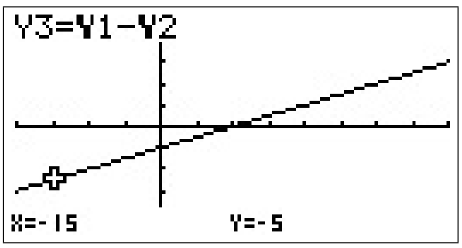


A quicker way to do this is to set Y3 = Y1 − Y2 (Y for function names is in the VARS GRPH menu) and scroll until Y3 is ±5 (-5 in the right-hand figure above).

(b) You can do the same thing on the graph using Trace and Y·CAL, as before. This takes a little longer than the table. You could suggest graphing Y3 = Y1 − Y2, and ask what to look for. The V-Window will need changing to reveal both values.





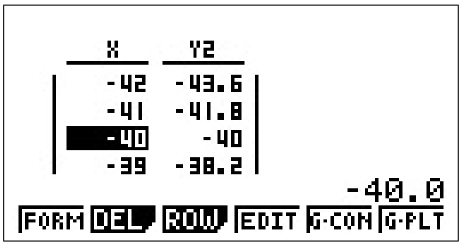


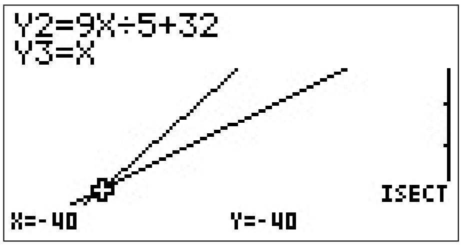
(c) Algebraically you have to solve $\left|(2x+30)-(9x/5+32)\right|\leqslant 5$ for x.
Simplifying gives $\left|x-10\right|\leqslant 25$ or $-15\leqslant C\leqslant 35$.

7. There is a temperature at which the numeric value is the same in degrees Celsius and in degrees Fahrenheit. What is this temperature? How far out, in degrees Fahrenheit, is the rough rule in this case?

The answer is $C=F=-40$.

(a) Here the tables wins hands down! You want to find where X (degrees Celsius) is the same as Y2, the actual rule for degrees Fahrenheit. Scrolling will find this eventually (below left). You may need to change *Start* and *End* in SET.





(b) Graphically will require some explanation and sophistication on the part of your students. You have to find where the graph of the actual rule (Y2) intersects with the graph of $y=x$.
Set Y3 = X, change the V-Window and use Trace or ISCT (above right).

(c) Algebraically, start with the actual rule $F=\frac{9}{5}C+32$, and set $F=C$, giving

$$C=\frac{9}{5}C+32.$$

Solving for C gives $C=-40$.

For a temperature of -40°C, the rough rule gives a value of -50°F, 10°F too low.

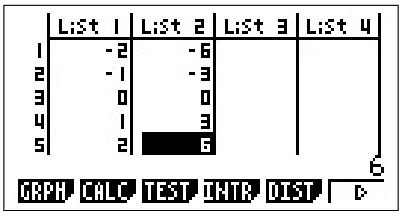
17 What's My Line?

Years 9, 10, Levels 1, 2; Strand: Algebra; Sub-strand: Coordinate Geometry – Straight Lines.
Author: Margie Smith.
These worksheets investigate the connection between a table of values, the line on a number plane and the equation of the line.

What's My Line? Worksheet 1

1. Press MENU 2 (STAT). Enter the following data into lists List 1 (X values) and List 2 (Y values).

X	−2	−1	0	1	2
Y	−6	−3	0	3	6



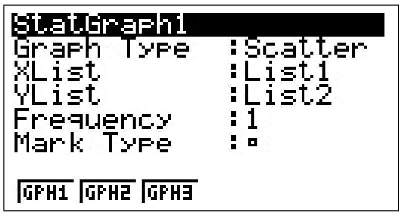
Question: What is the rule for this relationship?

