Mathematics on a Casio 9860/CG20/CG50

Volume 2: Calculus

Chapters 9 – 15

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Contents

Mathematics on a Casio 9860/CG20/CG50

Volume 1 of this book contains the basic topics: *Graphics Calculators and Mathematics*; *Getting Started*; *Coordinate Geometry*; *Inequalities and Linear Programming*; *Fitting Curves to Data – Calculator Functions*; *Population Modelling 1 – Exponential Growth*; *Financial Mathematics 1 – Compound Interest*; and *Probability and Statistics 1 – Descriptive Statistics.*

Volume 1 Supplement: Activities for Years 9 and 10 contains extra activities for *Coordinate Geometry* and *Probability and Statistics 1.*

Volume 3 of this book contains more advanced topics, relevant to students and teachers of Specialist Mathematics and first-year university Mathematics courses. The topics in Volume 3 are: *Sequences and Series*; *Probability and Statistics 2 – Probability Distributions and Hypothesis Testing*; *Matrices and Vectors*; *Population Modelling 3: Matrix Models*; *Financial Mathematics 2 – TVM Calculations*; and *Complex Numbers.*

Calculator versions

The Casio graphics calculator models CG20 AU and CG50 AU are basically the same as the 9860 used here (except for a higher-resolution colour screen). This is probably true of all Casio graphics calculators one level below the ClassPad. There may be minor differences in how the screen looks and in the menus but they all do the same calculations.

Calculations, screenshots and figures were done on a Casio fx-9860G AU PLUS. The calculator programs were also written on this calculator, and converted to run on the other calculators. The programs are available at *www.canberramaths.org.au* under *Resources.*

Reference

Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS by Barry Kissane.[1] This book is a real bible on everything a graphics calculator can do and how to do it. A must-have for teachers of Years 10 – 12 using Casio calculators. Still very relevant but sadly now hard to find. Happily, a new electronic version is on the horizon.

Meanwhile, a shorter version (no Finance) is in *Learning Mathematics with Graphics Calculators* by Barry Kissane and Marian Kemp, available at *www.canberramaths.org.au Resources Graphics Calculators* in the *Articles* folder.

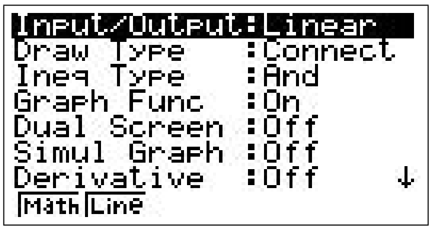
[1] *Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS* by Barry Kissane. The Mathematical Association of Western Australia, 2003, ISBN 1 876583 24 X.

9 Functions and their Graphs

9.1 Introduction

9.1.1 Setting up for graphing

Press [MENU] [5] for GRAPH mode. Press [SET UP] ([SHIFT] [MENU]).



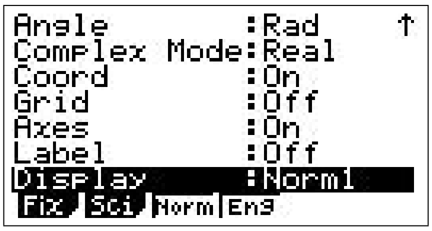
Input/Output: In *Linear* mode, commands are typed on one line, with arguments in brackets. In *Math* mode, the calculator tries to display commands in mathematical notation, with small boxes in the relevant positions for the inputs. *Linear* is used here.

Draw Type: *Connect* means graphs are a continuous line whereas *Plot* provides a set of points (those calculated) which are not connected. *Connect* is better in most cases.

Graph Func: *On* means the equation of the function is displayed *while* its graph is being drawn. Harmless, so leave On.

Simul Graph: *On* means that, if two or more functions are being drawn, they will be drawn simultaneously; *Off* means they are drawn sequentially. Unless you are simulating a race, leave set on *Off*.

Derivative: *On* means that, when a graph is being traced, (an approximation to) the value of the derivative at a point will be displayed as well as the function value. Leave *Off* unless required.



Angle: *Rad* (radians) is the appropriate setting for Mathematics.

Coord: *On* means the coordinates of the cursor wil be displayed when tracing a graph.

Axes: *On* means Cartesion axes will be drawn on plots where appropriate.

Display: *Fix* allows you to set the number of decimal places in numerical output; useful in financial calculations where the numbers are in dollars and cents. *Sci* displays all numbers in scientific notation. *Norm1* displays numbers smaller than 0.01 in scientific notation, *Norm2* numbers smaller than 0.000000001 (10^{-9}). Toggle between the two with [Norm]. *Eng* displays numbers similar to scientific notation but adjusted so that the exponent is always a multiple of 3. *Norm1* preferred unless you like counting zeros.

9.1.2 Australian Curriculum

References are given here to the corresponding topics in the Australian Curriculum. Specific references are given to the texts *Nelson Senior Maths Methods 11* (NSM11) and *Nelson Senior Maths Methods 12* (NSM12) used in the ACT.

The material here in Sections 2 − 5 is directly relevant to the topic *Functions and graphs*, Chapter 4 in NSM11.

Section 9.3.1 here on exponential and logarithmic functions is relevent to part of the topic *Derivatives, exponential and trigonometric functions*, Chapter 1 in NSM12, and to the topic *Logarithmic functions*, Chapter 7 in NSM12.

Section 9.3.4 here on periodic functions is relevent to the topic *Trigonometric functions and graphs*, Chapter 6 in NSM11.

9.2 Functions

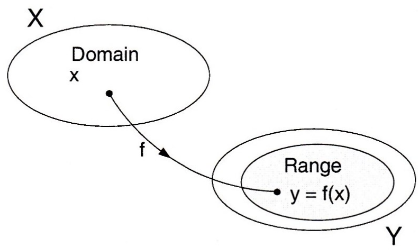
Many common relationships involve two variables in such a way that the value of one of the variables depends on the value of the other. Consider the relationship between the area of a circle A and its radius r. This can be expressed by the equation $A = \pi r^2$: the value of A depends on the value of r. We call A the **dependent variable** and r the **independent variable**.

Of particular interest are relationships in which to every value of the independent variable there corresponds *one and only one* value of the dependent variable. We call this type of correspondence a **function**.

9.2.1 Definition

A **function** f from a set X into a set Y is a corresondence that assigns to each element x in X exactly one element y in Y. We call y the **image** of x under f, and denote it by $f(x)$. The **domain** of f is the set X, and the **range** of f consists of all images of elements in X.

If, to each value in its range, there corresponds exactly one value in its domain, the function is said to be **one-to-one**. Moreover, if the range of f consists of all of Y, the function is called **onto**.



Remark: To begin with we work only with functions whose domains and ranges are sets of real numbers. We call such functions **real-valued functions** of a **real variable**.

Functions can be specified in a variety of ways. One way is an equation involving the dependent and independent variables. To evaluate a function described by an equation, we generally isolate the dependent variable on the left-hand side of the equation.

For example, the equation $x+2y=1$, written as

$$y = \frac{1-x}{2},$$

describes y as a function of x, and we can write this function as

$$f(x) = \frac{1-x}{2}.$$

This functional notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time providing a name 'f' for the function. The symbol $f(x)$ is read as 'f of x'. To denote the value of the dependent variable when $x = a$, we use the symbol $f(a)$. For example, the value of f when $x=3$ is

$$f(3) = \frac{1-3}{2} = \frac{-2}{2} = -1.$$

In an equation that defines a function, the role of the variable x is simply that of a placeholder. For example, the function

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\ ) = 2(\ )^2 - 4(\ ) + 1,$$

where parentheses are used instead of x. Therefore, to evaluate $f(-2)$, we simply place -2 in each set of parentheses:

$$f(-2) = 2(-2)^2 - 4(-2) + 1 = 2(4) + 8 + 1 = 17.$$

Remark: Although we generally use f as a convenient function name and x as the independent variable, we can use any symbols. For example, the following equations all define the same function:

$$\begin{aligned} f(x) &= x^2 - 4x + 7 \\ f(t) &= t^2 - 4t + 7 \\ g(s) &= s^2 - 4s + 7. \end{aligned}$$

Example 1 *Evaluating a function*

For the function $f(x)=x^2-4x+7$, evaluate

(a) $f(3a)$.

We begin by writing the equation for f in the form

$$f(\ ) = (\ )^2 - 4(\ ) + 7.$$

Then, $f(3a) = (3a)^2 - 4(3a) + 7 = 9a^2 - 12a + 7.$

(b) $f(b-1)$

$f(b-1) = (b-1)^2 - 4(b-1) + 7 = b^2 - 2b + 1 - 4b + 4 + 7 = b^2 - 6b + 12.$

(c) $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.

This one takes a bit longer.

$$\begin{aligned}
\frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{\big((x+\Delta x)^2 - 4(x+\Delta x) + 7\big) - \big(x^2 - 4x + 7\big)}{\Delta x} \\
&= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 4x - 4\Delta x + 7 - x^2 + 4x - 7}{\Delta x} \\
&= \frac{2x\Delta x + (\Delta x)^2 - 4\Delta x}{\Delta x} \\
&= 2x - 4 + \Delta x.
\end{aligned}$$

Exercises 1(a)–(f); 3(a)–(e) (Section 9.4)

9.2.2 Intervals, domains and ranges

Let $a<b$. Then x lies in the **open interval** (a, b) if $a<x<b$. x lies in the **closed interval** $[a, b]$ if $a \leqslant x \leqslant b$. $(a, b]$ means $a<x\leqslant b$; $[a, b)$ means $a\leqslant x<b$.

The domain of a function may be explicitly described together with the function, or it may be implied by the equation used to define the function. The implied domain is the set of all real numbers for which the equation is defined. For example, the function

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \leqslant x \leqslant 5$$

has the explicitly defined domain $\{x : 4 \leqslant x \leqslant 5\}$. On the other hand, the function

$$g(x) = \frac{1}{x^2 - 4}$$

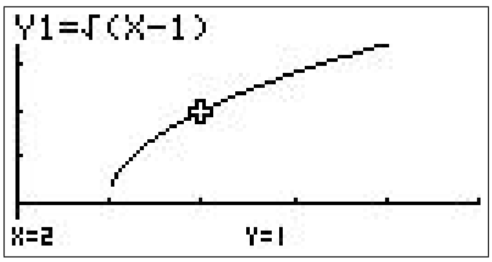
had the implied domain $\{x : x \neq \pm 2\}$.

Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function $f(x)=\sqrt{x+2}$ has the implied domain $\{x : x\geqslant -2\}$.

Example 2 *Finding the domain and range of a function*

Determine the domain and range of the function f defined by $f(x)=\sqrt{x-1}$.

$\sqrt{x-1}$ is not defined for $x-1<0$, that is for $x<1$, so we must have $x \geqslant 1$. Therefore, the domain is the interval $[1, \infty)$.
To find the range, we observe that $f(x)$ is never negative. Moreover, as x takes on the various values in the domain, $f(x)$ takes on all non-negative numbers. The range therefore is $[0, \infty)$.

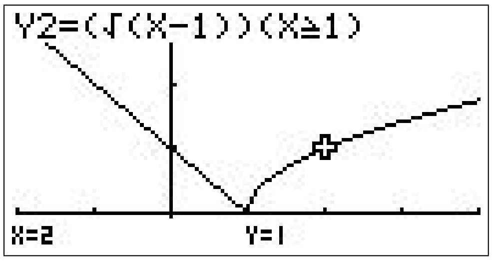


V-Window $[0, 5, 1] \times [-1, 4, 1]$

Example 3 *A function defined by more than one formula*

$$f(x) = \begin{cases} \sqrt{x-1} & \text{if } x \geqslant 1 \\ 1-x & \text{if } x < 1 \end{cases}$$

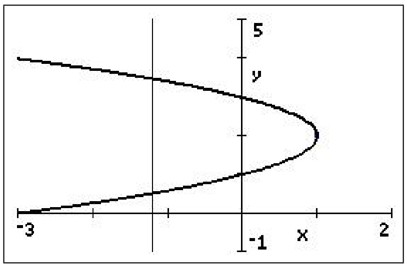
As f is defined for $x \geqslant 1$ and $x < 1$, the domain of the function is the set of all real numbers $(-\infty, \infty)$. For $x \geqslant 1$, the function behaves as in the previous example. For $x < 1$, $1-x$ is positive, and therefore the range of the function is the interval $[0, \infty)$. The graph of the function helps verify our conclusions. A function defined in several parts, like this one, is called a *piecewise* function.



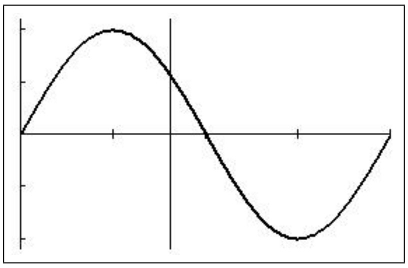
V-Window $[-2, 4, 1] \times [-0.5, 3, 1]$

Remember that the graph of the function $y = f(x)$ consists of all points $\big(x, f(x)\big)$, where x is the directed distance from the y axis and $f(x)$ the directed distance from the x axis.

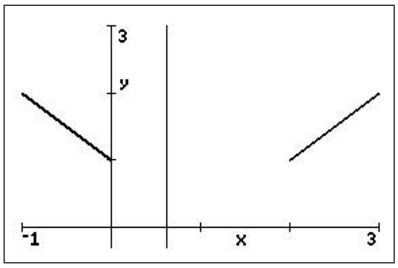
By definition of a function f, there is exactly one y value for each x value in the domain of f. It follows that a vertical line can intersect the graph of a function at most once. This provides us with a convenient visual test for functions. For example, in the left-hand figure below, the graph is not that of a function because a vertical line intersects the graph twice.



Not a function of x



A function of x



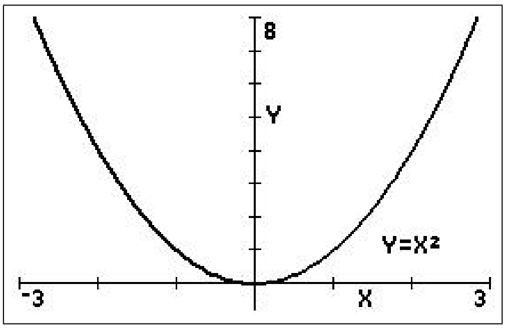
A function of x

Vertical-line test for functions

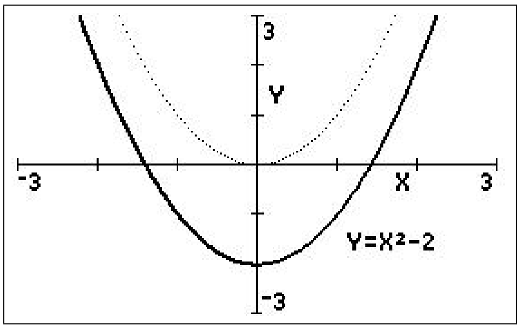
Exercise 5 (Section 9.4)

9.2.3 Basic transformations

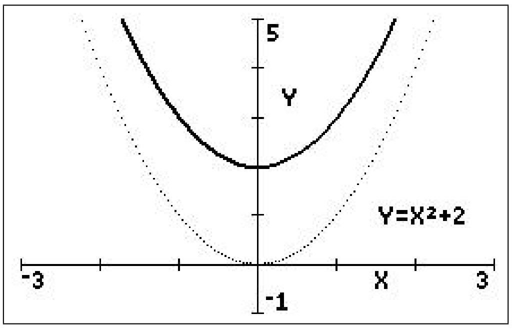
Functional notation lends itself well to describing transformation of graphs in the plane. Some families of graphs all have the same basic shape. For example, consider the graph of $y=x^2$, as shown below left. Compare this graph with the other graphs. The graph of $y=x^2$ is shown as a thin dotted line in these.



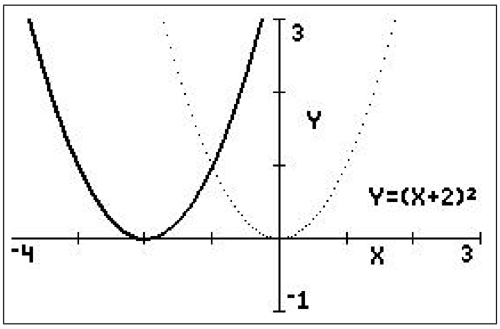
Original graph



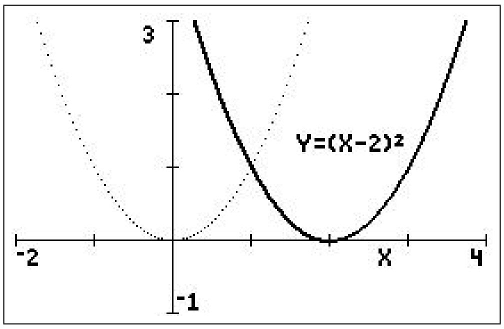
Vertical shift downward



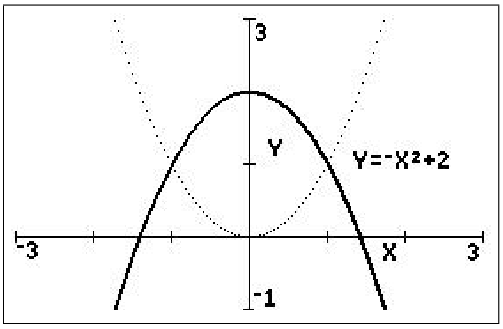
Vertical shift upwards



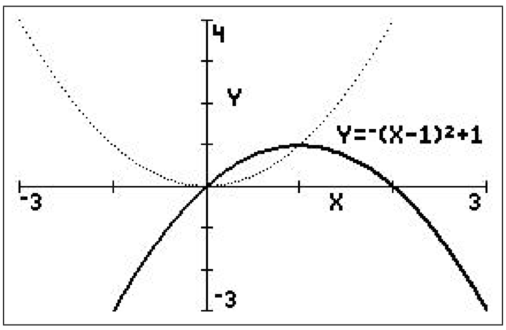
Horizontal shift to the left



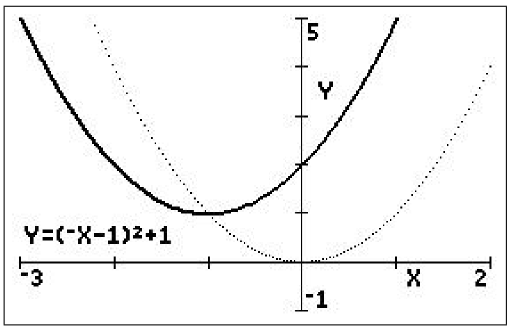
Horizontal shift to the right



Reflection in x axis and vertical shift



Reflection in x axis, vertical, horizontal shifts



Vertical, horizontal shifts, reflection in y axis

Each of the graphs above is a **transformation** of the graph of $y = x^2$. The three basic types of transformations involved in these seven graphs are: (1) *horizontal shifts*; (2) *vertical shifts*; and (3) *reflections*.

Basic types of transformations $(c > 0)$

Original graph	$y = f(x)$
Horizontal shift c units to the right	$y = f(x-c)$
Horizontal shift c units to the left	$y = f(x+c)$
Vertical shift c units downward	$y = f(x)-c$
Vertical shift c units upward	$y = f(x)+c$
Reflection about the x axis	$y = -f(x)$
Reflection about the y axis	$y = f(-x)$

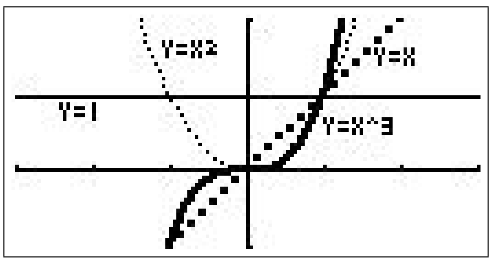
9.2.4 Classifications and combinations of functions

The modern notion of a function is due to the efforts of many different mathematicians who lived in the seventeenth and eighteenth centuries. Of particular note was Leonhard Euler (1707–1783), to whom we are indebted for the functional notation $y = f(x)$. The modern definition of a function was given by the German mathematician Peter Gustav Dirichlet (1805–1859). Dirichlet made many contributions to Mathematics and, together with Cauchy, Riemann and Weierstrauss, developed much of the rigour present in Calculus today.

By the end of the eighteenth century, mathematicians and scientists had concluded that most real-world phenomena can be represented by mathematical models taken from a basic collection of functions called **elementary functions**. Elementary functions are divided into three categories: (1) algebraic; (2) trigonometric; and (3) logarithmic and exponential.

Power functions

Power functions are of the form $f(x) = ax^b$, where a and b are constants. The first few of these with b zero or a positive integer ($a = 1$) are well-known: $f(x) = x^0 = 1$, a constant; $f(x) = x^1 = x$, a linear function; $f(x) = x^2$, a quadratic function; $f(x) = x^3$, a cubic function; and so on; all have domain $(-\infty, \infty)$. The figure below shows the graphs of these functions. The graphs of all of them, except the first, pass through the origin.

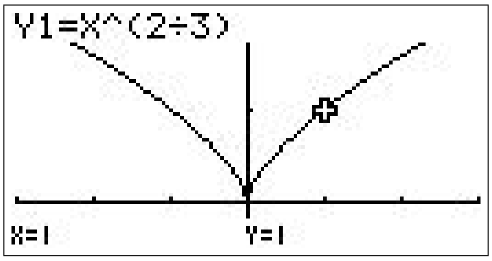


V-Window $[-3, 3, 1] \times [-1, 2, 1]$

Power functions with exponent b an even integer are even functions (Section 9.2.4), with range $[0, \infty)$ (except x^0), those with b an odd integer are odd functions, with range $(-\infty, \infty)$.