Y = ___X + ___

2. Press F1 (GRPH), F6 (SET).

Set up your screen as shown in the figure.

Press EXIT.



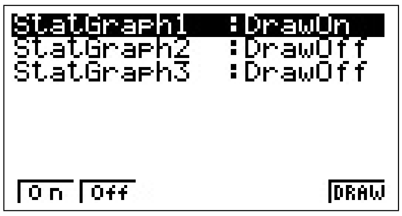
3. Press F4 (SEL) and turn StatGraph1 On.

Press F6 (DRAW); check you can see all the points.

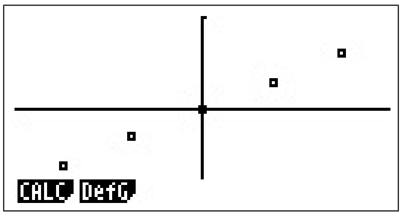
If you can't see all the points, press SET UP (SHIFT MENU). Set StatWind to Auto.

Press EXIT, then F1 twice to replot.

You can also adjust the window using V-Window.



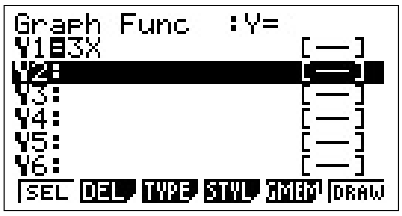
You should have a graph something like this →



4. Press OPTN F1 (PICT), F1 (STO) F1 1 EXE. Press SET UP (SHIFT MENU) and set Background to Pic1. Press EXIT.

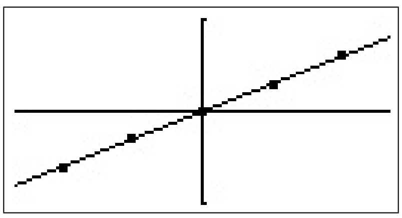
5. Press MENU 5 (GRAPH) and enter the relationship that you found in Question 1 into Y1.

Remember that X is the X,θ,T key.



6. Press F6 (DRAW) to draw the graph. Check that the graph runs through all your points.

If it doesn't, your rule is incorrect!!!!
Try again.



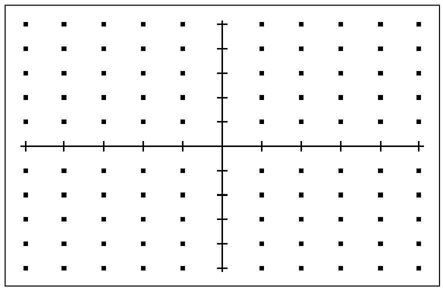
What's My Line? Worksheet 2

Using your graphics calculator, enter the following tables and determine the equation (rule) of the line that connects these points. Adjust the V-Window where necessary.
Sketch and label each pair of lines. Let the distance between tick marks in the figures below be 1 or 2 as required.

(a)

X	−2	−1	0	1	2
Y	−5	−2	1	4	7

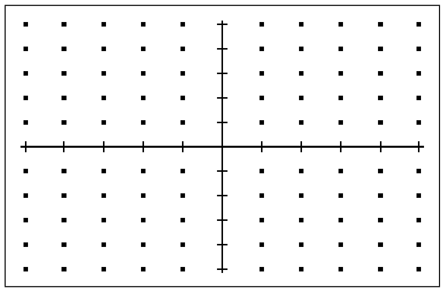
Y = ___________



(b)

X	−2	−1	0	1	2
Y	−8	−5	−2	1	4

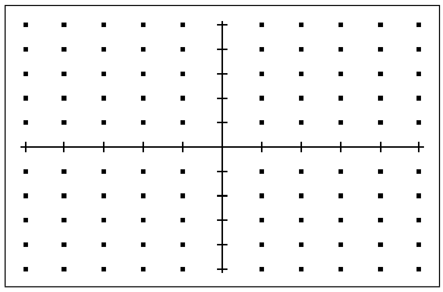
Y = ___________



(c)

X	−2	−1	0	1	2
Y	−5.5	−2.5	0.5	3.5	6.5

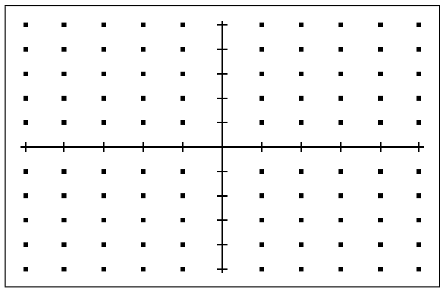
Y = ________



(d)

X	−2	−1	0	1	2
Y	−9	−6	−3	0	3

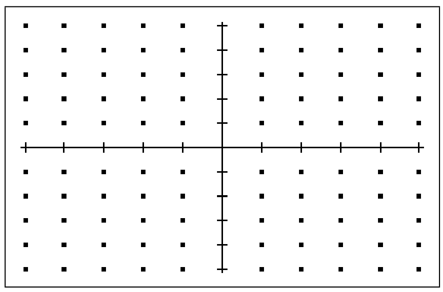
Y = ___________



(e)

X	−2	−1	0	1	2
Y	6	3	0	−3	−6

Y = ___________

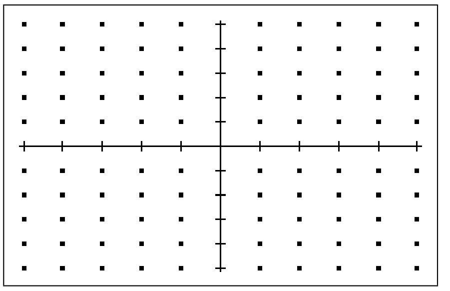


PTO

(f)

X	−2	−1	0	1	2
Y	7	4	1	−2	−5

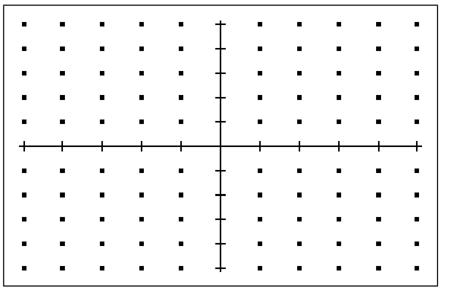
Y = ________



(g)

X	−2	−1	0	1	2
Y	4	1	−2	−5	−8

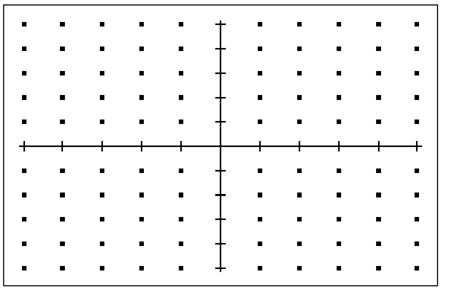
Y = ________



(h)

X	−2	−1	0	1	2
Y	6.5	3.5	0.5	−2.5	−5.5

Y = ________



18 Which Fuel?

Years 9 & 10, Level 1, Strand: Algebra; Sub-strand: Coordinate Geometry.

From 45 Single Lessons, University of Melbourne, 1996. Downloaded from the Casio Australia website. Modified by Peter McIntyre.

An application of linear functions to choosing whether to use petrol or LPG in your car.

A client asks her financial advisor whether it is better to install a liquid-petroleum-gas (LPG) tank in her recently acquired car or to stick with petrol. The advisor researches the cost of each option for her car and notes the following facts:

	Petrol	**LPG**
Installation cost ($)	0	2500
Average distance travelled (km/yr)	20,000	20,000
Fuel economy (litres/100 km)	10	12.5
Fuel cost ($/litre)	1.20	0.55

Construct a function $C(t)$ giving the cumulative cost of each option over t years.

Note that we are ignoring other costs such as depreciation and maintenance but we can assume these costs are the same for both options.

Make sure the units in your functions are correct.