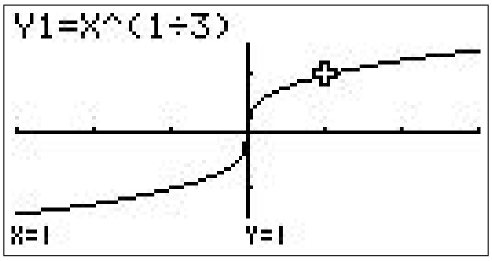
If b is a fraction, say $b = c/d$, roots of numbers come into play: $f(x) = x^{\frac{c}{d}} = \sqrt[d]{x^c}$, that is $f(x)$ is the dth root of x^c.

For example, $f(x)=x^{\frac{1}{2}}=\sqrt{x}$, $f(x)=x^{\frac{2}{3}}=\sqrt[3]{x^2}$. The graph of $f(x)=x^{\frac{2}{3}}$ has a cusp at $x=0$.



V-Window $[-3,3,1]\times[-1,2,1]$

The graph of $f(x)=x^{\frac{1}{3}}=\sqrt[3]{x}$ has a vertical point of inflection at $x=0$.



V-Window $[-3,3,1]\times[-2,2,1]$

Polynomial functions

The most common type of elementary function is a **polynomial function**.

Definition *polynomial function*

Let n be a non-negative integer. The function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function** of degree n. The numbers a_i, $i=0,\dots,n$ are called **coefficients**, with $a_n \neq 0$ the **leading coefficient** and a_0 the **constant term** of the polynomial function.

Remark: It is common practice to use subscript notation for coefficients of general polynomial functions but, for polynomial functions of low degree, we often use the following simpler forms.

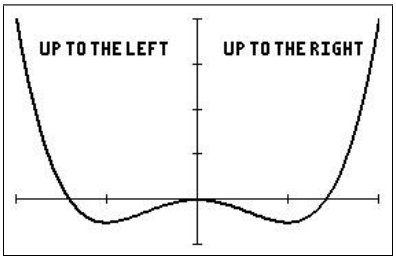
Zeroth degree	$f(x)=a$	constant
First degree	$f(x)=ax+b$	linear
Second degree	$f(x)=ax^2+bx+c$	quadratic
Third degree	$f(x)=ax^3+bx^2+cx+d$	cubic

Although the graph of a polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the left or right. Whether the graph eventually rises or falls is determined by the function's degree (odd or even) and by the sign of the leading coefficient, as indicated in the figures below.

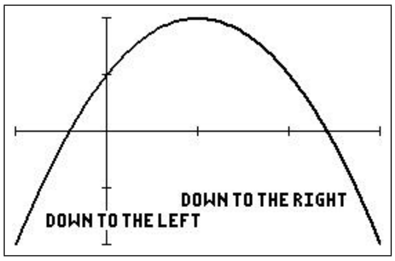
Leading-coefficient test for polynomial functions

Note that this test only determines the ultimate behaviour of the graphs, i.e. as $x \to \pm\infty$.

$\boldsymbol{a_n > 0}$

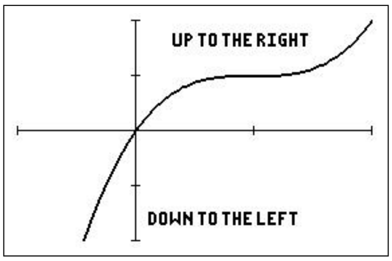


$\boldsymbol{a_n < 0}$

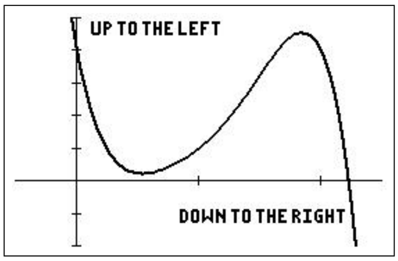


Polynomial functions of *even* degree

$\boldsymbol{a_n > 0}$



$\boldsymbol{a_n < 0}$



Polynomial functions of *odd* degree

Rational functions

Just as a rational number is the quotient of two integers, a **rational function** is the quotient of two polynomials. Specifically, a function f is rational if it has the form

$$f(x) = \frac{P(x)}{Q(x)}, \quad Q(x) \neq 0,$$

where P and Q are polynomials.

Polynomial functions and rational functions are two examples of a larger class of functions called **algebraic functions**. An algebraic function is one that can be expressed in terms of finitely many sums, differences, multiples, quotients and radicals (roots) involving x^n. For example, the following functions are algebraic:

$$f(x) = \sqrt{x+1} \quad \text{and} \quad g(x) = x + \frac{1}{\sqrt{x+1}}.$$

Functions that are not algebraic are called **transcendental**.

Combinations of functions

Two functions can be combined in various ways to create new functions. For example, if

$$f(x) = 2x-3 \quad \text{and} \quad g(x) = x^2+1,$$

we can form the functions

$$\begin{aligned} f(x)+g(x) &= (2x-3)+(x^2+1) = x^2+2x+2 && \text{sum} \\ f(x)-g(x) &= (2x-3)-(x^2+1) = -x^2+2x-4 && \text{difference} \\ f(x)g(x) &= (2x-3)(x^2+1) = 2x^3-3x^2+2x-3 && \text{product} \\ \frac{f(x)}{g(x)} &= \frac{2x-3}{x^2+1} && \text{quotient} \end{aligned}$$

Composite functions

Two functions can be combined in yet another way to form what is called a composite function.

Definition *composite function*

Let f and g be functions such that the range of g is the domain of f. The function given by $f \circ g(x) = f\big(g(x)\big)$ is called the **composite** of f with g.

It is important to realise that the composite of f with g may not be equal to the composite of g with f. This is illustrated in the following example.

Example 4 *Composition of functions*

Given $f(x)=2x-3$ and $g(x)=x^2+1$, find $f\big(g(x)\big)$ and $g\big(f(x)\big)$.

$$f\big(g(x)\big) = 2\big(g(x)\big)-3 = 2\big(x^2+1\big)-3 = 2x^2-1.$$

$$g\big(f(x)\big) = \big(f(x)\big)^2+1 = (2x-3)^2+1 = 4x^2-12x+10.$$

Note: $f\big(g(x)\big) \neq g\big(f(x)\big)$, i.e. $f \circ g \neq g \circ f$.

Exercises 1(g), (h); 3(f)–(i) (Section 9.4)

Even and odd functions

A function is even if its graph is symmetric with respect to the y axis; a function is odd if its graph is symmetric 'with respect to the origin'.

The algebraic equivalents, and a means for testing functions, are given in the following

Definition *even and odd functions*

The function $y=f(x)$ is *even* if $f(-x)=f(x)$.

The function $y=f(x)$ is *odd* if $f(-x)=-f(x)$.

Remarks: Except for trivial cases such as $f(x)=0$, the graph of a function cannot have symmetry with respect to the x axis, as it then fails the vertical-line test for the graph of a function.

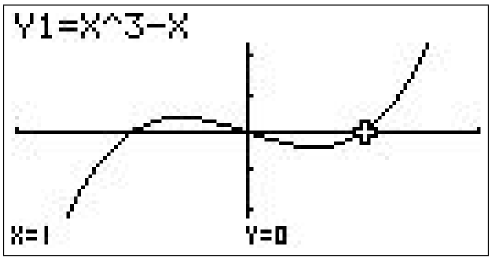
Example 5 *Even and odd functions*

Determine algebraically whether the following functions are even, odd or neither. Graph the function to confirm your answer.

(a) $f(x)=x^3-x$.

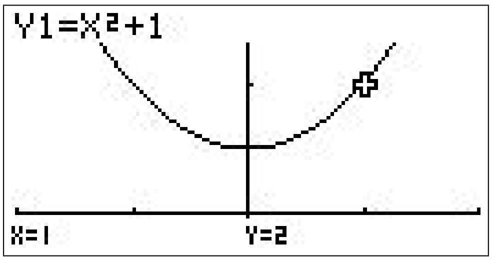
This function is odd because

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3-x) = -f(x).$$



(b) $f(x)=x^2+1$.

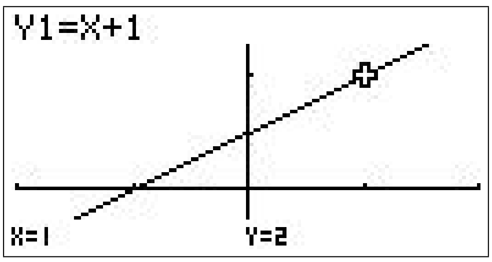
This function is even because $f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$.



(c) $f(x)=x+1$.

This function is neither even nor odd because

$$f(-x) = (-x) + 1 = -x + 1 \neq f(x) \text{ or } -f(x).$$



Most functions are neither even nor odd.

Any function can always be split into an even part and an odd part:

$$f(x) \;=\; \frac{1}{2}\Big(f(x)+f(-x)\Big) \;+\; \frac{1}{2}\Big(f(x)-f(-x)\Big) \;=\; g(x) \;+\; h(x).$$

Then,

$$g(-x) \;=\; \frac{1}{2}\big(f(-x)+f(x)\big) \;=\; g(x), \quad \text{so } g \text{ is even, and}$$

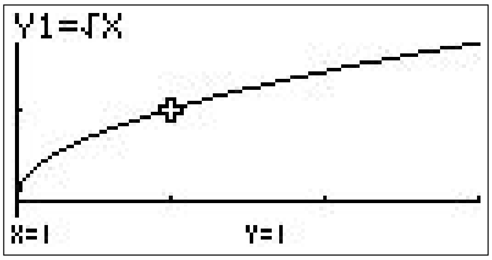
$$h(-x) \;=\; \frac{1}{2}\big(f(-x)-f(x)\big) \;=\; -\frac{1}{2}\big(f(x)-f(-x)\big) \;=\; -h(x), \quad \text{so } h \text{ is odd.}$$

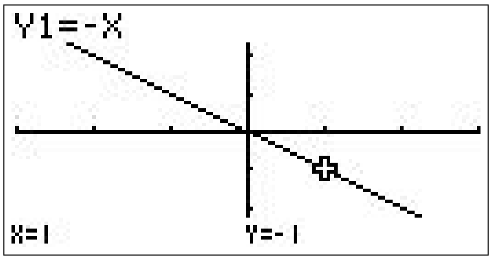
For example, if $f(x)=x+1$, then $g(x)=1$ and $h(x)=x$.

Exercise 6 (Section 9.4)

Monotonic functions

$f(x)$ *increases* (strictly) *monotonically* in an interval (a,b) if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all pairs of numbers x_1, x_2 in (a,b). For example, $f(x) = \sqrt{x}$ is a (strictly) monotonically increasing function on the interval $[0,\infty)$.





$f(x)$ *decreases* (strictly) *monotonically* in an interval (a,b) if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$. For example, $y=-x$ is a (strictly) monotonically decreasing function on the interval $(-\infty,\infty)$.

Absolute value

The absolute value of $g(x)$ or mod $g(x)$, written $|g(x)|$, is **defined** as

$$|g(x)| = \begin{cases} g(x) & \text{if} \quad g(x) \geq 0 \\ -g(x) & \text{if} \quad g(x) < 0. \end{cases}$$

In graphing absolute-value functions, the key point is the x value at which $g(x) = 0$; the defintion of the function is different either side of this value.

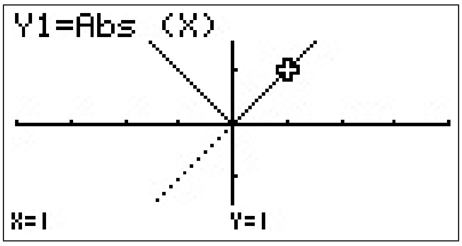
Example 6

Sketch the graph of $f(x) = |x|$ for $-2<x<2$.

Here, $f(x)=0$ when $x=0$. Then $f(x)=-x$ for $-2<x<0$ and $f(x)=x$ for $0 \leqslant x < 2$. The left-hand part of the graph of $|f(x)|$, i.e. $f(x)=-x$, is just the graph of $f(x)=x$ reflected in the x axis.

PTO

This provides a general method for plotting absolute values: plot the whole function, then reflect the part of the function below the x axis in the x axis.



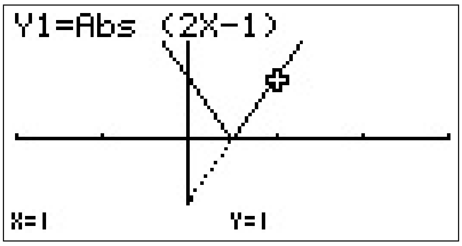
V-Window $[-4, 4, 1] \times [-2, 2, 1]$

The dotted line is $y = x$. *Abs* is the calculator function for absolute value. The graph here is called *piecewise linear* because each part of it is linear.

Example 7

Sketch the graph of $f(x) = |2x-1|$ for $-2 < x < 2$.

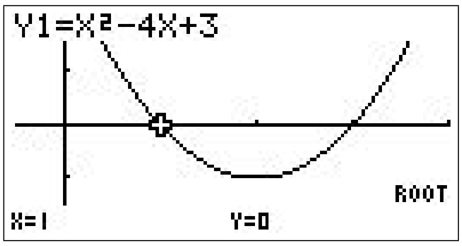
The cross-over here is when $2x-1 = 0$, i.e. $x = \frac{1}{2}$. Then $f(x) = -(2x-1)$ for $-2 < x < \frac{1}{2}$ and $f(x) = 2x-1$ for $\frac{1}{2} \leqslant x < 2$. The left-hand part of the graph, $x < \frac{1}{2}$, again is just the graph of $f(x) = 2x-1$ reflected in the x axis. The dotted line in the figure is $y = 2x-1$.

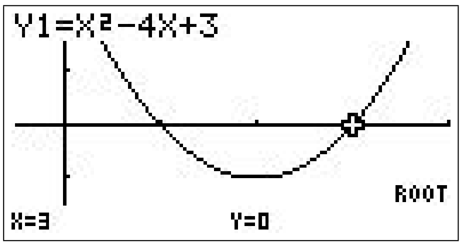


V-Window $[-2, 3, 1] \times [-1.5, 2, 1]$

9.2.5 Zeros of a function

When the graph of a function crosses the x axis (the value of the function is zero), we say that the function has a zero at that point. For example, the function $f(x) = x^2 - 4x + 3$ has a zero at $x = 1$, as $f(1) = 0$, and at $x = 3$, as $f(3) = 0$.





Zeros of a function

Algebraically, we find the zeros of a function by setting $f(x) = 0$ and solving for x. For the example here, we have

$$\begin{aligned} x^2 - 4x + 3 &= 0. \\ \therefore\ (x-1)(x-3) &= 0 \qquad \text{factorise the quadratic.} \\ \therefore\ x &= 1 \text{ or } 3. \end{aligned}$$

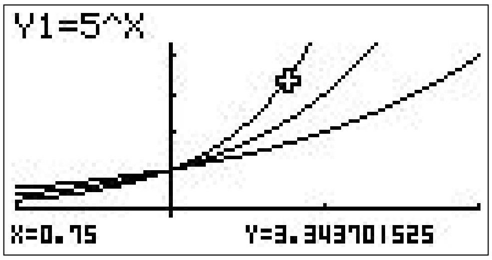
Alternatively, use the quadratic formula to find the zeros.

9.3 Some useful and interesting functions

9.3.1 Exponential and logarithmic functions

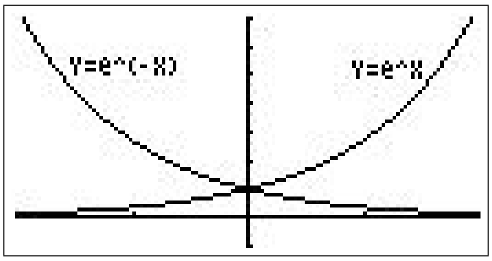
The general exponential function is of the form $f(x) = ab^x$, where a and b are constants. Compare these with power functions in Section 9.2.4.

Thel exponential functions $f(x) = 5^x$ (top curve), $f(x) = 3^x$ (middle curve) and $f(x) = 2^x$ (bottom curve) are shown in the figure below. All exponential functions pass through the point $(0, 1)$.



V-Window $[-1, 2, 1] \times [-1, 5, 1]$

Exponential functions occur frequently in modelling. Unless the model involves a discrete process such as bacteria dividing, for which $f(x) = 2^x$ is an obvious choice, the exponential function $f(x) = ae^{bx}$ is often used, where $e = 2.7182818\ldots$. This function represents growth when b is positive, decay when b is negative.[2]

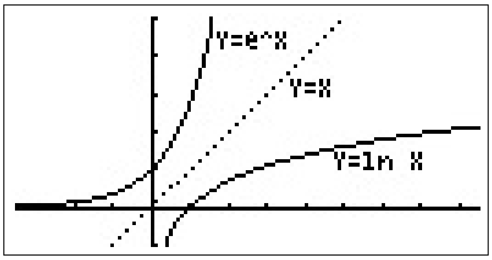


V-Window $[-2, 2, 1] \times [-1, 7, 1]$

Logarithmic functions occur sometimes in their own right but often because they are inverses of the corresponding exponential fuctions: e^x and $\ln(x)$ are inverses, so that

$$e^{\ln(x)} = x \quad \text{and} \quad \ln(e^x) = x.$$

The graphs of e^x and $\ln(x)$ are reflections of each other in the graph of $y = x$, as are all functions and their inverses.

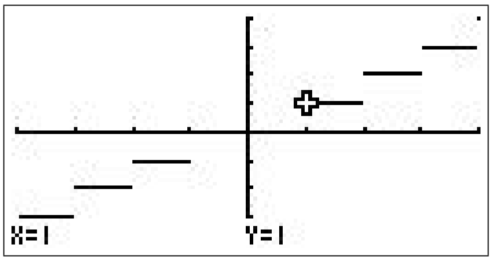


V-Window $[-3.5, 8.5, 1] \times [-1, 5, 1]$

[2] $f(x) = e^x$ has a unique place in Calculus, being the only function which is its own derivative.

9.3.2 Greatest-integer function

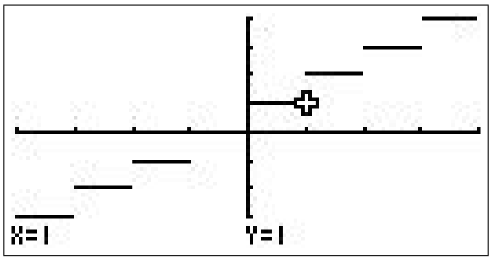
The **greatest-integer function** or **floor function** $\lfloor x$ is defined to be the greatest integer less than or equal to x. If $x = 1$, then $\lfloor x = 1$; if $x = 1.2$, then $\lfloor x = 1$, etc. The graph of the greatest-integer function is montonic increasing and piecewise. The left-hand end of each interval is a closed circle (included), the right-hand end an open circle (not included).



V-Window $[-4, 4, 1] \times [-4, 4, 1]$

9.3.3 Least-integer function

The **least-integer function** or **ceiling function** $\lceil x$ is defined to be the least integer greater than or equal to x. If $x = 1$, then $\lceil x = 1$; if $x = 1.2$, then $\lceil x = 2$, etc. The graph of the least-integer function is also montonic increasing and piecewise. The right-hand end of each interval is a closed circle (included), the left-hand end an open circle (not included).



V-Window $[-4, 4, 1] \times [-4, 4, 1]$

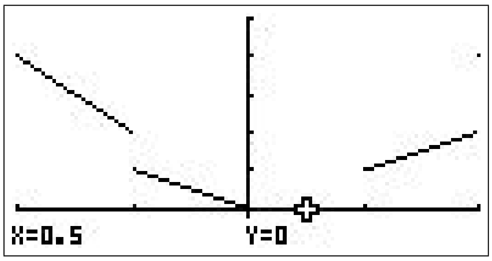
Examples of the greatest-integer function and least-integer function occur frequently in practice. In computing using integer arithmetic, the number 5.67 would be truncated to 5, an example of the greatest-integer function, The height of a staircase is an example of the least-integer function, and so are postal charges: e.g. letters up to 20 g cost \$1, letters over 20 g and up to 50 g cost \$1.50, etc.

Example 8

Sketch the graph of $f(x) = x\lfloor x$ for $-2 < x < 2$.

$\lfloor x$ is constant between integers, so that $f(x)$ is a series of straight-line segments of different slopes. The left-hand end of each interval is a closed circle (included), the right-hand end an open circle (not included).

x	$\lfloor x$	$x\lfloor x$
−2	−2	4
−1.5	−2	3
−1	−1	1
−0.5	−1	0.5
0	0	0
0.5	0	0
1	1	1
1.5	1	1.5
2	2	4



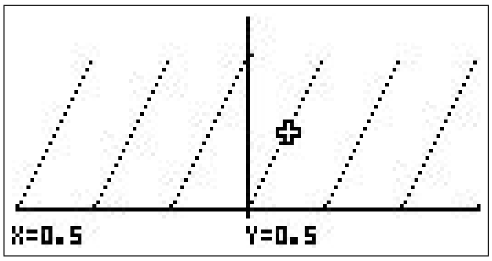
V-Window $[-2, 2, 1] \times [-0.5, 5, 1]$

9.3.4 Periodic functions

Example 9

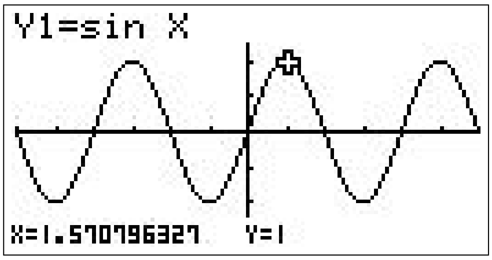
Sketch the graph of $f(x) = x - \lfloor x$ for $-3 < x < 3$.