Enter both functions into your graphics calculator.
Graph with $0 < \mathrm{X} < 5$; $0 < \mathrm{Y} < 10,000$.

1. After how many years and months does the cumulative cost of the LPG option become cheaper? How much has it cost to this time?

2. If our motorist only travels 10,000 km a year, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

3. If the fuel cost for LPG doubled, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

4. If the installation cost for LPG doubled, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

5. Can we analyse this problem in a general way?

Let $\$I$ be the installation cost of the LPG option, D km be the total distance travelled, e_p and e_l be the fuel economies in litres/100 km for petrol and LPG respectively, f_p and f_l be the fuel costs in \$/litre for petrol and LPG respectively.

(a) Using your previous calculations of $C(t)$ as a guide, write down $C(t)$ for each option for this general case.

(b) By equating C for the two options, find the time at which the cumulative cost of both options is the same. Verify your answer in Question 1 by substituting the appropriate values into the expression you found here.

(c) Show that your results in Q2 and Q4 are true for any values of D and I.

(d) With the price of petrol at \$1.20, what is the minimum cost of LPG for which LPG never becomes the cheaper option? Assume e_p and e_l take the values given in the table above.

Hint: Think in terms of the graphs when this happens.

Notes for Teachers

A client asks her financial advisor whether it is better to install a liquid-petroleum-gas (LPG) tank in her recently acquired car or to stick with petrol. The advisor researches the cost of each option for her car and notes the following facts:

	Petrol	**LPG**
Installation cost ($)	0	2500
Average distance travelled (km/yr)	20,000	20,000
Fuel economy (litres/100 km)	10	12.5
Fuel cost ($/litre)	1.20	0.55

Construct a function $C(t)$ giving the cumulative cost of each option over t years.

Note that we are ignoring other costs such as depreciation and maintenance but we can assume these costs are the same for both options.

Make sure the units in your functions are correct.

The only tricky bit here is making sure the fuel costs are converted to litres/km. The units for each term guide us to the correct formulas.

Petrol

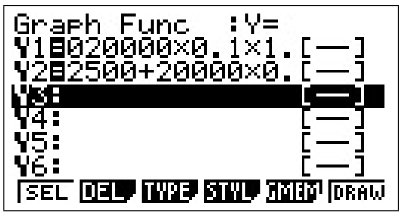
$$\begin{aligned} C_p(t)\,(\$) &= 20,000\left(\frac{\text{km}}{\text{yr}}\right)\times\frac{10}{100}\left(\frac{\ell}{\text{km}}\right)\times 1.20\left(\frac{\$}{\ell}\right)\times t\ (\text{yr}) \\ &= 20,000\times 0.1\times 1.2\,t\ (\$). \qquad \text{Note units `cancel' on RHS to give \$.} \end{aligned}$$

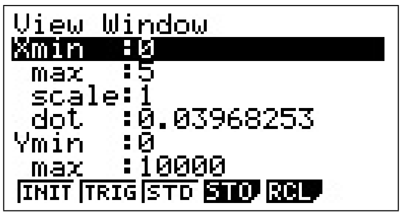
LPG

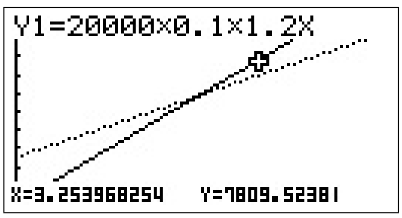
$$\begin{aligned} C_l(t)\ (\$) &= 2500\,(\$)\ +\ 20,000\left(\frac{\text{km}}{\text{yr}}\right)\times\frac{12.5}{100}\left(\frac{\ell}{\text{km}}\right)\times 0.55\left(\frac{\$}{\ell}\right)\times t\,(\text{yr}) \\ &= 2500+20,000\times 0.125\times 0.55\,t\ (\$). \end{aligned}$$

Enter both functions into your graphics calculator; graph with $0 < \text{X} < 5$; $0 < \text{Y} < 10,000$. The independent variable t becomes X in the calculator version of the equations.