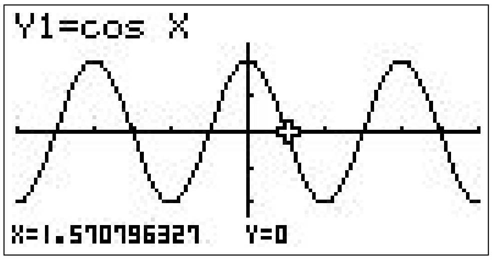
x	$\lfloor x$	$x - \lfloor x$
−3	−3	0
−2.75	−3	0.25
−2.5	−3	0.5
−2.25	−3	0.75
−2	−2	0
−1.75	−2	0.25
−1.5	−2	0.5
−1.25	−2	0.75
−1	−1	0
−0.75	−1	0.25
−0.5	−1	0.5
−0.25	−1	0.75
0	0	0
⋮	⋮	⋮

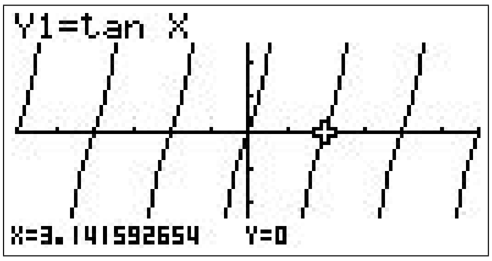


V-Window $[-3, 3, 1] \times [-0.25, 1.25, 1]$

This is an example of a **periodic function**, one that repeats itself at regular intervals. Mathematically, $f(x+a) = f(x)$, where the constant a is called the period. The function $x - \lfloor x$ has period 1. The trigonometric functions $\sin(x)$, $\cos(x)$ and $\tan(x)$ are periodic functions with period 2π (figures over the page; Radian mode; View Window TRIG).







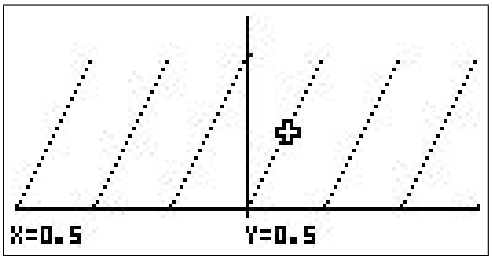
$\tan(x)$ has vertical asymptotes at odd multiples of $\pi/2$.

Example 10

Sketch the graph of $f(x)$ for $-3 < x < 3$, where

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ f(x+1) & \text{for all } x. \end{cases}$$

x	$f(x)$
⋮	⋮
−1	0
−0.75	0.25
−0.5	0.5
−0.25	0.75
0	0
0.25	0.25
0.5	0.5
0.75	0.75
1	0
1.25	0.25
1.5	0.5
1.75	0.75
⋮	⋮



V-Window $[-3, 3, 1] \times [-0.25, 1.25, 1]$

This is the same as Example 9, which shows that there may be more than one way of expressing a function.

Exercise 4(b) (Section 9.4)

9.4 Exercises

Solutions in Section 9.5

1. If $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{2x+5}{4x-3}$, find

(a) $f(2)$

(b) $f(0)$

(c) $f(x^2)$

(d) $g(t)$

(e) $g(s)$

(f) $f\big(\tan(\theta)\big)$

(g) $f\big(g(x)\big)$

(h) $g\big(f(x)\big)$

2. Sketch the graph of the following functions. Check with your calculator. If the domain is not stated, assume the maximum possible domain.

(a) $y = \dfrac{x}{x-1}$

(b) $y = |x-1| + 1$

(c) $y = \dfrac{x}{x^2+1}$

(d) $y = \lfloor x, \; -3 \leqslant x \leqslant 3$

(e) $y = x + \lfloor x, \; -2 \leqslant x \leqslant 2$

(f) $y = \lfloor x^2, \; -2 \leqslant x \leqslant 2$

(g) $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$

(h) $y = x|x|$

3. If $f(x) = |x|$ and $g(x) = x^2$, find

(a) $f(-1)$

(b) $g(2)$

(c) $f(x^2)$

(d) $g\big(\cos(t)\big)$

(e) $g\left(\dfrac{1}{x}\right)$

(f) $f\big(g(x)\big)$

(g) $g\big(f(x)\big)$

(h) $f(x+y)$

(i) $f\big(g(x+y)\big)$

4. Sketch the following functions. Check with your calculator.

(a) $f(x) = |x+1| + |2x-1|; \; -2 < x < 2.$

Hint: Consider the x values at which each term in the function equals 0. Either use these values to draw up a table of function values or work out the expression for the function in each of three intervals of the x axis.

(b) $f(x) = x^2, \; 0 \leqslant x < 1$ and $f(x+1) = f(x)$ for all x. Plot the function for $-2 < x < 2$.

PTO

5. Find the range and domain of the following functions. Sketch their graphs, using your calculator if necessary.

(a) $f(x) = (x-1)^{\frac{1}{2}}$

(b) $f(x) = (1-x)^{\frac{1}{2}}$

(c) $f(x) = x^2$

(d) $f(x) = 4-x^2$

(e) $f(x) = (9-x^2)^{\frac{1}{2}}$

(f) $f(x) = (25-x^2)^{\frac{1}{2}}$

(g) $f(x) = \dfrac{1}{|x|}$

(h) $f(x) = |x-2|$

(i) $f(x) = \dfrac{|x|}{x}$

(j) $f(x) = (x^2-4)^{\frac{1}{2}}$.

6. Determine algebraically if the following functions are even, odd or neither. Confirm your answer by graphing the function, using your calculator if necessary.

(a) $f(x) = 4-x^2$

(b) $f(x) = x^3$

(c) $f(x) = x(4-x^2)$

(d) $f(x) = 1+x^3$

(e) $f(x) = 4x-x^2$

(f) $f(x) = x^{\frac{2}{3}}$. *Hint*: $x^{\frac{a}{b}} = \left(x^a\right)^{\frac{1}{b}}$

9.5 Solutions to the exercises

1. If $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{2x+5}{4x-3}$,

(a) $f(2) = 3$

(b) $f(0) = -1$

(c) $f(x^2) = \dfrac{x^2+1}{x^2-1}$

(d) $g(t) = \dfrac{2t+5}{4t-3}$

(e) $g(s) = \dfrac{2s+5}{4s-3}$

(f) $f\big(\tan(\theta)\big) = \dfrac{\tan(\theta)+1}{\tan(\theta)-1}$

(g) More complicated!

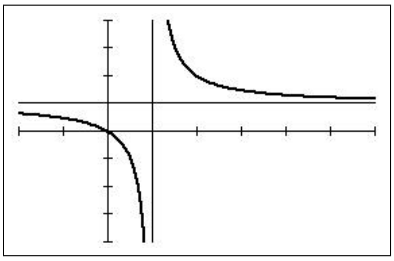
$$\begin{aligned}
f\big(g(x)\big) &= f\left(\frac{2x+5}{4x-3}\right) \\
&= \frac{\left(\frac{2x+5}{4x-3}\right)+1}{\left(\frac{2x+5}{4x-3}\right)-1} \qquad \times \frac{4x-3}{4x-3} \\
&= \frac{2x+5+(4x-3)}{2x+5-(4x-3)} \\
&= \frac{6x+2}{-2x+8} \\
&= \frac{3x+1}{-x+4}
\end{aligned}$$

(h) Again, a bit of algebra.

$$\begin{aligned}
g\big(f(x)\big) &= g\left(\frac{x+1}{x-1}\right) \\
&= \frac{2\left(\frac{x+1}{x-1}\right)+5}{4\left(\frac{x+1}{x-1}\right)-3} \qquad \times \frac{x-1}{x-1} \\
&= \frac{2(x+1)+5(x-1)}{4(x+1)-3(x-1)} \\
&= \frac{7x-3}{x+7}.
\end{aligned}$$

2. Sketch the graph of the following functions. Check with your calculator. If the domain is not stated, assume the maximum possible domain.

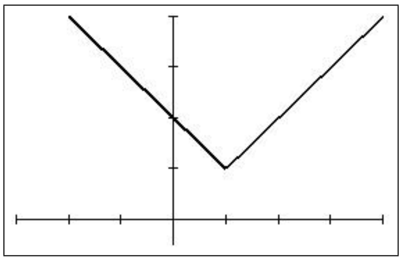
(a) $y = \dfrac{x}{x-1}$ Y1 = X ÷ (X−1)



V-Window $[-2, 6, 1] \times [-4, 4, 1]$

Domain all x except $x = 1$. Note the horizontal asymptote at $y = 1$ and the vertical asymptote at $x = 1$.

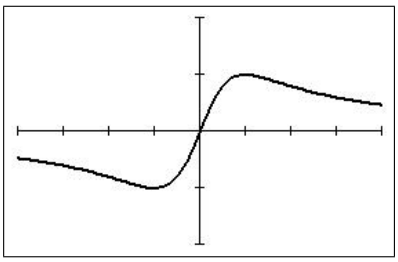
(b) $y = |x-1| + 1$ Y1 = Abs(X−1) + 1 *Abs* is in the OPTN NUM menu



V-Window $[-3, 4, 1] \times [-0.5, 4, 1]$

Domain all x.

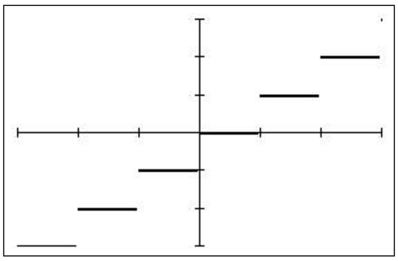
(c) $y = \dfrac{x}{x^2+1}$ Y1 = X ÷ (X²+1)



V-Window $[-4, 4, 1] \times [-1, 1, 0.5]$

Domain all x.

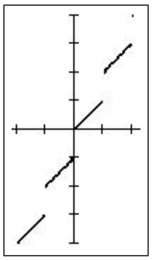
(d) $y = \lfloor x, \; -3 \leqslant x \leqslant 3$ Y1 = Int(X) *Int* is in the OPTN NUM menu



V-Window $[-3, 3, 1] \times [-3, 3, 1]$

Domain all x. The left-hand end of each line interval is included (closed circle), the right-hand end is not (open circle).

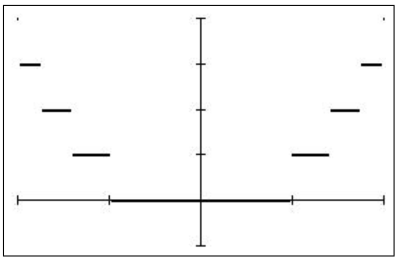
(e) $y = x + \lfloor x, \; -2 \leqslant x \leqslant 2$ Y1 = X + Int(X)



V-Window $[-2, 2, 1] \times [-4, 4, 1]$

Domain all x. The left-hand end of each line interval is included (closed circle), the right-hand end is not (open circle).

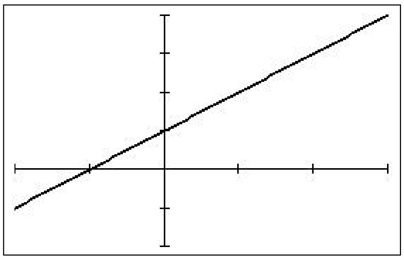
(f) $y = \lfloor x^2, \; -2 \leqslant x \leqslant 2$ Y1 = Int(X^2)



V-Window $[-2, 2, 1] \times [-1, 4, 1]$

Domain all x. The end of each line interval closer to the y axis is included (closed circle), the other end is not (open circle).

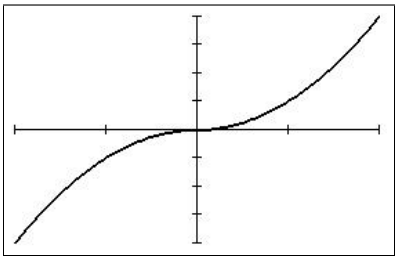
(g) $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1 \\ 2 & x=1 \end{cases}$ just plot Y1 = (X²−1) ÷ (X−1); Trace to X = 1.



V-Window $[-2, 3, 1] \times [-2, 4, 1]$

Domain all x. The first part of the function has a hole discontinuity at $x=1$; the second part 'fills in' that hole to make the function a continuous straight line.

(h) $y = x|x|$ Y1 = X Abs(X)



V-Window $[-2, 2, 1] \times [-4, 4, 1]$

Domain all x. The part of the curve with $x \geqslant 0$ is just the graph of $y=x^2$; the part with $x<0$ is the graph of $y=-x^2$, that is the graph of $y = x^2$ reflected in the x axis.

3. If $f(x) = |x|$ and $g(x) = x^2$,

(a) $f(-1) = 1$

(b) $g(2) = 4$

(c) $f(x^2) = x^2$

(d) $g\big(\cos(t)\big) = \cos^2(t)$

(e) $g\left(\dfrac{1}{x}\right) = \dfrac{1}{x^2}$

(f) $f\big(g(x)\big) = f(x^2) = |x^2| = x^2$

(g) $g\big(f(x)\big) = g(|x|) = |x|^2 = x^2$

here, $f \circ g = g \circ f$

(h) $f(x+y) = |x+y|$

(i) $f\big(g(x+y)\big) = |x+y|^2 = (x+y)^2$

4. Sketch the following functions. Check with your calculator.

(a) $f(x) = |x+1| + |2x-1|;\ -2<x<2$ Y1 = Abs(X+1) + Abs(2X−1)

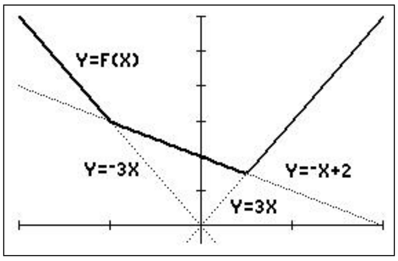
Algebraically: the first term is 0 at $x=-1$, the second at $x=\frac{1}{2}$. Therefore, divide the x axis into three intervals: $-2<x<-1$; $-1<x<\frac{1}{2}$; and $\frac{1}{2}<x<2$.

For $-2<x<-1$, $|x+1|=-x-1$ and $|2x-1|=-2x+1$, so that $f(x) = -x-1-2x+1 = -3x$.

For $-1<x<\frac{1}{2}$, $|x+1|=x+1$ and $|2x-1|=-2x+1$, so that $f(x) = x+1-2x+1 = -x+2$.

For $\frac{1}{2}<x<2$, $|x+1|=x+1$ and $|2x-1|=2x-1$, so that $f(x) = x+1+2x-1 = 3x$.

Check by plotting these three functions against the plot of Y1 = Abs(X+1) + Abs(2X−1).

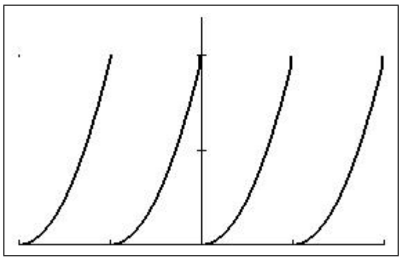


V-Window $[-2, 2, 1]\times[-0.5, 6, 1]$

Domain all x. A piecewise-linear function.

(b) $f(x) = x^2,\ 0\leqslant x<1$ and $f(x+1) = f(x)$ for all x. Plot the function for $-2\leqslant x<2$.

Plot the basic function $y = x^2$ on the interval $0 \leqslant x < 1$, then repeat the graph at x intervals of 1.



V-Window $[-2, 2, 1]\times[0, 1.2, 0.5]$

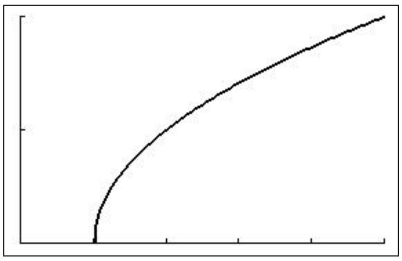
Domain all x. This is a periodic function of period 1. The left-hand end of each part of the graph is included (closed circle), the right-hand end is not (open circle).

You could also use the PERIODIC program with Y1 = X∧2, PERIOD 1, PHASE SHIFT 0. Change the V-Window after the graph has been plotted, press [MENU] [5] [6] to replot the graph.

5. Find the range and domain of the following functions. Sketch their graphs, using your calculator when necessary.

(a) $f(x) = (x-1)^{\frac{1}{2}}$ Y1 = √(X−1) or Y1 = (X−1)∧(1/2)

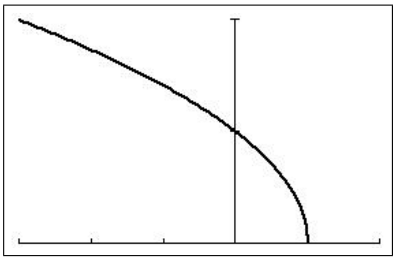
Domain $[1, \infty)$; range $[0, \infty)$.



V-Window $[0, 5, 1] \times [0, 2, 1]$

(b) $f(x) = (1-x)^{\frac{1}{2}}$ Y1 = √(1−X)

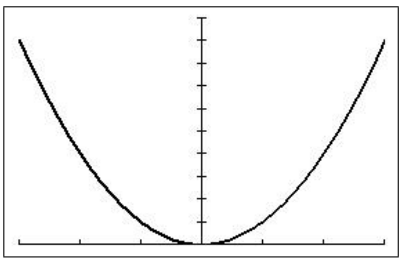
Domain $(-\infty, 1]$; range $[0, \infty)$.



V-Window $[-3, 2, 1] \times [0, 2, 1]$

(c) $f(x) = x^2$ Y1 = X^2

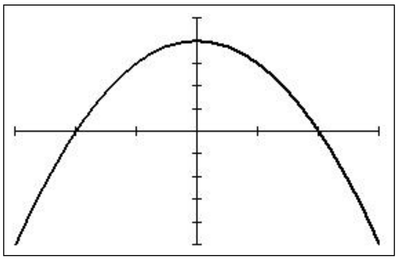
The basic parabola. Domain $(-\infty, \infty)$; range $[0, \infty)$.



V-Window $[-3, 3, 1] \times [0, 10, 1]$

(d) $f(x) = 4-x^2$ Y1 = 4−X^2

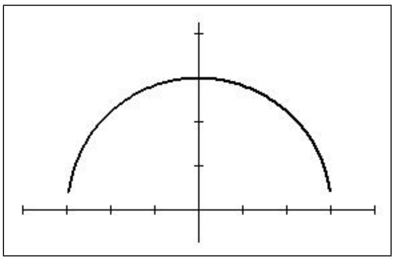
A head-up parabola. Domain $(-\infty, \infty)$; range $(-\infty, 4]$.



V-Window $[-3, 3, 1] \times [-5, 5, 1]$

(e) $f(x) = (9-x^2)^{\frac{1}{2}}$ Y1 = √(9−X^2)

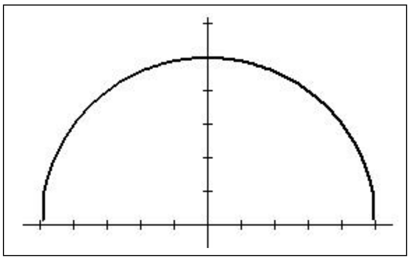
A semi-circle, centre the origin, radius 3. Domain $[-3, 3]$; range $[0, 3]$.



V-Window $[-4, 4, 1] \times [-0.7, 4.2, 1]$

(f) $f(x) = (25-x^2)^{\frac{1}{2}}$ Y1 = √(25−X^2)

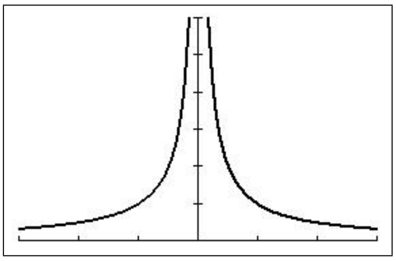
A semi-circle, centre the origin, radius 5. Domain $[-5, 5]$; range $[0, 5]$.



V-Window $[-5.5, 5.5, 1] \times [-0.7, 6.2, 1]$

(g) $f(x) = \dfrac{1}{|x|}$ Y1 = 1 ÷ Abs X

Domain $(-\infty, 0)$, $(0, \infty)$; range $(0, \infty)$.

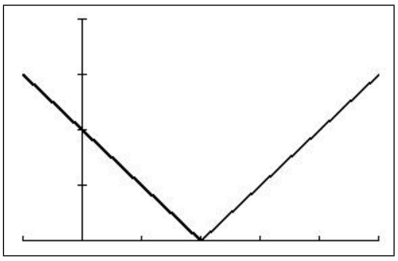


V-Window $[-3, 3, 1] \times [0, 6, 1]$

Vertical asymptote at $x = 0$.

(h) $f(x) = |x-2|$ Y1 = Abs (X−2)

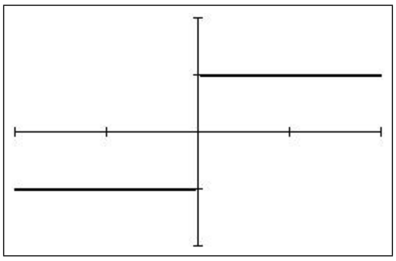
Domain $(-\infty, \infty)$; range $[0, \infty)$.



V-Window $[-1, 5, 1] \times [0, 4, 1]$

(i) $f(x) = \dfrac{|x|}{x}$ Y1 = Abs X ÷ X

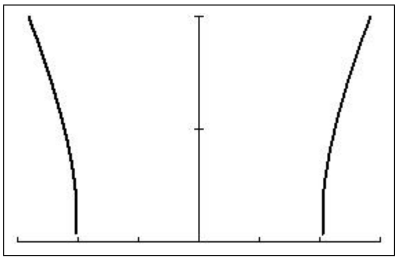
Domain $(-\infty, 0)$, $(0, \infty)$; range $-1, 1$.



V-Window $[-2, 2, 1] \times [-2, 2, 1]$

(j) $f(x) = (x^2 - 4)^{\frac{1}{2}}$ Y1 = √(X²−4)

Domain $(-\infty, -2]$, $[2, \infty)$; range $[0, \infty)$.

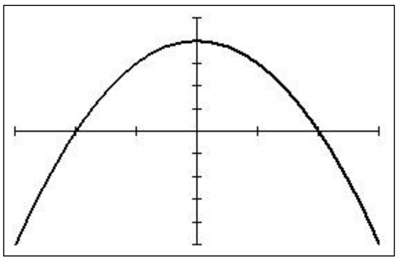


V-Window $[-3, 3, 1] \times [0, 2, 1]$

6. Determine algebraically if the following functions are even, odd or neither. Confirm your answer by graphing the function, using your calculator if necessary.

(a) $f(x) = 4 - x^2$ Y1 = 4−X²

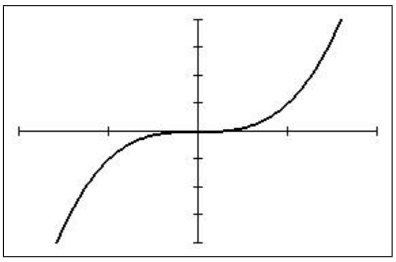
$f(-x) = 4 - (-x)^2 = 4 - x^2 = f(x)$. An even function.



V-Window $[-3, 3, 1] \times [-5, 5, 1]$

(b) $f(x) = x^3$ Y1 = X^3

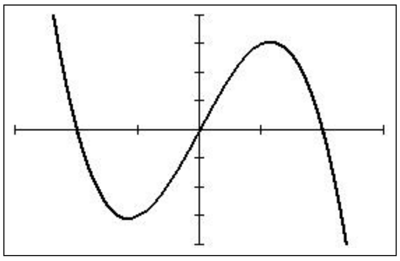
$f(-x) = (-x)^3 = -x^3 = -f(x)$. An odd function.



V-Window $[-2, 2, 1] \times [-4, 4, 1]$

(c) $f(x) = x(4-x^2)$ Y1 = X(4−X^2)

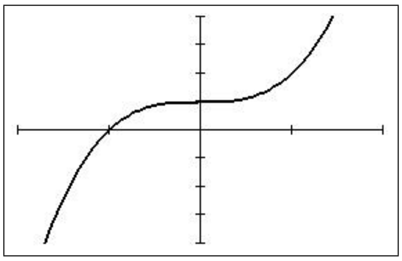
$f(-x) = (-x)(4-(-x)^2) = -x(4-x^2) = -f(x)$. An odd function.



V-Window $[-3, 3, 1] \times [-4, 4, 1]$

(d) $f(x) = 1 + x^3$ Y1 = 1 + X^3

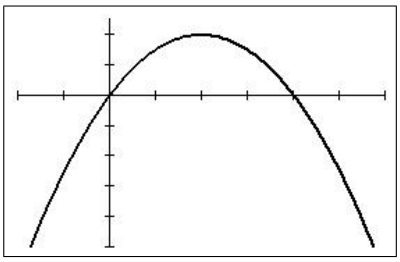
$f(-x) = 1 + (-x)^3 = 1 - x^3 \neq f(x)$ or $-f(x)$. Neither even nor odd.



V-Window $[-2, 2, 1] \times [-4, 4, 1]$

(e) $f(x) = 4x - x^2$ Y1 = 4X − X^2

$f(-x) = 4(-x)-(-x)^2 = -4x - x^2 \neq f(x)$ or $-f(x)$. Neither even nor odd.

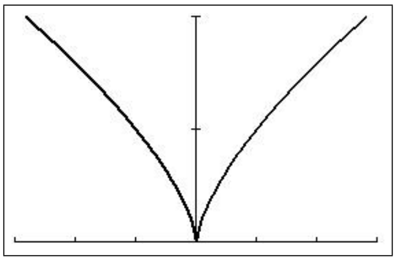


V-Window $[-2, 6, 1] \times [-10, 5, 2]$

(f) $f(x) = x^{\frac{2}{3}}$ Y1 = X^(2/3)

Write f as $f(x) = (x^2)^{\frac{1}{3}}$.

$f(-x) \;=\; \left((-x)^2\right)^{\frac{1}{3}} \;=\; (x^2)^{\frac{1}{3}} \;=\; x^{\frac{2}{3}} \;=\; f(x)$. An even function.



V-Window $[-3, 3, 1] \times [0, 2, 1]$

9.6 Doing and saving the graphs

Casio 9860

I used Casio FA-124 (PC only) to capture the screens.[3]

Start FA-124.

Connect your calculator to a computer using the USB cable; press F2 for screen capture and EXIT as instructed.

Generate the figure (graph or diagram) on your calculator.

Click on the small camera icon on the left-hand screen of FA-124 (the sixth icon of nine). You will get a window on your computer saying Screen Capturing . . . .

Press SHIFT 7 (CAPTURE) on your calculator and the screen should be transferred to your computer.

Press *File*, then *Save As* to save the screen as a .bmp file to wherever you want to save it.

Casio CG20/50

Download the *Screen Receiver* (PC and Mac) from *edu.casio.com/forteachers/er/software*, install and open it. In the *Help* menu at the top of the screen, there is the manaul; this tells you how it all works. You can save the figure as .jpg or .png (open Preferences in the File menu).

Cropping the figure

On a PC, if you want to crop the figure or save the file as something other than a .bmp file, open the file with (Microsoft) *Paint* (right click on the file name and select *Open with*).

To crop the figure, click on the down arrowhead under *Image*, then the down arrowhead under *Select* and on *Rectangular selection.* With the cursor, starting at a corner of the part of the figure you wish to retain, pull out a box to cover the rest of it. Click on the *Crop* icon (top right-hand corner of the *Image* box).

[3]If you have the original Utilities CD that came with the calculator, use this. Otherwise, *FA-124* is available at *edu.casio.com/forteachers/er/software*. Download the manual there and read the part about installing FA-124 on your particular operating system. Good luck!

Save the figure by clicking on the down arrow in the box to the left of *Home*. You can also save it at this stage as a .jpg file or various other types.

On a Mac, open the saved figure with *Preview*, click on *Tools* and make sure *Automatic Selection* or *Rectangular Selection* is ticked. Pull out a box with the cursor around the part you want, then click on the *Crop* command in *Tools* to crop the figure. Save it as usual.

You can also Export (File menu) the figure in a different format using *Preview*.

10 Graph and Calculus Operations

10.1 Introduction

10.1.1 Australian Curriculum

References are given here to the corresponding topics in the Australian Curriculum. Specific references are given to the texts *Nelson Senior Maths Methods 11* (NSM11) and *Nelson Senior Maths Methods 12* (NSM12) used in the ACT.

The material here in Sections 10.4 and 4.3 – 4.7 is generally relevant to the topics *Rates of change* (Chapter 10 in NSM11), *Properties of derivatives* (Chapter 10 in NSM11), *Applications of derivatives* (Chapter 12 in NSM11), *Applications of derivatives* (Chapter 3 in NSM12), and *Integration and areas* (Chapter 4 in NSM12).

10.1.2 Graphics calculators and Calculus

Graphics calculators lend themselves very nicely to demonstrating the important visual aspects of Calculus, graphs of functions, tangent lines, areas under curves, etc, without the off-putting effort required to do these by hand. They can also calculate numerically (approximate) many of the quantities that arise — derivatives and definite integrals, maximum and minimum values, etc.[4]

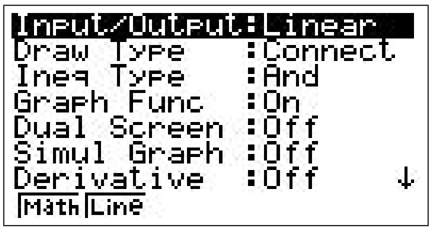
They can be used at a number of levels.

- As a sophisticated scientific calculator with many built-in functions.
- As a basic graph plotter — what does the graph of $y = e^x$ look like?
- As an advanced graph plotter able to plot Cartesian, parametric, polar and sequence graphs.
- To investigate 'what if' questions, for example what happens if you change the parameters a and b in the equation $y = ax^2 + b$?
- To do (numerically) many of the basic calculations in Calculus, such as finding the slope at a point on a graph, definite integrals, maxima and minima, solutions of equations, intersection points of graphs (simultaneous equations), etc.
- To illustrate graphically, perhaps by way of a program, some of the concepts of Calculus. Two examples are showing how a secant line tends to a tangent line in the appropriate limit and how we can approximate the area under a graph by the areas of some rectangles. With sufficient ingenuity, almost anything can be done here, the only limitation being the small screen of the calculator.
- To automate, using the built-in operations or programs, some of the calculations that arise in Calculus and other areas of Mathematics: numerical integration methods; fitting curves to data points; statistical functions for organising, analysing and displaying data; probability calculations; matrices; and so on.

[4]There are, of course, 'calculators' that go even further and do things symbolically — CAS calculators such as the Casio ClassPad.

10.2 Setting up

Press MENU 5 for GRAPH mode. Press SET UP (SHIFT MENU).



Input/Output: In *Linear* mode, commands are typed on one line, with arguments in brackets. In *Math* mode, the calculator tries to display commands in mathematical notation, with small boxes in the relevant positions for the inputs.

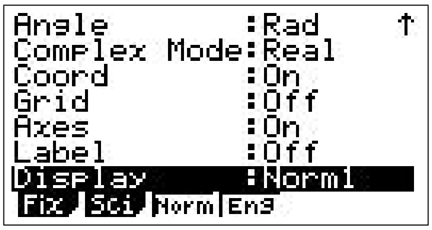
Linear is used here.

Draw Type: *Connect* means graphs are a continuous line whereas Plot provides a set of points (those calculated) which are not connected. *Connect* is better in most cases.

Graph Func: *On* means the equation of the function is displayed *while* its graph is being drawn. Harmless, so leave On.

Simul Graph: *On* means that, if two or more functions are being drawn, they will be drawn simultaneously; *Off* means they are drawn sequentially. Unless you are simulating a race, leave set on *Off*.

Derivative: *On* means that, when a graph is being traced, (an approximation to) the value of the derivative at a point will be displayed as well as the function value. Leave *Off* unless required.



Angle: *Rad* (radians) is the appropriate setting for Mathematics.

Coord: *On* means the coordinates of the cursor wil be displayed when tracing a graph.

Axes: *On* means Cartesion axes will be drawn on plots where appropriate.

Display: *Fix* allows you to set the number of decimal places in numerical output; useful in financial calculations where the numbers are in dollars and cents. *Sci* displays all numbers in scientific notation. *Norm1* displays numbers smaller than 0.01 in scientific notation, *Norm2* numbers smaller than 0.000000001 (10^{-9}). Toggle between the two with Norm. *Eng* displays numbers similar to scientific notation but adjusted so that the exponent is always a multiple of 3.

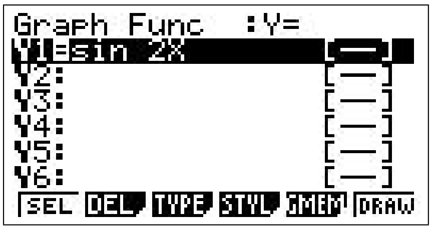
Norm1 preferred unless you like counting zeros.

10.3 Graph operations

10.3.1 Graph $f(x) = \sin(2x)$ for $0 < x < \pi$

- Press MENU 5: set Y1 = sin 2X and press EXE. The independent variable X is the X, θ, T key in the fourth row of keys.

  Note the highlighted = sign, which means the function will be plotted when you press DRAW. Use F1 (SEL) to toggle the function off/on.

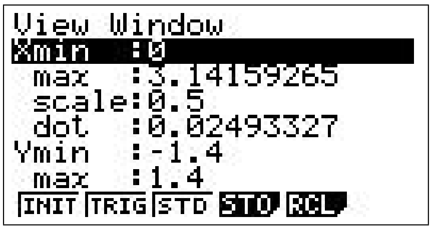


- Press SHIFT F3 (V-Window): specify the viewing window.
  - For X, suitable values here are:
    Xmin = 0 Xmax = π Xscale = 0.5.

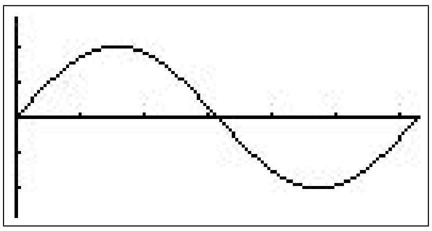
    Press EXE or the down arrow to move between values, *including the last.*

    π is SHIFT EXP.

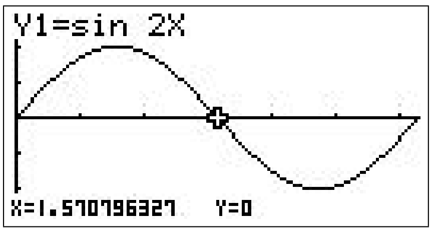
    Xscale is the distance between tick marks on the X axis (cosmetic only: 0 gives no tick marks).
  - Suitable Y values are: Ymin = −1.4 Ymax = 1.4 Yscale = 0.5.
  - Note the difference between the subtract key − and the change-sign key (–).
  - Press EXIT to return to the *Graph Func* screen.



- Press F5 (DRAW): graph the function.



- Press F1 (Trace): move the cursor along the graph with the left/right arrows; the coordinates of the point on the graph are shown at the bottom (and at the point on a CG50).

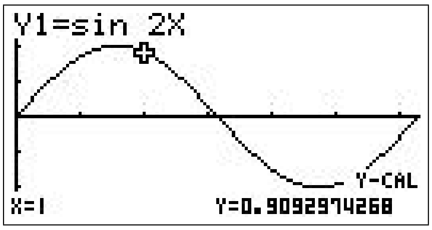


10.3.2 Estimate $f(1)$

- **On the GRAPH screen** MENU 5
  - Press F6 to redraw the graph if necessary.
  - Press F5 (G-Solv) F6 (▷) F1 (Y·CAL).

    Type the X value, 1 EXE, to move to the desired point on the graph. Note the coordinates at the bottom of the screen.

    Alternatively, use the left/right arrows to move the cursor along the curve in Trace (but note the problem that arises when trying to reach X=1). The up and down arrows move between functions if there is more than one graphed.

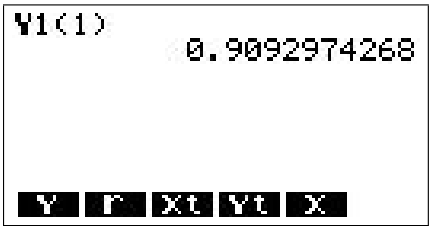


- **On the RUN screen** MENU 1

  Type in **Y** 1 (1) EXE.

  Y is VARS F4 F1.

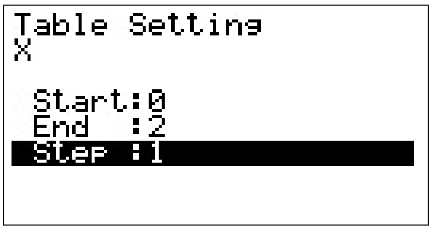
  You can't just type the letter Y.



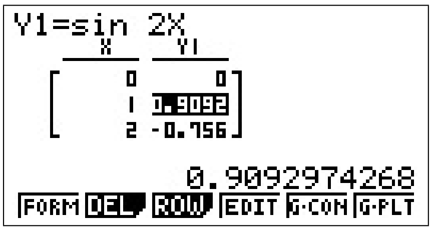
- **On the TABLE screen** MENU 7
  - Set the table 'window' using F4 (SET): Start = 0, End = 2, Step = 1;

    press EXE after each.

    Press EXIT to return to *Table Func.*



- Press F6 (TABL). Use the arrow keys to move around the table.

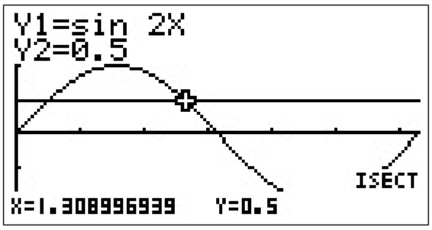


Answer: $f(1) = 0.90930$, rounded to 5 significant digits.

10.3.3 Solve $\sin(2x) = 0.5$ for $0 \leqslant x \leqslant \pi$

- **On the GRAPH screen** MENU 5
  - Graph Y1 and Y2 = 0.5.
  - Use *ISCT*, F5 in the G-Solv menu. The calculator will find the first intersection; press the right arrow to find further intersections (the second one is shown in the figure).

    Note that, because the right-hand side of the equation here is a constant, we could have used *X·CAL* in the G-Solv menu. The intersection method here can be used no matter what the right-hand side.



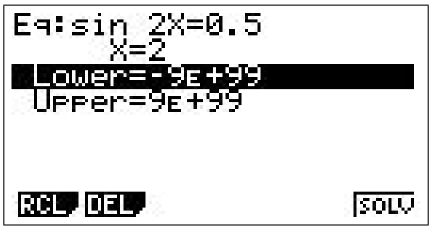
- **On the EQUATION screen** MENU A
  - Use the *Equation Solver*: F3.

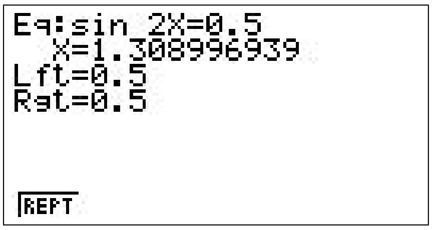
    Enter the equation as

    $$\sin 2X = 0.5$$

    and press EXE. Enter a guess for X (this will determine which of the two possible values you find), press EXE and F6 (SOLV). Press F1 to repeat the calculation with a different guess.

    You can also use the *Lower* and *Upper* bounds to narrow down the search.



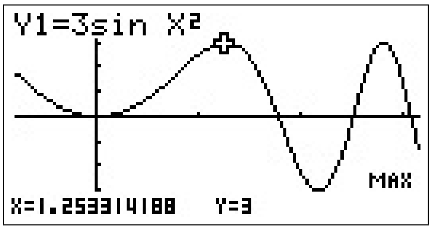


- *Answer*: the curves intersect at $x = 0.26180$ (Guess: X = 0) and $x = 1.3090$ (Guess: X = 2), both rounded to 5 significant digits.

Note: The graphical method has the advantage that you can see not only the solution you are looking for but also any other nearby solutions that may confuse the Equation Solver.

10.3.4 Find the value of the first positive maximum of $f(x) = 3\sin(x^2)$

- **On the GRAPH screen** [MENU] [5]
  - Set Y1 = 3 sin X². Turn off or delete all other functions. Set a [V-Window] of $[-\pi/4, \pi, 1] \times [-4, 4, 1]$ and plot the function.
  - Select *MAX* in the [G-Solv] menu. The calculator will find the first (local) maximum from the left on the screen. Press the right arrow to find successive maxima until you reach the one you want.
  - Press [MENU] [1] and type 2X² [EXE] on the RUN screen. You should recognise the first 6 digits of the number.

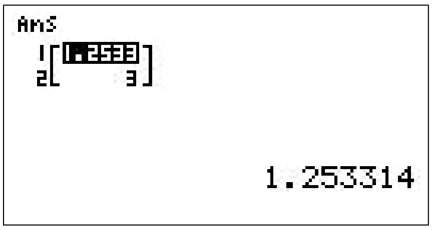


- **On the RUN screen** [MENU] [1]
  - FMax (Y1, 1, 1.5).

    FMax: [OPTN] [F4] (CALC) [F6] [F2].

    Y: [VARS] [F4] (GRPH) [F1].

    The last two inputs are the bounds for the search.



- *Answer*: the maximum value $f(x) = 3$ occurs at $x = 1.2533$, rounded to 5 significant digits.

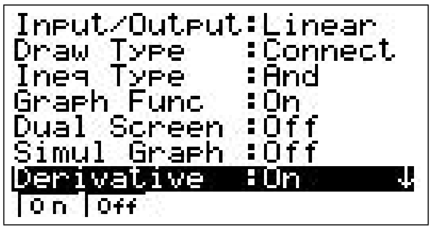
Again the graphical method has the advantage that you can see not only the (local) maximum you are looking for but also any other nearby maxima that may confuse *MAX*.

The [G-Solv] operations *ROOT*, *MIN* and *Y·ICPT* work in the same way as *MAX*.

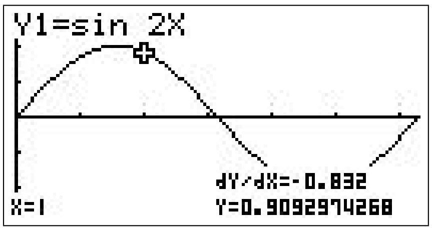
10.4 Calculus operations

Set $f(x) = \sin(2x)$.

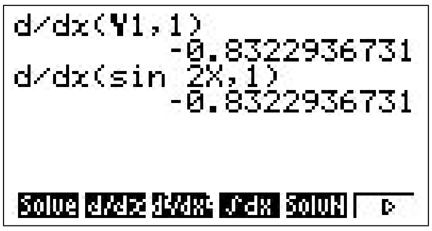
10.4.1 Estimate $f'(1)$



- **On the GRAPH screen** [MENU] [5]
  - Press [SET UP] ([SHIFT] [MENU]) and set *Derivative On*. Press [EXIT].
  - Press [F6] to redraw the graph.
  - Press [Trace] and use the arrow keys to move the cursor along the curve; the value of the derivative is shown at each point. Type in an X value, then [EXE] to go to a particular X value.