If you enter the functions in the form above, it makes it easier to change numbers later.
Yscale is 1000.







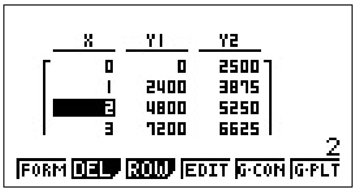
1. After how many years and months does the cumulative cost of the LPG option become cheaper? How much has it cost to this time?

Clearly the petrol option is cheaper to start with because there is no conversion cost, that is $C_p(0) < C_l(0)$. With the saving in fuel cost, eventually the LPG option becomes cheaper. We need to find when the two costs are the same.

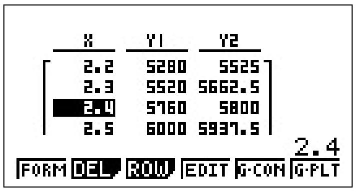
In mathematical terms, we need to solve $C_p(t) = C_l(t)$ for t. We can do this in three ways: numerically using a table, graphically or algebraically.

(a) Generate a table of Y1 and Y2 with Start = 0, End = 10 and Step = 1 (left-hand figure below). Scroll down to find when Y1 = Y2.

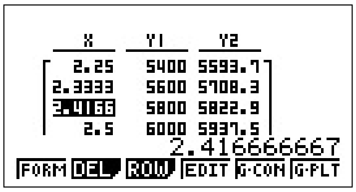
They are equal some time between $t = 2$ (Y1 < Y2) and $t = 3$ (Y1 > Y2), that is some time in the third year.



Step = 1



Step = 0.1

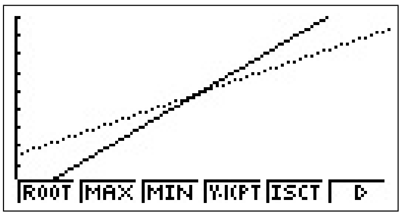


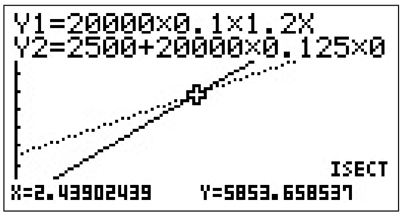
Step = 1/12

To find the answer more accurately, we need to reduce Step, with Start = 2. The natural thing to do is reduce Step to 0.1: we find that Y1 = Y2 between $t = 2.4$ and $t = 2.5$ (middle figure above). We then need to convert decimal years to months, giving the bounds as somewhere between 2 years 4.8 months and 2 years 6 months, still not accurate enough.

A better approach is to set Start = 2 and Step = 1/12 (type it in just like that): each division in the table corresponds to a month (you need to count down to find out which month — right-hand figure above). We then find that equality occurs between the end of month 5 (Y1 < Y2) and the end of month 6 (Y1 > Y2) of the second year, so the answer is after 2 years 6 months, realising that equality is sometime in the sixth month.

(b) On the graph, use [Trace] to find the approximate intersection point. This takes a little longer than the table. Faster is using ISCT. Make sure the intersection point appears on the screen, select ISCT from the [G-Solv] menu. Read off the coordinates of the point of intersection at the bottom of the screen.





The intersection point is (to 4 significant digits) (2.439, 5853), and converting the time (X) in decimal years to years and months, we find the two options cost the same after 2 years 5.3 months, or in the sixth month of the second year, as we found above.

(c) Algebraically we set $C_p(t) = C_l(t)$, giving

$$\begin{aligned} 20,000 \times 0.1 \times 1.20\,t &= 2500 + 20\,000 \times 0.125 \times 0.55\,t. \\ \therefore\ t &= \frac{2500}{20,000\big(0.1 \times 1.20 - 0.125 \times 0.55\big)} \\ &= 2.439, \end{aligned}$$

the same answer we found in (b).