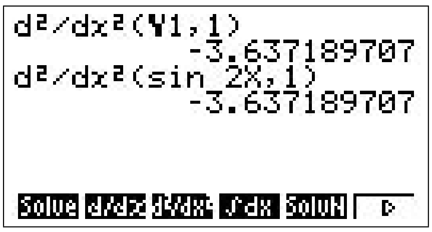
- **On the RUN screen** [MENU] [1]
  - d/dx (Y1, 1).
    d/dx: [OPTN] [F2] (CALC) [F2].
    Y: [VARS] [F4] (GRPH) [F1].
    Or just type in the function.



- *Answer*: $f'(1) = -0.83229$, rounded to 5 significant digits.
- *Accuracy*: can be adjusted in *d/dx* by an optional third argument, though it is not clear exactly how or if this works.

10.4.2 Estimate $f''(1)$

- **On the RUN screen** [MENU] [1]
  - d^2/dx^2 (Y1, 1).
    d^2/dx^2: [OPTN] [F3] (CALC) [F2].
    Y: [VARS] [F4] (GRPH) [F1]. Alternatively, just type in the function.



- *Answer*: $f''(1) = -3.637$, rounded to 4 significant digits.
- *Accuracy*: can be adjusted by an optional third argument, though it is not clear exactly how or if this works.

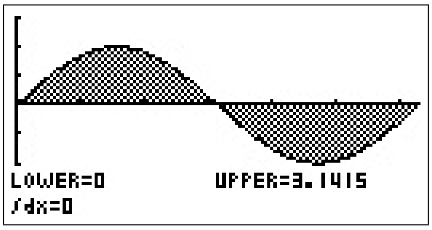
10.4.3 Graphing derivative functions

See Section 10.5.6.

10.4.4 Estimate $\int_0^\pi \sin(2x)\,dx$

- **On the GRAPH screen** MENU 5
  - Press F6 to redraw the graph.
  - Select $\int dx$: G-Solv F6 F3.

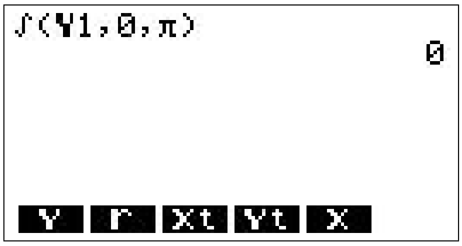
    Type in the lower and upper integration limits, followed each time by EXE. Alternatively, move the cursor respectively to the lower and upper integration limits when prompted, followed each time by EXE.



- **On the RUN screen** MENU 1
  - $\int$ (Y1, 0, π).

    $\int$: OPTN F4 F4 ($\int dx$).

    Y: VARS F4 (GRPH) F1.



- *Answer*: $\int_0^\pi \sin(2x)\,dx = 0$.
- *Accuracy*: can supposedly be adjusted in $\int dx$ by an optional fourth argument.

10.4.5 Graphing definite integrals

See Section 10.5.6.

10.5 Activities

10.5.1 A classic problem

A hare and tortoise compete in a one-kilometre race. The distance each competitor has travelled from the starting point is given by a formula. In time t **minutes**, the distance in **metres** travelled by the hare is given by $H(t) = \frac{500}{3}(2\sqrt{t} + \sqrt[3]{t})$, while the distance in **metres** travelled by the tortoise is given by $T(t) = 100t + 250\sqrt{t}$.

Press [MENU] [5] (GRAPH) and enter the formulas for H and T in Y1 and Y2 respectively. You have to use X ([X,θ,T]) as the independent variable. The cube root is [SHIFT] [(].

Set your View Window ([SHIFT] [F3]) so that the two graphs will go from the bottom left to the top right of the screen. *Hints*: The race takes about 5 minutes. How far is the race?

If you select *Simul Graph On* in the [SET UP] menu of your calculator before graphing, you will get a real-time view of the race.

Answer the following questions, *writing down the steps you took.*

[Trace] ([F1]) and, in [G-Solv], ISCT (intersection), Y·CAL (find y given x) and X·CAL (find x given y) will be helpful. Press the right arrow after an ISCT operation to find further values.

You may need to increase *Ymax* or ZOOM IN before using ISCT or X·CAL in Questions 2 and 3 so that the function formulas do not obscure the point you are interested in.

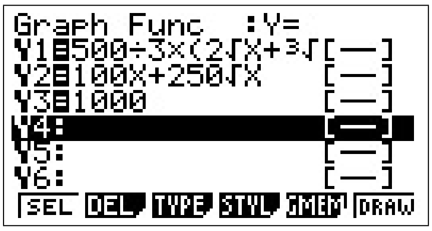
1. Who gets to the halfway point first? How long does it takes them? Verify your answer algebraically.

2. What is the time and distance at which the two runners are neck and neck?

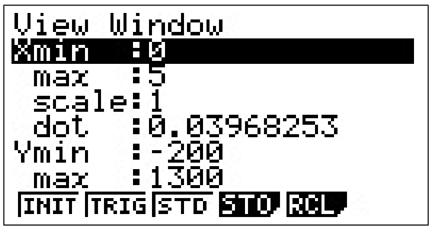
3. Who wins the race, by what time margin and by what distance margin?

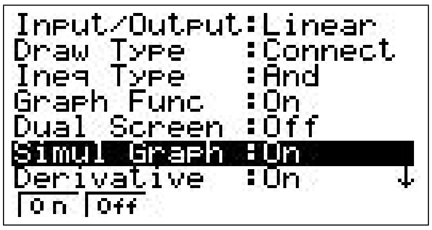
Notes for teachers

The questions in this version have been written in general terms deliberately for a good class. For a less-advanced class, students may need to be led a little through each question. For example: *What equation do we need to solve to answer this question? What does this mean about the graphs of each side of the equation? How do we solve this equation on the calculator?* and so on.



Press [MENU] [5] (GRAPH) and put the equation for the hare in Y1 and that for the tortoise in Y2. You might like to discuss with the class how to write the formulas in a suitable form for the calculator. Time t becomes X on the calculator.

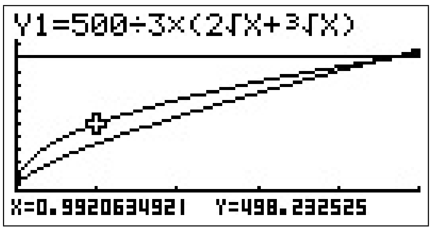


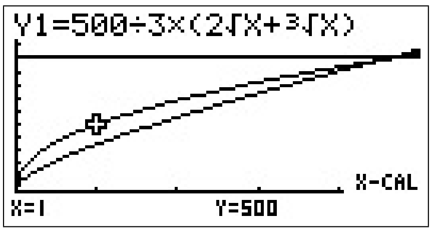


Then set the View Window: discuss first with the class what each axis represents and suitable scales. The Y scale is the distance run in metres, so that $0 \leqslant \mathrm{Y} \leqslant 1000$.

To avoid function labels and coordinates hiding relevant points on the graphs, we increase Ymax to 1300 and decrease Ymin to -200. The winner is the competitor whose graph first reaches $\mathrm{Y} = 1000$ (assuming *Simul Graph On*), shown by the horizontal line. *Yscl* is 100 here.

The time (X) scale has to be guessed. The race takes a little under 5 minutes, so $0 \leqslant \mathrm{X} \leqslant 5$ gives a good view.
Press [EXIT] to return to the *Graph Func* screen.





1. Press [F6] (DRAW), then [F1] (Trace): use the up/down arrows to see which graph is which.

   The hare clearly reaches the halfway point (500 m) first. Note that we cannot Trace to the exact point.

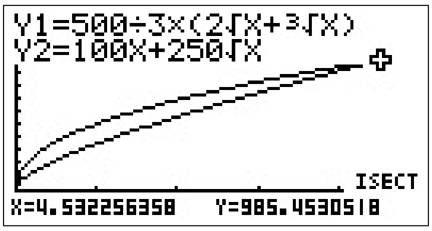
   To find more accurately how long the hare took, solve $H(t) = 500$ for t using *X·CAL* ([G-Solv] [F6] [F2]). Select the appropriate curve (Y1) using the up/down arrow keys and press [EXE].
   Type in the Y value (500) and press [EXE].

   The value for t is 1 minute, a value we can confirm algebraically to be exact by substituting $t = 1$ into the equation for the hare. Note that it is easy to **verify** that $t = 1$ is a solution, but tricky to **solve** $H(t) = 500$ algebraically (it turns into a cubic equation).

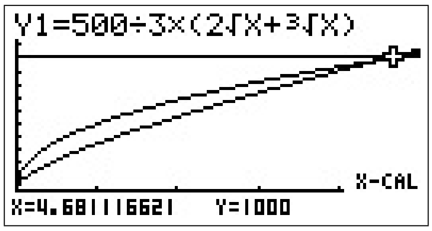
   The hare reaches the halfway point first in a time of 1 minute.

2. To find when they are neck and neck, we have to solve $H(t) = T(t)$, that is find the intersection (*ISCT*) of Y1 and Y2 (algebraically, this turns into a quartic equation).

We obtain $t = 4.53$ minutes and a distance of 985 m, both rounded to 3 significant digits.

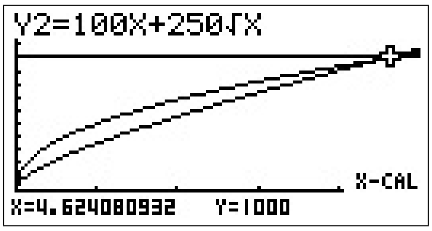


The hare and tortoise are neck and neck after about 4.53 minutes or about 4 minutes 32 seconds, 985 metres from the start.



3. To find the winner, we have to determine the time at which each competitor reaches the finish (1000 m).

Using X·CAL on the appropriate curves, we find that the hare finishes at $t = 4.681$ minutes and the tortoise finishes at $t = 4.624$ minutes.



To find the distance margin, calculate $H(4.624)$, the position of the hare when the tortoise finishes, using Y·CAL: $H(4.624) = 994.45$ m, rounded to five significant digits.

The tortoise wins the race by a margin of 0.057 minutes or 3.42 seconds. The distance margin is 5.55 m.

10.5.2 The best shape for a can

When manufacturers are designing their packaging, they must keep in mind the amount of product that has to fit inside and the amount of material it will take to make the package. Consider the humble soft-drink can. The standard volume is 375 mL or 375 cm^3. Any number of cans can be designed that will hold this volume of liquid, but they will vary in shape and therefore in the amount of material needed to make the can (and therefore cost).

The formula for the volume of a cylinder (measured here in cm^3) in terms of radius r and height h (both measured in cm), is

$$V = \pi r^2 h.$$

Rearrange the volume formula to make h the subject and let the volume be 375 cm^3.

$$h =$$

Press [MENU] [7] (TABLE) and enter the formula for h as Y1, with X to represent the radius r. As a check, enter the volume formula: Y2 = πX^{2}Y1(X) Y1 is [VARS] [F4] [F1] [1]. Then press [EXIT] twice. Press [F5] (TABL). *You may get ERROR for Y1 and Y2 if X = 0 is in your table. Why?*	
Let's specify which X values we want. Press [F1] (FORM) to go back to the formula, press [F4] (SET) and set *Start* = 1, *End* = 10 and *Step* = 1. Press [EXIT] and then [F6] again. *Do you get the correct value for the volume in Y2?*	
Write down the formula for the surface area of a cylinder, including the ends. The surface area determines the amount of material needed to make the can. *Why?* Press [F1] and enter the formula for surface area in Y2 in terms of X (radius) and Y1 (height).	SA =
View the table of values again. *What do you notice about the values of the surface area?*	

Press F1, then F5 and set a new starting value and a smaller step to find the minimum surface area and corresponding radius (radius accurate to 1 decimal place).

minimum surface area = radius =

Now graph the surface area as a function of radius: press MENU 5 (GRAPH), set a V-Window (SHIFT F3) of $[0, 12.6, 2] \times [-150, 1000, 100]$ and press EXIT. Turn off Y1 by pressing F1. Press F6 (DRAW) to plot the graph.

Draw your graph here with scales on the axes.

Use Trace (F1) and the cursor to find an approximate value for the minimum.

Write down your values for the radius, height, ratio of height to radius, surface area and circumference of the can when the surface area is a minimum.

How does this compare with a soft-drink can? Why might there be differences?
How does the theory fit with the shapes of other cans?

Notes for teachers

There are similar maximum/minimum activities to suit all levels of Years 9 and 10. The activities all follow the same outline as this particular one. This way of doing the problem allows for numeric, graphic and sometimes algebraic approaches. Algebraic approaches without Calculus usually rely on the function to be minimised/maximised being a quadratic, which is not the case here.[5]

Before you start this activity, you might like to discuss with the class the different shapes of cans one finds on a supermarket shelf (bring in a few examples). Are they just scaled versions of each other? To quantify the shape, measure the ratio of height to radius for different cans. This should be the same if they are scaled versions. *What range of values of h/r do you find?*

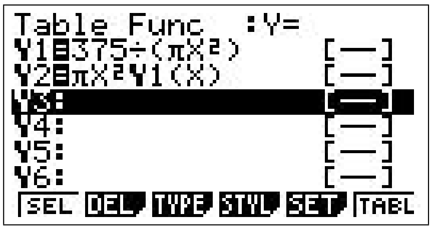
The question then arises: *What is the reason for the manufacturer choosing a particular shape?* This activity explores one possible explanation, that of minimising the amount of metal used to make a can.

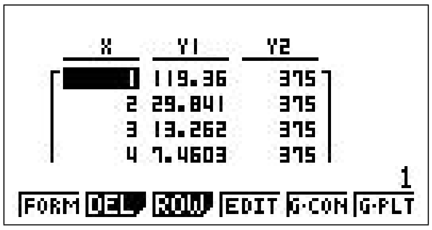
The total volume of the metal, assuming the walls are of uniform thickness, is just the surface area times the thickness. Minimum surface area therefore means minimum volume of metal.

The height of the cylinder is given by $\text{Y1} = 375 \div (\pi \text{X}^2)$, where X is the radius r.

As a check, enter the volume formula $\text{Y2} = \pi \text{X}^2 \text{Y1}(\text{X})$. The (X) with Y1 is not necessary but it emphasises that Y1 is a function of X.

Press [F6] (TABL). Note that Y2 = 375 (except at X = 0) as expected.





If your table starts at X = 0, you will see no value for Y1 at X = 0. This is because we are trying to divide by 0. This causes no value to be shown for Y2 as well, because Y2 is written in terms of Y1. This will only happen of course if the table starts at X = 0. We correct it by starting the table at X = 1 (in SET).

The surface area of a cylinder, including the ends, is given by

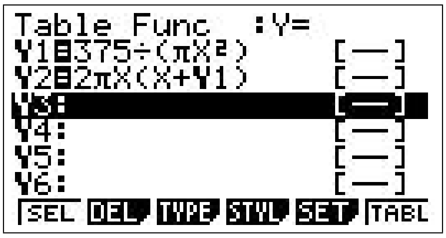
$$\text{SA} = 2\pi r^2 + 2\pi r h = 2\pi r(r + h).$$

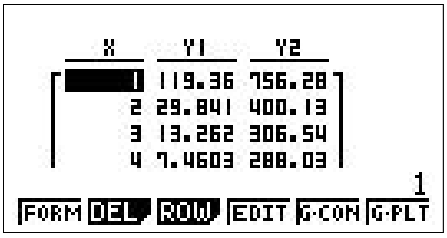
If you substitute for h, you find that

$$\text{SA} = 2\pi r^2 + \frac{750}{r},$$

so the function is not a quadratic. For the purposes of graphing the function though, it is easier to leave h in the formula, so that we set the surface area $\text{Y2} = 2\pi \text{X}\big(\text{X}+\text{Y1}\big)$ (figures over the page).

[5] I've included the exact results from Calculus below so this activity can be incorporated into a Calculus class too.

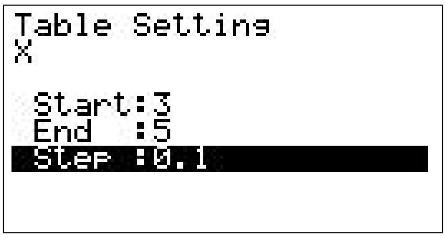


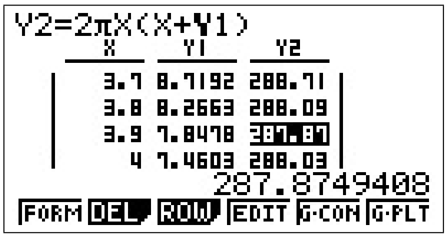


Look at the table of values again. What do you notice about the values of the surface area?

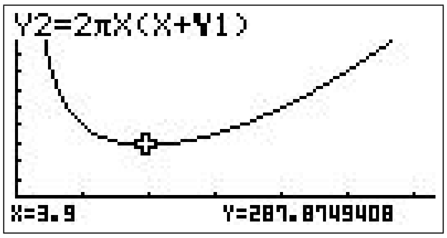
The surface area decreases, then increases as the radius increases. There is a (local) minimum.

With *Start* = 3, *End* = 5 and *Step* = 0.1 (below left), we find, on scrolling down, a radius of 3.9 cm for a minimum surface area of 287.9 cm^2, both values rounded to 1 decimal place.





Now graph the surface area Y2 as a function of radius using a V-Window of $[0, 12.6, 2] \times [-150, 1000, 100]$. Use Trace and the cursor to find the minimum surface area.



Note that, because of the choice of V-Window values, the cursor increments in steps of 0.1, so that eventually you will reach X = 3.9, and it will be clear that the answer is 3.9, not 3.8 or 4.0. With a different choice for Xmax, the X values in Trace will not be nice numbers.

Again, we obtain a value of $r = 3.9$ cm for the radius, giving a minimum surface area of 287.9 cm^2, both values rounded (and accurate) to 1 decimal place.

If your students have sufficiently developed Calculus skills, they could prove algebraically that the global minimum lies at $r = \sqrt[3]{375/2\pi} \approx 3.9$, with a corresponding height $h = \sqrt[3]{1500/\pi} \approx 7.8$. More generally, for a given volume V, it is not too hard to show that $h = 2r$ (height = diameter) for minimum surface area.

Collecting our results so far and calculating several more, we have (to 1 decimal place)

radius $r = 3.9$ cm height $h = 7.8$ cm ratio $h/r = 2$
surface area = 287.9 cm^2 circumference = 24.6 cm

Does it matter if the radius is not the exact minimum value? (follow-up)

If the radius varies by say 5%, 10%, etc from the minimum value, by what percentage does the surface area and therefore cost change? Alternatively, by how much does the radius need to change from its minimum value to make the surface area change by 5%? by 10%? Use the calculator graph and Trace. Discuss the difference between a flat minimum and a sharply pointed one.

How does this compare with a soft-drink can? Why might there be differences? How does the theory fit with the shapes of other cans?

Standard soft-drink cans have a radius of 3.25 cm and a height of 13 cm, so that $h/r = 4$. The surface area is about $332\,\text{cm}^2$ and circumference 20.4 cm.

Clearly, considerations other than minimum surface area are involved. These might be what circumference is comfortable for the average human hand, the wastage of material when cutting the ends and the cost of making the joins.

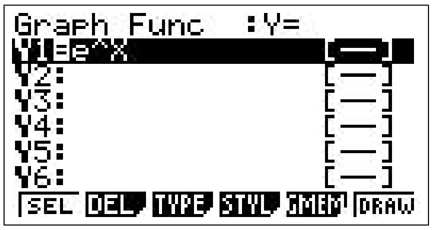
The article *The Best Shape for a Tin Can* by P.L. Roe, either in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993)[6] goes into why there might be differences between the theory here and the actual values. A good example of mathematical modelling.

[6]search for it online or look at *canberramaths.org.au Resources Graphics Calculators Articles*

10.5.3 Slope of a graph

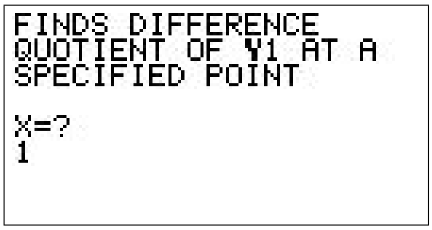
The program DIFFQPLT[7] illustrates graphically how the difference quotient (the slope of the secant line) approaches the derivative (the slope of the tangent line) at a point on a graph.

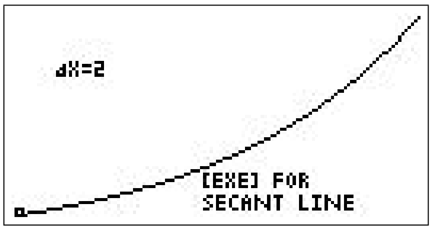
- Put the function $f(x)=e^x$ in Y1 (MENU 5). We wish to find the slope of the graph of f at $x=1$.



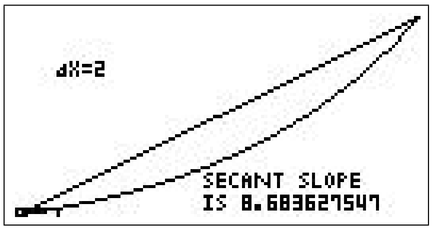
- Run the program: press MENU B , scroll down to DIFFQPLT and press F1 .
- Set X = 1 and ΔX = 2, pressing EXE after each.

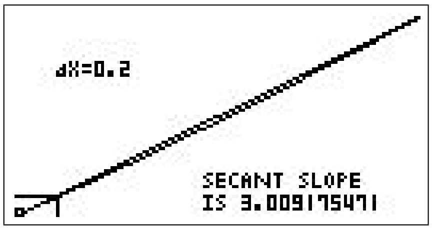
  The program will plot the function for $1 < X < 1+\Delta X$, with a small box at X = 1.





- Press EXE to see the secant line and its slope.





- Keep pressing EXE to repeat the process, but with ΔX reduced each time by a factor of 10. The box in the lower left-hand corner shows the area that will be graphed at the next step.
- When you've finished, select QUIT from the menu.

As we ZOOM IN more and more, the graph of f becomes straighter and straighter, so that it looks more and more like its tangent. The secant line, for which we can calculate the slope, approaches the tangent, so that the slope of the secant line is, for small enough ΔX, a good approximation to the slope of the tangent. Try the program with other functions in Y1.

[7] available at *canberramaths.org.au* under *Resources*

A convincing demonstration that even a nasty curve becomes straight if you magnify it enough (local linearity), provided the function is differentiable

Plot the graph of sin(20X) (Radian mode), using a V-Window of $[0, 5, 1] \times [-1, 1.5, 0.5]$. Put the cursor on one of the peaks and successively ZOOM IN ([F2] [F3]) at the maximum. The sharp peak eventually looks like a (horizontal) straight line (the tangent to the curve). The function is differentiable there.

A contrasting function on which to try the same process is $y = 1 - (x-1)^{2/3}$.
ZOOM IN successively at the peak. *Is this function differentiable at $x = 1$?*

What about $y = |x|$ (Abs X) at $x = 0$? *Abs* is [OPTN] [F6] [F4] (NUM) [F1].

10.5.4 Discovering the derivative by exploration

Modified from: Judy Broadwin, *Discovering the derivative by exploration*, TI-82/83 Activities for Calculus, TI Web site.

We put several different functions $f(x)$ in Y1 and try to discover their derivatives using the difference quotient

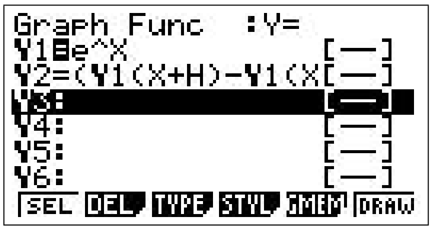
$$\text{DQ} = \frac{f(x+h) - f(x)}{h},$$

where h is a small number. We know that the derivative is the limit of DQ as $h \to 0$.

- Store 0.0001 in memory H: 0.0001 [→] [ALPHA] [H] [EXE].
- Enter the difference quotient into Y2 ([MENU] [5]) (watch brackets!):
  Y2 = (Y1(X+H)−Y1(X))/H.

  Y is in the [VARS] GRPH menu. You can't just type the letter [Y].
- Turn off Y2 by pressing [SEL].

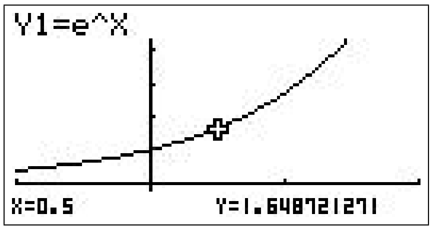
Exploration 1: Discover the derivative of $f(x) = e^x$

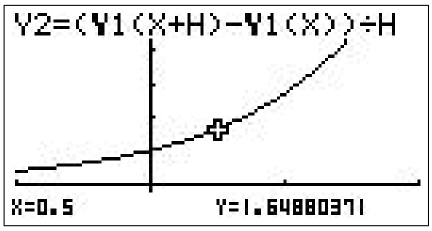


Set Y1 = e^X.

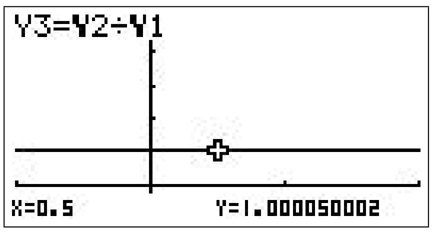
Set the [V-Window] to $[-1, 2, 1] \times [-1, 5, 1]$.

Press [EXIT], then [F6] (DRAW) and [F1] (Trace) .





Then turn off Y1 and graph Y2. *What do you see?*



Turn off Y2, set Y3 = Y2÷Y1 and graph it.

What do you see?

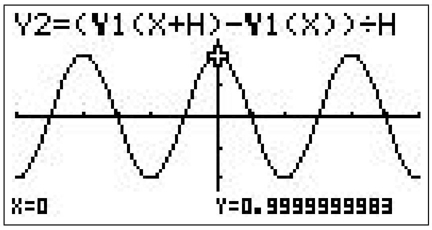
What is $\frac{d}{dx} e^x$*?*

Exploration 2: Discover the derivative of $f(x) = \sin(x)$

Set Y1 = sin X.
Turn Y1 off and Y2 on.

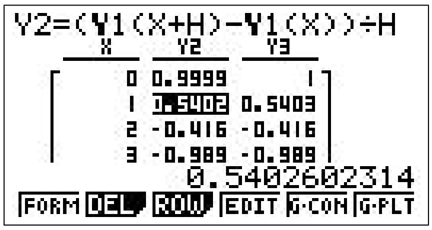
Press [V-Window] [F2] (TRIG), then [EXIT] and [F6] (DRAW).

Describe the graph of Y2.
Do you recognise the function?



Can you describe the relationship between the graph of the function and the graph of its difference quotient?

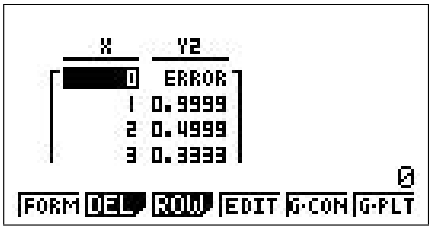
Set Y3 = cos X.
Compare Y2 and Y3 using [TABLE] ([MENU] [5]).
In [TABLE] SET, set Start = 0, End = 5 and Step = 1.
Why are the two functions not identical?



Exploration 3: Discover the derivative of $f(x) = \ln(x)$

Set Y1 = ln X and turn it off. Leave Y2 on.

In [TABLE] SET, set Start = 0, End = 10 and Step = 1.
Press [EXIT] [F6] (TABL).
Can you guess the derivative of $\ln(x)$*?*

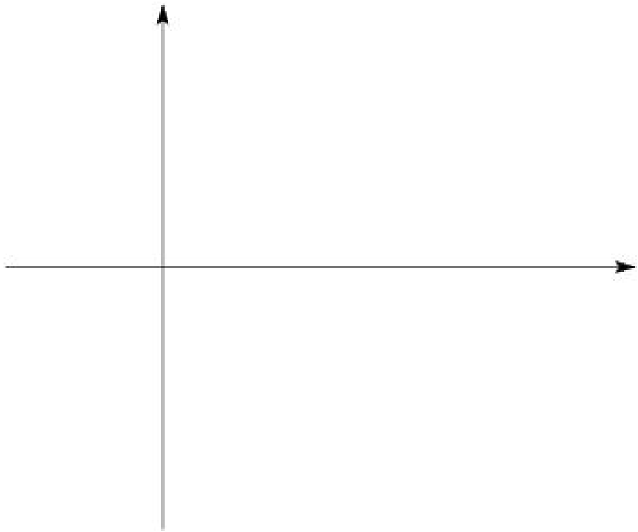


10.5.5 Derivatives and the shape of a graph

From: *This is I.T. Graphics-calculator activities for upper-secondary students* by Pat Forster, Alan Cadby and Gary Young, AAMT, 1998.

This investigation is to help you understand how the first and second derivatives of an equation can tell you the shape of its graph. You will also learn some new terminology to describe graphs.

1. Copy from your calculator the graph of $y = 5x^2 - 2x^5$ for $-1 \leqslant x \leqslant 2$, $-10 \leqslant y \leqslant 10$. Put scales on your graph.



2. Find and enter the equations for the first and second derivatives of $y = 5x^2 - 2x^5$ into your calculator as Y2 and Y3. Look at the values of these derivatives for different values of x using the calculator's table of values as follows.
Press [MENU] [7], then [F5] (SET) and set Start = −1; End = 2 and Step = 0.2.
Then press [EXIT] and [F6] (TABL).

(a) Use a red pen and put plus signs (+) along the sections of your graph above where dy/dx is positive. Similarly, put minus signs (−) and zero (0), where appropriate.

(b) Using a different-coloured pen, mark where d^2y/dx^2 is +, − or zero. (Scroll up the table in the X column.)

(c) The points on a graph where $dy/dx = 0$ are called **stationary points**. Fill in the table below.

Type of stationary point	**Co-ordinates**	$\frac{dy}{dx}$ **(+, −, 0)**	$\frac{d^2y}{dx^2}$ **(+, −, 0)**
Maximum			
Minimum			

3. **(a)** Sections of a graph where $\frac{d^2y}{dx^2}$ is positive are described as **concave up**.

If $\frac{d^2y}{dx^2}$ is negative the curve is **concave down**.

For what values of x is your graph concave up? ....

For what values of x is your graph concave down? ....

Is the graph at the maximum point concave up or concave down? ....

Is the graph at the minimum point concave up or concave down? ....

(b) A point at which a curve is concave down (d^2y/dx^2 negative) on one side and concave up (d^2y/dx^2 positive) on the other side, *or vice versa*, is called a **point of inflection**. The sign of the second derivative must *change* as we pass through a point of inflection because the graph changes concavity.

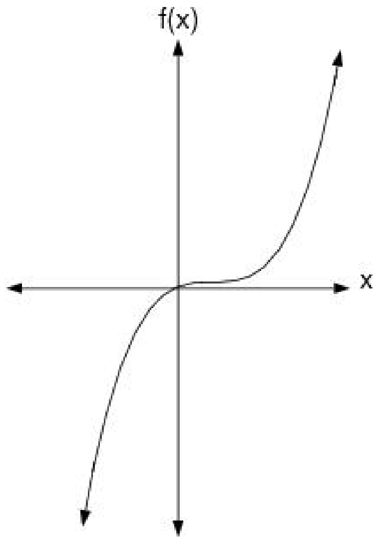
Use the table of values to find the point of inflection of the function here: change *Start*, *End* and *Step* to zoom in on the zero of Y3. *Why do we look for a zero? Is this zero a point of inflection?* Write its co-ordinates on your graph.

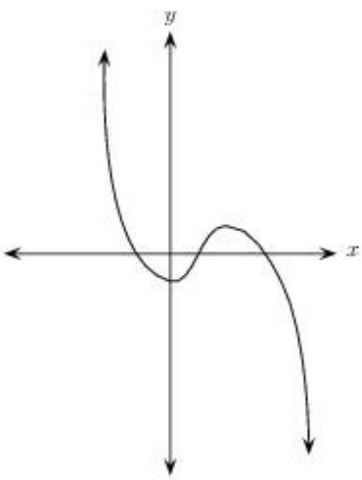
(c) *Is $x=0$ a point of inflection of the function $y=x^4$?* Give reasons.

(d) Points of inflection can be the steepest section of a graph — look at the graph of $y=\sqrt[3]{x-1}$ at $x=1$. They can also be stationary points (graph is horizontal) — look at the graph of $y=x^3+1$ at $x=0$. However, in general, the slope at a point of inflection can be any value.

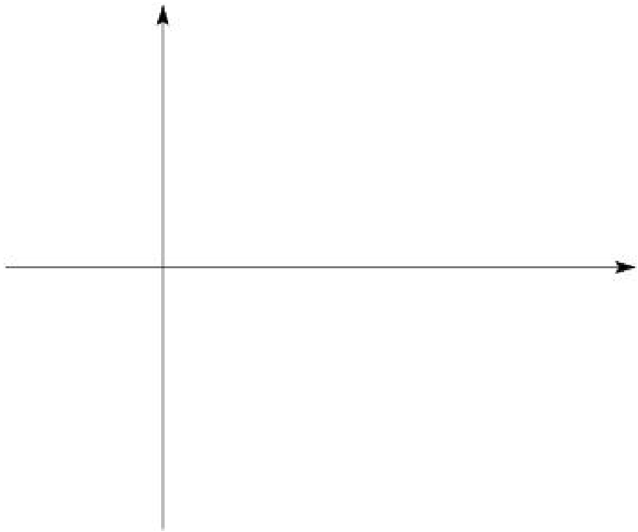
Put a cross on the points of inflection in each of the following two graphs. *Is the point of inflection in either graph a stationary point?*

....





4. Sketch a graph with the given properties.



(a) Endpoints $(-2, 1)$ and (7,6).

(b) Stationary point at $(1, -5)$ where $\dfrac{d^2y}{dx^2}$ is positive.

(c) Stationary point at (3,2) where $\dfrac{d^2y}{dx^2} = 0$.

(d) $\dfrac{d^2y}{dx^2} = 0$ at $(2, -1)$, $\dfrac{d^2y}{dx^2}$ is negative between $x = 2$ and $x = 3$, and $\dfrac{d^2y}{dx^2}$ is positive for $x > 3$.

(e) Classify the points in (b) – (d) as a maximum, minimum or point of inflection.

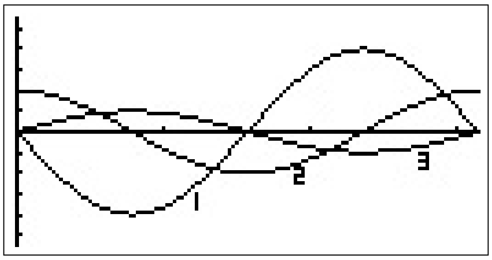
(f) For what values of x is the curve concave up?

10.5.6 Graphing derivatives and anti-derivatives

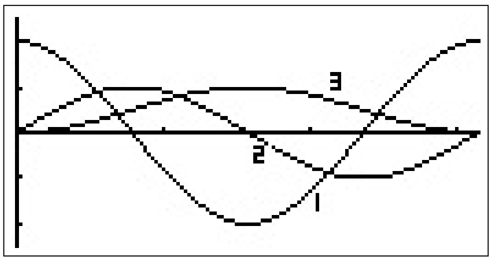
Based on: *How do you graph derivatives and anti-derivatives?* by John Maloney, Eightysomething! 7(1), 1997.

Here are two questions to challenge your students' understanding of the concepts of derivative and anti-derivative.

Question 1: The figure shows graphs of a function, its derivative and its second derivative. *Which curve is which?*



Question 2: The figure shows graphs of a function, its derivative and an anti-derivative. *Which curve is which?*

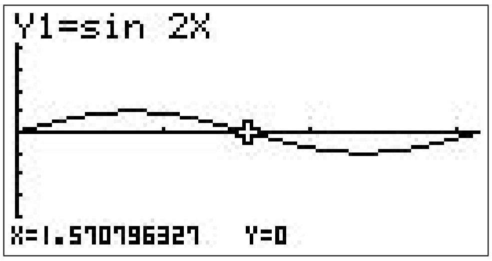


Let's investigate how to graph a function, and its first two derivatives. Press [SET UP] and make sure that the *Func Type* is Y= and that *Angle* is Rad.

Press [MENU] [5]: set Y1 = sin 2X.

Set [V-Window] to $[0, \pi, 1] \times [-5.5, 5.5, 1]$.

Press [EXIT], then [F6] (DRAW) and [F1] (Trace).

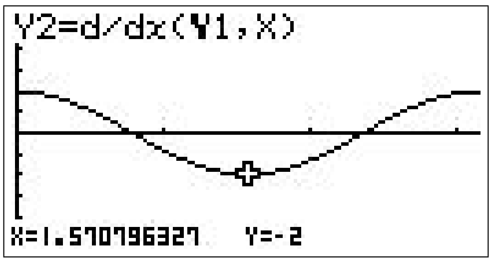


Graphing derivatives

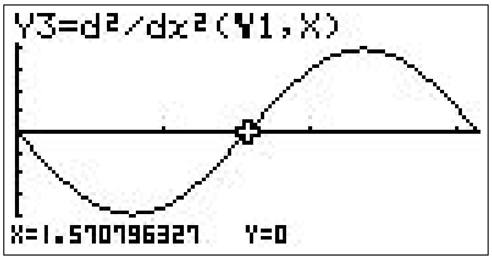
To graph the first derivative, set Y2 = d/dx (Y1, X).
d/dx is [OPTN] [F2] [F1]; Y is [VARS] [F4] [F1].

Y1 is the function we are differentiating and the X is the value at which we wish to calculate the derivative — this value is set by the grapher as it plots successive points on the graph.

Press [EXE] to store Y2, move the cursor to Y1 and press [F1] to turn it off, then press [F6] to plot the graph of Y2.



Set Y3 = d^2/dx^2 (Y1, X) for the second derivative of Y1. d^2/dx^2 is [OPTN] [F2] (CALC) [F2].



With Y1, Y2 and Y3 turned on, you should obtain the graph of the first question above. Students, of course, must use the relationship between a function and its derivative to answer the question.

Graphing an anti-derivative

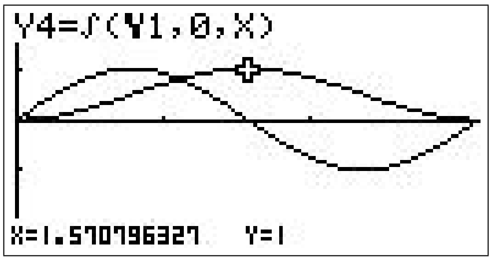
To plot an anti-derivative of Y1, set Y4 = $\int$ dx (Y1, 0, X).

$\int$ dx is [OPTN] [F2] [F3]; Y is [VARS] [F4] [F1].

Y1 is the function we are integrating,[8] 0 and X are the integration limits, with the value of X set by the grapher as it plots successive points on the graph.

The function we are plotting is therefore, in proper math notation, $y(x) = \int_0^x \sin(2t)\,dt$, with t a dummy integration variable.[9]

Turn off Y2 and Y3, set Ymin = −2.5, Ymax = 2 and press [DRAW].



Note: The lower integration limit 0 is arbitrary. Change it to other numbers to show students that a function has (infinitely) many (related) anti-derivatives. The one graphed here is the one that passes through the origin.

With Y1, Y2 and Y4 turned on, you should obtain the graph of the second question above.

The Chain Rule

With Y1 = sin 2X, set

Y5 = X^2

Y6 = Y1(Y5)

Y7 = d/dx(Y1, Y5) ∗ d/dx(Y5, X)

Y8 = d/dx(Y6, X).

Plot and compare the graphs of Y7 and Y8 using a V-Window of $[-3, 3, 1] \times [-12, 12, 4]$. Use [Trace] and the up/down arrows to identify the two curves. Explain what you see.

[8] We could have put the function in directly as sin 2X. The advantage of the way we have set up the functions Y2 − Y4 is that we can see the derivatives and anti-derivative of whatever function is in Y1.

[9] Casio calculators only allow X to be the variable. Here it plays two different roles: as a (dummy) integration variable; and as the independent variable (and integration limit) of the function $y(x)$.

Answers to the questions

Question 1 *One possible answer*

Curve 3 is initially positive, so that it cannot be the derivative of either Curve 1 or Curve 2, both of which have negative slopes initially. Therefore, Curve 3 must be the function.

The initial slope of Curve 3 is positive, so that Curve 1, which is initially negative, cannot be the first derivative of Curve 3. Therefore, Curve 2, which is initially positive, must be the first derivative of Curve 3.

Curve 1, initially negative, must be the derivative of Curve 2, and therefore the second derivative of Curve 3.

Question 2 *One possible answer*

Curve 3 cannot be the derivative of either Curve 1 or Curve 2 because it does not pass through zero when either of these curves has a maximum/minimum. Therefore, Curve 3 must be either the function or an anti-derivative.

If Curve 3 were the function, Curve 2 could be its derivative (although its behaviour at either end doesn't look correct), but Curve 1 could not be an anti-derivative because it is decreasing initially, whereas the anti-derivative of (positive) Curve 3 must be increasing. Therefore, Curve 2 must be the function and Curve 3 an anti-derivative.

The anti-derivative of Curve 2 must increase where Curve 2 is positive and decrease where Curve 2 is negative. Curve 3 has these properties. Because the area of Curve 2 below the x axis is equal to the area above the x axis, the final y value of Curve 3 must be equal to its initial y value.

Curve 1 must therefore be the derivative of Curve 2. *Check*: when Curve 2 has a maximum or minimum, Curve 1 is zero; where Curve 2 is increasing, Curve 1 is positive; where Curve 2 is decreasing, Curve 1 is negative.

11 Numerical Integration

11.1 Introduction

Finding the area under a curve was one of the challenges and triumphs of Calculus. Starting with rectangles under and touching the curve, the sum of the areas of these rectangles, known as a Riemann sum, approximates the area under the curve. The limit of the Riemann sums as the number of rectagles tends to infinity and the width of the rectangles tends to zero is known as the definite integral. The Fundamental Theorem of Calculus allows this definite integral to be found exactly in terms of the anti-derivative of the function being integrated.

However, in some cases, the anti-derivative cannot be found or is very complicated. Going back one step to the rectangles, and calculating the sum of their areas with more and more, thinner and thinner rectangles gives us a simple approximation to the integral and some idea of the error in the approximation; this is known as the **Left-endpoint Rule** if the top left-hand corner of each rectangle touches the curve, the **Right-endpoint Rule** if the top right-hand corner of each rectangle touches the curve and the **Midpoint Rule** if the middle of the top of each rectangle lies on the curve.