The cost to this time is somewhere between \$5,823 and \$5,938 from the table (use a smaller Step for a more accurate value), \$5,854 from the graph and \$5,854 from the algebraic result, found by evaluating Y1 or Y2 at X = 2.439.

2. If our motorist only travels 10,000 km a year, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

Change the 20,000 to 10,000 in Y1 and Y2, and recalculate the time at which both options cost the same using one of the three methods above. We find $t = 4.878$, twice the previous time.

Halving the distance travelled doubles the pay-back time for the LPG option.

3. If the fuel cost for LPG doubled, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

Change the cost of LPG to \$1.10 in Y2 and replot the graphs or look at the table. The slope of the LPG graph is now greater than that of the petrol graph, so the two graphs never intersect when $t \geqslant 0$. The LPG option is never cheaper.

4. If the installation cost for LPG doubled, does this mean the LPG option takes twice as long to become the cheaper option? Explain what you did.

Change the 2,500 to 5,000 in Y2 and recalculate the time at which both options cost the same using one of the three methods above. We find again $t = 4.88$, twice the previous time.

Doubling the LPG installation cost doubles the pay-back time for the LPG option.

5. Can we analyse this problem in a general way?

Let $\$I$ be the installation cost of the LPG option, D km be the total distance travelled, e_p and e_l be the fuel economies in litres/100 km for petrol and LPG respectively, f_p and f_l be the fuel costs in \$/litre for petrol and LPG respectively.

(a) Using your previous calculations of $C(t)$ as a guide, write down $C(t)$ for each option for this general case.

Petrol: $C_p(t) = De_pf_pt/100$.

LPG: $C_l(t) = I + De_lf_lt/100$.

(b) By equating C for the two options, find the time at which the cumulative cost of both options is the same. Verify your answer in Question 1 by substituting the appropriate values into the expression you found here.

Setting $C_P(t) = C_L(t)$, we have

$$\begin{aligned} De_pf_pt/100 &= I + De_lf_lt/100. \\ \therefore\ t &= \frac{100I}{D\left(e_pf_p - e_lf_l\right)}. \end{aligned}$$

Using $I = 2500$, $D = 20\,000$, $e_p = 10$, $f_p = 1.20$, $e_l = 12.5$ and $f_l = 0.55$, we find $t = 2.439$, the value we found in Q1.

(c) Show that your results in Q2 and Q4 are true for any values of D and I.

The fact that t is proportional to I and inversely proportional to D shows that the results in Q2 and Q4 are true for any values of D and I.

(d) With the price of petrol at \$1.20, what is the minimum cost of LPG for which LPG never becomes the cheaper option? Assume e_p and e_l take the values given in the table above. *Hint*: Think in terms of the graphs when this happens.

LPG eventually becomes cheaper if the the graphs of $C_P(t)$ and $C_L(t)$ intersect for some positive t. The condition that they don't intersect for some positive t is that the two graphs are parallel, that is they have the same slope, or that the LPG graph has a larger slope.

The slope of $C_P(t)$ is $De_pf_p/100$, whereas the slope of $C_L(t)$ is $De_lf_l/100$. These slopes are the same when

$$\begin{aligned} \frac{De_pf_p}{100} &= \frac{De_lf_l}{100}, \quad \text{or} \\ e_pf_p &= e_lf_l, \quad \text{giving} \\ f_l &= \frac{e_pf_p}{e_l} \\ &= \frac{10 \times 1.20}{12.5} \\ &= 0.96, \end{aligned}$$

using the given values for e_p and e_l. When the cost per litre of LPG is 96c or more, the LPG option is never cheaper.

The same result can be seen from the expression for t in (b): when the lines are parallel, t is infinite, corresponding to the denominator being 0.

More generally,

$$f_l = \frac{e_p}{e_l} f_p = \frac{10}{12.5} f_p = 0.8 f_p,$$

so that the LPG option is never cheaper if the price of LPG is greater than or equal to 80% of the price of petrol.