One finds, however, that many rectangles are needed to find the integral to any degree of accuracy in most cases, and looks to methods that converge faster. Rather than approximating the function on each sub-interval (rectangle base) by a horizontal straight line (the top of the rectangle), a polynomial of degree 0, we can try approximating the function by a general straight line, a polynomial of degree 1, giving the **Trapezoidal Rule**, or by a parabola approximating the function over two sub-intervals, a polynomial of degree 2, giving **Simpson's Rule**. Simpson's Rule is the workhorse of numerical integration in many cases.

To improve on Simpson's Rule, we move away from a geometric approach; Numerical Analysis gives us methods such as Gauss quadrature, which are significantly more accurate than Simpson's Rule for the same number of function evaluations but they still calculate a weighted sum of function values.

All these methods are easy to implement on a graphics calculator; a simple program in all cases allows us to vary easily the number of sub-intervals/function evaluations so that the error can be quantified. That is the subject here. Without a program, the hand calculations for even the simplest method are too lengthy to allow any sort of meaningful use (especially in knowing accuaracy) or exploration.

11.1.1 Australian Curriculum

The material here is relevant to the topic *Integration and areas*, Chapter 4 in *Nelson Senior Maths Methods 12*, used in the ACT.

11.2 Methods and programs

The numerical-integration methods here all calculate approximations to the definite integral

$$\int_a^b f(x)\,dx. \tag{1}$$

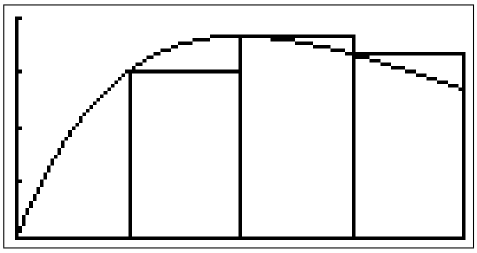
11.2.1 Geometric methods

These methods rely on the geometric fact that the definite integral, Eq. (1), is equal to the (signed) area between the graph of the function f and the x axis, between $x=a$ and $x=b$.

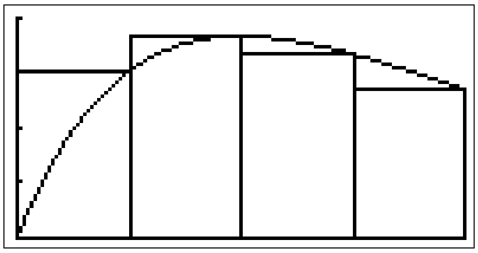
The integration interval $[a, b]$ is partitioned (divided) by $N+1$ equally spaced points into N sub-intervals each of length h, where $h=(b-a)/N$. The function is then approximated by a polynomial of order 0, 1 or 2 on each sub-interval.

Left- and Right-Endpoint Rules

Here rectangles, with bases the sub-intervals of width h, are drawn with the top left-hand corner of each rectangle touching the graph of f (Left-Endpoint Rule) or the top right-hand corner of each rectangle touching the graph of f (Right-Endpoint Rule), as shown in the figures below with 4 rectangles/sub-intervals. The integral is approximated by the sum of the (signed) areas of the N rectangles.



Left-Endpoint Rule



Right-Endpoint Rule

Another way of looking at this is that the function is approximated on each sub-interval by a horizontal straight line, i.e. a polynomial of degree 0. Clearly, the larger the number of rectangles N (and the smaller h), the better the approximation on each sub-interval and for the integral.

This is the basic method, and gives rise to Riemann sums (the sum of the (signed) rectangle areas) and the definition of the definite integral as the limit of the sum of the areas of the rectangles as $N \to \infty$ and $h \to 0$.

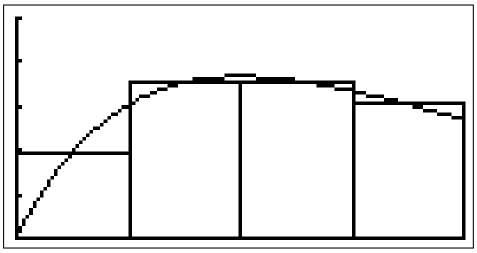
Program: NINTGRPH draws the rectangles for the Left-Endpoint Rule or Right-Endpoint Rule and calculates the sum of their areas as an approximation to the integral of the function in Y1 over a given integration range. You specify the number of rectangles/sub-intervals N.

Program: NUMINT calculates and displays (without graphics) the approximations to the integral of Y1 over a specified integration range using the Left- and Right-endpoint Rules with N (specified) rectangles/sub-intervals.

Activity 1 (Section 11.4.1) goes into more detail on the Left- and Right-Endpoint Rules.

Midpoint Rule

The same as the Left- and Right-Endpoint Rules but the midpoint of the top of each rectangle lies on the the graph of f, as shown in the figure below with 4 rectangles/sub-intervals. Again, the integral is approximated by the sum of the (signed) areas of the N rectangles. In general, this is more accurate than the Left- and Right-Endpoint Rules for a given N.



Midpoint Rule

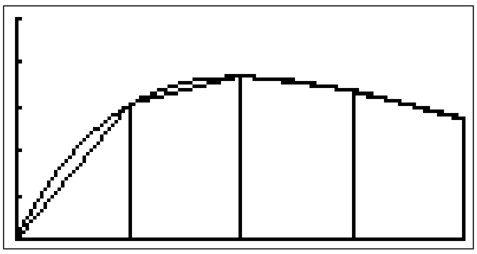
Here, the function is also approximated on each sub-interval by a horizontal straight line, a polynomial of degree 0.

Program: NINTGRPH draws the rectangles for the Midpoint Rule and calculates the sum of their areas as an approximation to the integral of the function in Y1 over a given integration range. You specify the number of rectangles/sub-intervals N.

Program: NUMINT calculates and displays (without graphics) the approximation to the integral of Y1 over a specified integration range using the Midpoint Rule with N (specified) rectangles/sub-intervals.

Trapezoidal Rule

Here, a straight line joins the points on the graph of f corresponding to each endpoint of a sub-interval, giving a trapezium, as shown in the figure below with 4 trapeziums/sub-intervals. The integral is approximated by the sum of the (signed) areas of the N trapeziums. The Trapezoidal Rule is the mean of the Left- and Right-Endpoint Rules.



Trapezoidal Rule

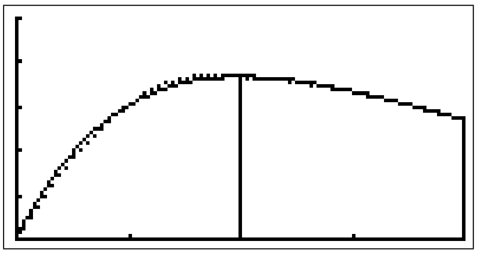
Here, the function is approximated on each sub-interval by a straight line, a polynomial of degree 1.

Program: NINTGRPH draws the trapeziums and calculates the sum of their (signed) areas as an approximation to the integral of the function in Y1 over a given integration range. You specify the number of sub-intervals (trapeziums) N.

Program: NUMINT calculates and displays (without graphics) the approximation to the integral of Y1 over a specified integration range using the Trapezoidal Rule with N (specified) trapeziums/sub-intervals.

Simpson's Rule

Here, a parabola or quadratic function, a polynomial of degree 2, is fitted to 3 points on the graph of the function corresponding to the endpoints of two adjacent sub-intervals. The integral is then approximated by the sum of the (signed) areas between the $N/2$ parabolas and the x axis. The number N of sub-intervals must therefore be even. Simpson's Rule is, in general, more accurate than the methods above for a given (even) number of sub-intervals.



Simpson's Rule showing two parabolas over 4 sub-intervals

Program: NINTGRPH draws the parabolas and calculates the sum of their areas as an approximation to the integral of the function in Y1 over a given integration range. You specify the number of sub-intervals N. If N is odd, the programs uses N+1.

Program: NUMINT calculates and displays (without graphics) the approximation to the integral of Y1 over a specified integration range using Simpson's Rule with 2N sub-intervals to ensure an even number.

Activity 2 (Section 11.4.2) goes into more detail about and compares the Left-Endpoint Rule, the Trapezoidal Rule and Simpson's Rule.

PTO

11.2.2 Gauss quadrature

Unlike the geometric methods of Section 11.2.1, Gauss quadrature methods do not have a simple geometric interpretation. The term Gauss quadrature covers several methods in which a definite integral is approximated as the weighted sum of n function values, with the weights w_i and the x values at which the function is evaluated x_i specified for a given method and a given order n;[10] the x_i are not evenly spaced, in general, unlike in the geometric methods. The geometric methods above also fit into this category because their formulas follow the same format.

Reference: Hooper, M, Dharmasena, V and Cui, M., *Exploring Gaussian quadrature with students: Part 1 — A forgotten idea*, Australian Senior Maths Journal, 32 (2), 23–31 (2021).

Theory

The basic Gauss quadrature sum is of the form

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

where the w_i are the specified weights, the x_i the specified x values at which the function is evaluated and n the number of these points (the order of the method).

For the general integral, a change of variables gives

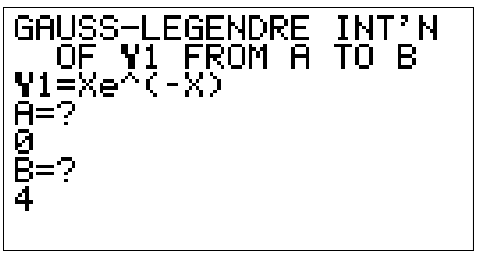
$$\int_{a}^{b} f(x)\,dx \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right).$$

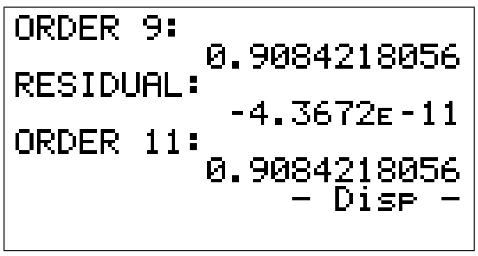
Gauss-Legendre quadrature

Here the w_i and x_i are related to the Legendre functions P_n. The x_i for Gauss-Legendre quadrature of order n are the zeros of P_n: $P_n(x_i)=0$. The weights x_i are given by

$$w_i = \frac{2(1-x_i^2)}{\big(nP_{n-1}(x_i)\big)^2}.$$

Program: GLQUAD calculates the Gauss-Legendre quadrature of orders 9 and 11 for the function in Y1 over the interval [A, B], where you specify A and B. Comparison of the two results gives you some idea of the accuracy. The residual can be added to the first value to give an answer to 14 digits. This is a non-adaptive algorithm with a fixed number of points.





Gauss-Legendre quadrature of orders 9 and 11

[10]These can be obtained from a number of sources. I used the website *keisan.casio.com* under *Professional/ Numerical integration*. This website also does the calculations.

Calculator numerical integrator $\int$dx

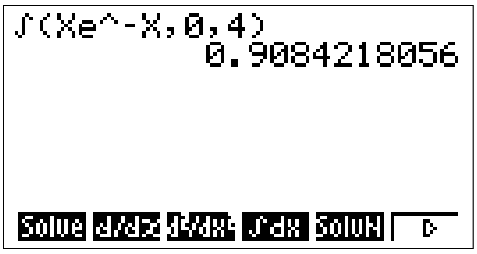
From the RUN screen, press OPTN CALC F4 for $\int$dx.

The numerical-integration command is of the form

$$\int(f(\mathrm{X}), a, b[,tol]).$$

X is always the variable we are integrating the function f with respect to, a and b are the integration limits and the optional argument tol is the tolerance. The default value is 10^{-5}.

Here, we are integrating $f(\mathrm{X}) = \mathrm{X}e^{-\mathrm{X}}$ between $\mathrm{X} = 0$ and $\mathrm{X} = 4$.



As usual, Casio gives no indication of the method used — it is almost certainly some type of adaptive Gauss quadrature — or how the tolerance works; you just have to trust the results. To be fair, whatever the method, it seems to be very accurate.

If in doubt, try a range of values for the tolerance, usually decreasing negative powers of 10, to see if there is any change in the digits you are interested in although, in my experience, changing the tolerance does not seem to make any difference, even with nastier functions.

11.3 Comparison of methods

The table below gives values for $\displaystyle\int_0^4 xe^{-x}\,dx$ using the different methods with 11 points (function evaluations). Values are rounded to 6 significant digits.

LER	0.879810
RER	0.909115
MID	0.915361
TRP	0.894462
SIM	0.908006
GLE	0.908422
$\int$dx	0.908422

The exact answer is $1 - 5e^{-4} = 0.908422$ to 6 significant digits. Unrounded, this answer and the last two in the table differ only in the 14th decimal place.

11.4 Activities

11.4.1 Rectangles, area and the definite integral

The program NINTGRPH illustrates graphically how the area under a graph can be approximated by the areas of rectangles. As the number of rectangles covering the area increases, we obtain a better approximation to the area.

We approximate $\int_0^1 e^x\,dx$ by drawing rectangles (the sum of the areas is a *Riemann sum*).

- Press [MENU] [5] and put the function $f(x)=e^x$ in Y1.
- Set a [V-Window] of $[0,1,0.2]\times[0,3,1]$.
- Press [MENU] [B], scroll down to NINTGRPH and press [F1] to start the program.
- Set the integration limits $A=0$ and $B=1$.
- Set the number of sub-intervals/rectangles $N=5$.
- Choose the Left-Endpoint Rule (LER in the table below). The program will plot the function and draw in 5 rectangles, each rectangle touching the curve at its top-left corner. In this case, the area of the rectangles clearly underestimates the area under the graph.
- Press [EXE] to see the area of the rectangles as an approximation to the area under the graph.
- Press [EXE] and set $N=5$ again, but this time choose the Right-Endpoint Rule (RER). Now we obtain an overestimate of the area under the curve.
- Repeat the above steps, doubling the number of rectangles each time. Fill in the table below, rounding your answers to 3 decimal places.[11]

N	LER	RER	MEAN
5			
10			
20			
40			
80			

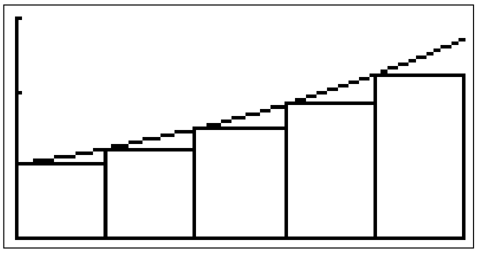
Your best estimate: $\int_0^1 e^x\,dx \approx$ ____________.

- When you've had enough, select QUIT in the RULE menu.

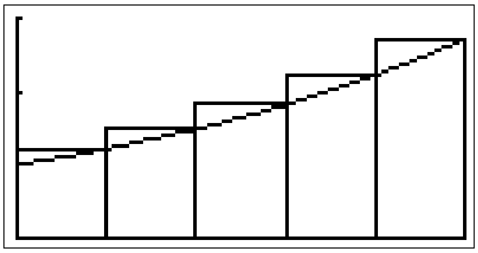
[11]The mean of the two estimates is equivalent to the Trapezoidal-Rule approximation to the area, a more accurate approximation for a given N than either the Left- or Right-Endpoint Rules.

Notes for teachers

Answers *rounded to 3 decimal places.*



Left-Endpoint Rule



Right-Endpoint Rule

N	LER	RER	MEAN
5	1.552	1.896	1.724
10	1.664	1.806	1.720
20	1.676	1.762	1.719
40	1.697	1.740	1.718
80	1.708	1.729	1.718

Best estimate is 1.718 from the MEAN column. The fact that two successive values give 1.718 indicates that this is probably the exact answer rounded to 3 decimal places. MEAN is equivalent to the Midpoint Rule.

The exact answer is $e-1=1.718$ rounded to 3 decimal places.

11.4.2 Approximating definite integrals

Modified from an UNSW Canberra Maths lab, which is itself based on a lab in *Resources for Calculus, Volume 1: Learning by Discovery*, Anita Solow, editor, Mathematical Association of America Note 26, 1993.

In this lab, we shall be comparing several numerical approximations to

$$\int_0^1 (5x^4 - 3x^2 + 1)\, dx$$

with the exact answer obtained by algebraic integration. This will give us a feel for some of the methods of numerical integration, which we can then use for any function, including those which cannot be integrated algebraically.

Question 1 *Algebraic integration — the exact answer*

What is the exact value of this integral? You may not realise it, but you are using the *Fundamental Theorem of Calculus* to do this definite integral exactly.

Question 2 *The Left-Endpoint Rule*

One approach to numerical integration is to approximate the definite integral of $y = f(x)$ with $a \le x \le b$ by the areas of a number of rectangles under the curve. If the top left-hand corner of each rectangle touches the curve, we have the *Left-Endpoint Rule*; if the top right-hand corner of each rectangle touches the curve, we have the *Right-Endpoint Rule*.

As the number of rectangles in the interval $[a, b]$ gets larger and larger (covering the integration range $a \leqslant x \leqslant b$ with more and more rectangles), both rules give numbers closer and closer to the definite integral (exact answer).

(a) On Figure 1 (at the end of this Lab), sketch and shade in the rectangles for the Left-Endpoint-Rule approximation to the definite integral $\int_a^b f(x)\, dx$ with 4 rectangles.
Note: The function in Figure 1 is not the function in Question 1.

(b) Using your sketch in (a), explain why the Left-Endpoint Rule with 4 rectangles approximates the area under the graph as

$$h\big(f(x_0) + f(x_1) + f(x_2) + f(x_3)\big),$$

where $x_0 = a$, $x_4 = b$ and the width of each rectangle is $h = (b - a)/4$.

(c) Use the NINTGRPH program (instructions over) to estimate $\int_0^1 (5x^4 - 3x^2 + 1)\, dx$ using the Left-Endpoint Rule with the number of rectangles N = 4.

A suitable View Window is $[0, 1, 0.1] \times [0, 3, 0.5]$. Note that the integrand here is positive, so that the definite integral corresponds to the area under the graph of f.

(d) Now use NINTGRPH, doubling N until two successive answers for the Left-Endpoint Rule are the same when rounded to 2 decimal places. Write down the N value of the first of these two answers.

Question 3 *The Trapezoidal Rule*

The Left-Endpoint and Right-Endpoint Rules approximate the area under a function by rectangles. In many cases, for example the function in Figure 1 with the rectangles you drew in, this is not a good approximation. We get a better approximation by using trapeziums: both top corners of each trapezium touch the curve.

(a) On Figure 2 (at the end of this Lab), draw and shade in 4 trapeziums (N = 4), the total area of which approximates the definite integral $\int_a^b f(x)\,dx$.

The area of the trapezium in Figure 3 is $h(r+s)/2$. To see this result, split the trapezium into two regions — a triangle and a rectangle.

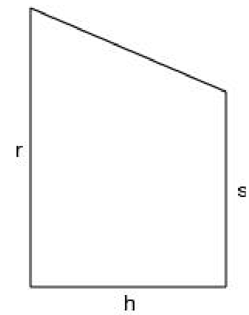


Figure 3

(b) Using your sketch in (a) and Figure 3, explain why the Trapezoidal Rule with 4 trapeziums approximates the area under the graph as

$$\frac{h}{2}\big(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\big),$$

where $x_0 = a$, $x_4 = b$ and the width of each trapezium is $h = (b-a)/4$.

(c) Evaluate T_4, the Trapezoidal Rule with 4 trapeziums, as an estimate of the integral $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using the NINTGRPH program.

How does this result compare with the Left-endpoint result and the exact answer?

(d) Now use NINTGRPH, doubling N, the number of trapeziums, until two successive answers for the Trapezoidal Rule are the same when rounded to 2 decimal places. Write down the N value of the first of these two answers.

Compare it with the Left-Endpoint value.

Question 4 *Simpson's Rule*

A picture of Simpson's Rule with 4 sub-divisions of the integration range is given in Figure 4 (at the end of this Lab). We want to estimate the area under the solid curve. We do this by fitting a parabola to each set of 3 successive points on the graph, covering 2 sub-intervals, and adding up the areas under the parabolas.

The dashed line in Figure 4 shows two parabolas: one through $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$; the other through $\big(x_2, f(x_2)\big)$, $\big(x_3, f(x_3)\big)$ and $\big(x_4, f(x_4)\big)$.

(a) On Figure 4, shade the area calculated by Simpson's Rule as an approximation to the definite integral $\int_a^b f(x)\,dx$ with two parabolas.

(b) Evaluate S_4, Simpson's Rule with 4 sub-intervals covering the integration interval, as an estimate of $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using NINTGRPH.

Compare your result with those from Questions 1−3.

(c) Now use the NINTGRPH program, doubling N, the number of sub-intervals, until two successive answers for Simpson's Rule are the same when rounded to 2 decimal places. Write down the N value of the first of these two answers.

Compare it with the values from Questions 2 and 3.

Question 5 *Comparing the methods*

Repeat your first (N = 4) and last calculations above for the three methods, this time keeping 5 decimal places. Put them in a summary table, together with the h value and the absolute value of the error $|E|$ for each entry (you know the exact answer).

The NUMINT program (no graphics) might be faster for this but note the N for Simpson's Rule (instructions below).

If $|E| = kh^m$, where k is a constant, find m for each method. *Hint*: Use the two sets of values of h and the corresponding errors to write down two equations for k and m; divide one equation by the other to obtain an equation for m. You'll need natural logs to isolate m. *Hint*: We are looking for integer values.

How many times do you have to double N in each method to increase the accuracy by a factor of 10?

What conclusions can you draw from your results regarding the different methods for estimating the definite integral? Which method would you choose to use? Why?

Programs

These programs calculate approximate values for $\displaystyle\int_A^B f(X)\, dX$.
The number N is an input to the program.

NINTGRPH approximates the integral using the *Left-Endpoint Rule*, the *Right-Endpoint Rule*, the *Trapezoidal Rule* or *Simpson's Rule* **with N sub-intervals** covering the interval [A, B] (N+1 if N is odd, for Simpson's Rule), and draws the corresponding approximations to the function on each sub-interval.

NUMINT (no graphics) approximates the integral using the *Left-Endpoint Rule* (**L**), the *Right-Endpoint Rule* (**R**), the *Trapezoidal Rule* (**T**) and the *Midpoint Rule* (**M**), **all with N sub-intervals**, and *Simpson's Rule* (**S**) **with 2N sub-intervals** to ensure an even number of sub-intervals.

Use: Type the function to be integrated into Y1.

- For NINTGRPH, first set a suitable View Window to display the function. Run the program and follow the prompts. Make sure B > A, otherwise things get mixed up.

  Press [EXE] after the graph is plotted to see the numerical approximation to the integral, and [EXE] again to do a new plot. When you've finished, select QUIT in the RULE menu.

- For NUMINT, run the program and follow the prompts. Press [EXE] repeatedly to obtain the respective answers, and finally to input a different number N of sub-intervals. A negative number of sub-intervals stops the program.

Numerical Integration Lab Figures

Figure 1: Left-Endpoint Rule

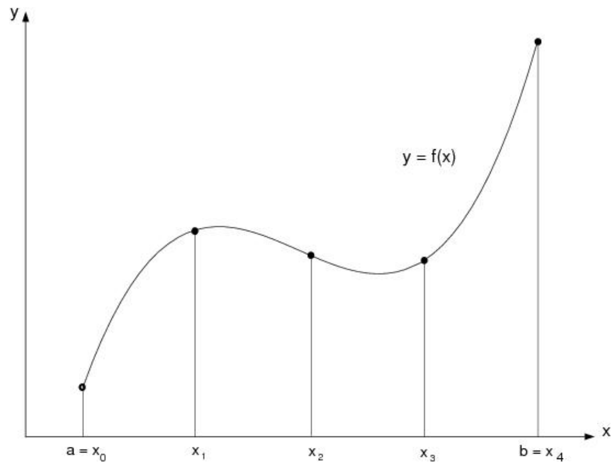


Figure 2: Trapezoidal Rule

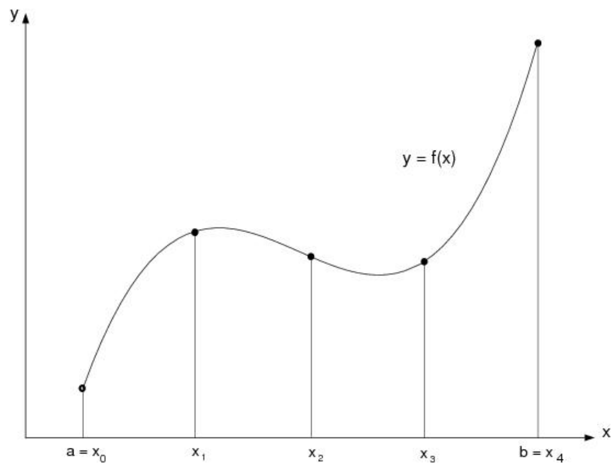
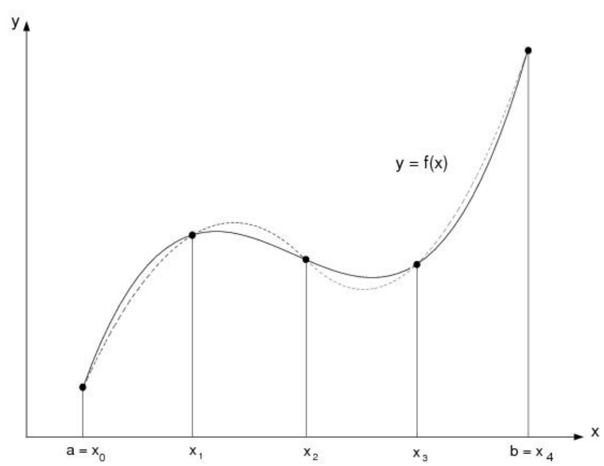


Figure 4: Simpson's Rule



Notes for teachers

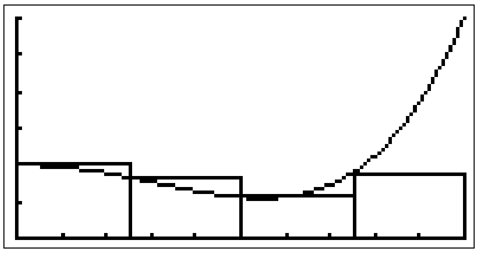
Question 1 *Algebraic integration — the exact answer*

$$\int_0^1 (5x^4 - 3x^2 + 1)\,dx = \Big[x^5 - x^3 + x\Big]_0^1 = 1.$$

Question 2 *The Left-Endpoint Rule*

(a) On Figure 1, draw and shade in the rectangles for the Left-Endpoint-Rule approximation to the definite integral $\int_a^b f(x)\,dx$ with 4 rectangles.

Done here with the function in Question 1 using the NINTGRPH program (no shading). The horizontal lines (tops of the rectangles) are the approximations to the function f on each sub-interval.



V-Window $[0,1,0.1] \times [0,3,0.5]$

(b) Using your sketch in (a), explain why the Left-Endpoint Rule with 4 rectangles approximates the area under the graph as

$$h\big(f(x_0) + f(x_1) + f(x_2) + f(x_3)\big),$$

where $x_0 = a$, $x_4 = b$ and the width of each rectangle is $h = (b-a)/4$.

The formula is just the sum of the areas of the four rectangles in (a) with h factored out.

(c) Use the NINTGRPH program to estimate $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using the Left-Endpoint Rule with the number of rectangles N = 4.

A suitable V-Window is $[0,1,0.1] \times [0,3,0.5]$.

Note that the integrand here is positive, so that the definite integral corresponds to the area under the graph of f.

See the figure in (a).

The Left-Endpoint Rule with N = 4 gives $\displaystyle\int_0^1 (5x^4 - 3x^2 + 1)\,dx \approx 0.822.$

(d) Now use NINTGRPH, doubling N until two successive answers from the Left-Endpoint Rule are the same when rounded to 2 decimal places. Write down the N value of the first of these two answers.

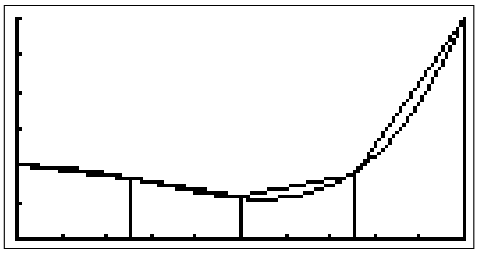
N	LER
4	0.82
8	0.89
16	0.94
32	0.97
64	0.98
128	0.99
256	1.00
512	1.00

The required N value is therefore 256. The last two values are the same as the exact answer rounded to 2 decimal places.

Question 3 *The Trapezoidal Rule*

(a) On Figure 2, draw and shade in the 4 trapeziums (N = 4), the total area of which approximates the definite integral $\int_a^b f(x)\,dx$.

Again done here with the function in Question 1 but without shading. The straight lines (tops of the trapeziums) are the approximations to the function f on each sub-interval.



V-Window $[0, 1, 0.1] \times [0, 3, 0.5]$

(b) Using your sketch in (a) and Figure 3, explain why the Trapezoidal Rule with 4 trapeziums approximates the area under the graph as

$$\frac{h}{2}\big(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\big),$$

where $x_0 = a$, $x_4 = b$ and the width of each trapezium is $h = (b-a)/4$.

The total area of the 4 trapeziums, each of width h is, using the given formula,

$$h\,\frac{f(0)+f(x_1)}{2} + h\,\frac{f(x_1)+f(x_2)}{2} + h\,\frac{f(x_2)+f(x_3)}{2} + h\,\frac{f(x_3)+f(x_4)}{2}$$
$$= \frac{h}{2}\Big(f(x_0) + 2hf(x_1) + 2hf(x_2) + 2hf(x_3) + hf(x_4)\Big).$$

(c) Evaluate T_4, the Trapezoidal Rule with 4 trapeziums, as an estimate of the definite integral $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using NINTGRPH.
How does this result compare with the Left-Endpoint result and the exact answer?

See the figure in (a). $T_4 = 1.07$, considerably closer to the exact answer 1 than the Left-Endpoint Rule value with N = 4 of 0.82.

(d) Now use NINTGRPH, doubling N, the number of trapeziums, until two successive answers are the same when rounded to 2 decimal places. Write down the N value of the first of these two answers. Compare it with the rectangle N value.

N	**TRAP**
4	1.07
8	1.02
16	1.00
32	1.00

The required N value is therefore 16, compared with the much larger value of 256 for the Left-Endpoint Rule. The last two values are the same as the exact answer rounded to 2 decimal places.

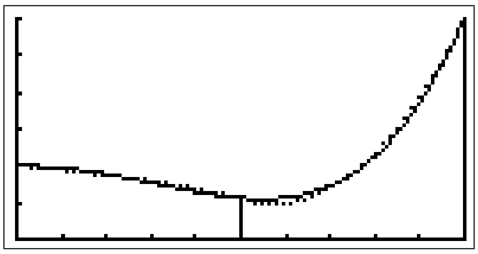
Question 4 *Simpson's Rule*

(a) On Figure 4, shade the area calculated by Simpson's Rule as an approximation to the definite integral $\int_a^b f(x)\,dx$.

Shade below the dotted curves.

(b) Evaluate S_4, Simpson's Rule with 4 sub-intervals, as an estimate of $\displaystyle\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ (Simpson's Rule) using NINTGRPH.

The dotted lines in the figure are the two parabolas, which are the approximations to the function on each pair of sub-intervals.



V-Window $[0, 1, 0.1] \times [0, 3, 0.5]$

Compare your result with those from Questions 1 − 3.

$S_4 = 1.003$, considerably closer to the exact answer 1 than the other two values.

(c) Now use NINTGRPH, doubling the number N of sub-intervals each time, until two successive answers are the same when rounded to 2 decimal places. Write down the N value of the first of these two answers. Compare it with the rectangle and trapezium values.

N	SIMP
4	1.00
8	1.00

The required N value is therefore 4, compared with 16 for the Trapezoidal Rule and the much larger value of 256 for the Left-Endpoint Rule. The two values here are the same as the exact answer rounded to 2 decimal places.

Question 5 *Comparing the methods*

If $|E| = kh^m$, where k is a constant, find m for each method.

$|E|$ in the table below is the difference between Value and 1, the exact answer.

Summary table

Method	**N**	$\boldsymbol{h}$	**Value** (5DP)	$\boldsymbol{|E|}$ (5DP)
Left Endpoint	4	0.25	0.82227	0.17773
Left Endpoint	256	0.0039	0.99611	0.00389
Trapezoidal	4	0.25	1.07227	0.07227
Trapezoidal	16	0.0625	1.00455	0.00455
Simpson	4	0.25	1.00260	0.00260
Simpson	8	0.125	1.00016	0.00016

Assume error $|E| = kh^m$, where k is a constant. Therefore, if we have values E_1, h_1, E_2 and h_2 for a method,

$$\frac{E_1}{E_2} = \frac{h_1^m}{h_2^m} = \left(\frac{h_1}{h_2}\right)^m.$$

Taking natural logs of both sides and solving for m, we have

$$m = \frac{\ln\left(\frac{E_1}{E_2}\right)}{\ln\left(\frac{h_1}{h_2}\right)}.$$

Substituting in the two values for E and h for each method, we get $m \approx 1$ for the Left-Endpoint Rule, $m \approx 2$ for the Trapezoidal Rule and $m \approx 4$ for Simpson's Rule.

How many times do you have to double N in each method to improve the accuracy by a factor of 10?

If you double N, you halve h. Therefore, for the Left-Endpoint Rule, you reduce the error by 0.5^1, i.e. you reduce the error by a factor of 2. For the Trapezoidal Rule, you reduce the error by 0.5^2, i.e. you reduce the error by a factor of 4. For Simpson's Rule, you reduce the error by 0.5^4, i.e. you reduce the error by a factor of 16.

To achieve an improvement in accuracy of 1 decimal place, you have to reduce the error by a factor of 10: therefore you have to double N (halve h) four times ($2^3 = 8 < 10$; $2^4 = 16 > 10$) using the Left-Endpoint Rule, twice ($4^2 = 16 > 10$) for the Trapezoidal Rule and only once ($16^1 = 16 > 10$) for Simpson's Rule.

What conclusions can you draw from your results regarding the different methods for estimating the definite integral? Which method would you choose to use? Why?

For a given N or h, Simpson's Rule gives the most accurate approximation to the definite integral. To calculate an approximation to a given accuracy, it will therefore be the fastest of the three methods.

12 Taylor Series

12.1 Introduction

Values of polynomial functions can be found by performing a finite number of additions and multiplications but other special functions such as $\ln(x)$, e^x, $\cos(x)$ and $\tan(x)$ cannot be evaluated as easily. Many functions can be approximated by polynomials, and the polynomial, instead of the original function, can be used for computations when the difference between the actual function and the polynomial approximation is sufficiently small. Calculators and computers use polynomials of some sort to calculate values of special functions.

Graphics calculators are ideal for this topic because they show the result graphically of all the algebra in calculating the Taylor polynomials and how the polynomials converge or otherwise to the function they are approximating. The interval of convergence of a series can also usually be found from the graphs.

The exercises and problems here go from the basic calculation up to several interesting approximate models for physical phenomena.

12.2 Approximating functions by polynomials

Various methods can be employed to approximate a given function by polynomials. One of the most widely used is the method of Taylor polynomials, named in honour of the English mathematician Brook Taylor (1685 – 1731), who introduced them in 1715. Taylor polynomials about $x = 0$ are also called Maclaurin polynomials after the Scottish mathematician Colin Maclaurin (1698 – 1746), who made extensive use of this special case of Taylor series in the 18th century.

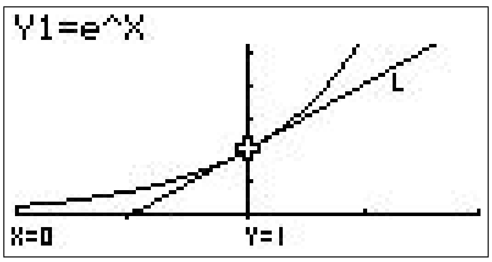
12.2.1 Linear approximation

The tangent line at a point on the curve of a function is a *linear approximation* to the function at that point; it has the *same value* and the *same first derivative* (slope) there as the function.

Example 1: Find the equation of the tangent line (graph L in the figure below) as a linear approximation to $f(x) = e^x$ near $x = 0$

Use the tangent-line approximation to estimate $e^{0.1}$.

Compare with the calculator value (presumably reasonably accurate).



V-Window $[-2, 2, 1] \times [-0.5, 3, 1]$

The function

$f(0) = e^0 = 1.$ $f'(x) = e^x$, so that $f'(0) = 1$.

The linear approximation

The general equation of a straight line is $y(x) = a_0 + a_1 x$, where a_0 and a_1 are constants.

The tangent has the same value as f at $x = 0$. Therefore,

$$y(0) = a_0 = f(0) = 1.$$

The tangent has the same slope as the graph of f at $x = 0$: $y'(x) = a_1$. Therefore,

$$y'(0) = a_1 = f'(0) = 1.$$

The equation of the tangent and the linear approximation to $f(x) = e^x$ centred at $x = 0$ is then

$$y(x) = 1 + x.$$

Therefore, $f(0.1) = e^{0.1} \approx y(0.1) = 1.1$.

The exact value is $f(0.1) = e^{0.1} = 1.105$, to four significant digits.

12.2.2 Quadratic approximation

Suppose we want a more accurate way of approximating $f(x) = e^x$ near $x = 0$. We will require that, at $x = 0$, the curve $f(x) = e^x$ and the approximating curve have *the same value, the same first derivative* (slope) and also the same rate of bending (concavity), that is *the same second derivative.*

The simplest type of function we might use for this approximation is a quadratic, with general formula[12]

$$P_2(x) \;=\; a_0 \;+\; a_1 x \;+\; a_2 x^2, \tag{1}$$

where we must determine the values of a_0, a_1 and a_2 to fit the quadratic to our function.

At $x = 0$ we want P_2 and f to have the same value, so we want

$$P_2(0) \;=\; f(0). \tag{2}$$

We want P_2 and f to have the same first derivative at $x = 0$, so

$$P_2'(0) \;=\; f'(0). \tag{3}$$

and we also want P_2 and f to have the same second derivative at $x = 0$, so

$$P_2''(0) = f''(0). \tag{4}$$

These three conditions will determine a_0, a_1 and a_2.

From Eq. (1),

$$\begin{aligned} P_2(0) \;&=\; a_0 \;=\; f(0) \qquad \text{from Eq. (2).} \\ \therefore\; a_0 \;&=\; f(0). \end{aligned}$$

[12] P for polynomial, 2 for degree 2. The linear approximation in Example 1 is $P_1(x)$.

Differentiating Eq. (1),

$$\begin{aligned} P_2'(x) &= a_1 + 2a_2x. \\ \therefore\; P_2'(0) &= a_1 = f'(0) \qquad \text{from Eq. (3)}. \\ \therefore\; a_1 &= f'(0). \end{aligned}$$

Continuing,

$$\begin{aligned} P_2''(x) &= 2a_2 \\ \therefore\; P_2''(0) &= 2a_2 = f''(0) \qquad \text{from Eq. (4)}. \\ \therefore\; a_2 &= \frac{f''(0)}{2}. \end{aligned}$$

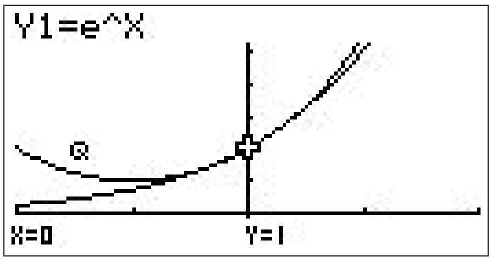
Therefore,

$$\begin{aligned} P_2(x) &= a_0 + a_1x + a_2x^2 \\ &= f(0) + f'(0)x + \frac{f''(0)}{2}\,x^2. \end{aligned}$$

This is a quadratic approximation to any function f centred at $x = 0$, provided that the function has finite first and second derivatives at $x=0$.

Exercise 1 *Solutions to the Exercises are on page 91.*

(a) Use the general result above to find the quadratic approximation (graph Q in the figure below) to $f(x)=e^x$ centred at $x=0$.



V-Window $[-2, 2, 1]\times[-0.5, 3, 1]$

(b) Use the approximation to estimate $e^{0.1}$. Compare with the calculator value and $P_1(0.1)$ (Example 1).

12.3 Taylor polynomials

12.3.1 Centred at $x=0$

As a rule, over a given interval the quadratic function will be a better approximation than the linear function. However, the figure on page 77 shows that even though we have matched up the function and the quadratic in terms of their values, slopes and concavity *at the point* $x=0$, they still bend away from one another for x *well away from* $x=0$. We can improve this by using approximating polynomials of higher degree.

Suppose that we want to approximate a function $f(x)$ near $x=0$ by a polynomial of degree n, that is,

$$f(x) \approx P_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_{n-1}x^{n-1} + a_nx^n. \tag{5}$$

We now need to find the values of the constants a_0, a_1, $\ldots$, a_n. To do this, we require that the values of the function and all of its n derivatives agree with those of the polynomial *at the point* $x=0$. Polynomials which have this property are called **Taylor polynomials**, in this case Taylor polynomials centred at $x=0$.

Taylor polynomial of degree n for $f(x)$ centred at $x=0$

$$f(x) \;\approx\; P_n(x) \;=\; f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

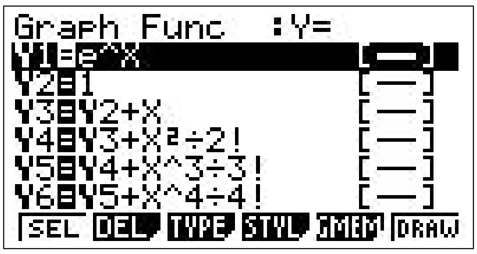
Note: $P_n(x)$ is just $P_{n-1}(x)$ plus an extra term in x^n.

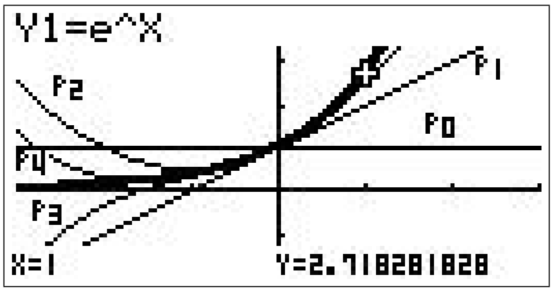
Example 2: Use the general formula to find the nth-degree Taylor polynomial centred at $x=0$ for $f(x)=e^x$.

For $f(x)=e^x$, $f(0)$ and all the derivatives evaluated at 0 are equal to 1. Therefore,

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \;=\; \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Plot f and $P_0 - P_4$ on your calculator. Y in the function names is VARS GRPH F1.





V-Window $[-3,3,1]\times[-2,4,1]$

Use each polynomial to approximate $e^{0.2}$, and compare with the calculator value. Underline the digits that are correct in each approximate value.

$e^{0.2} \approx P_0(0.2) = \underline{1}.$

$e^{0.1} \approx P_1(0.2) = \underline{1.2}.$

$e^{0.2} \approx P_2(0.2) = \underline{1.22}.$

$e^{0.2} \approx P_3(0.2) = \underline{1.221}\dot{3}.$

$e^{0.1} \approx P_4(0.2) = \underline{1.2214}.$

From the calculator, $e^{0.2} = 1.221403$ to 7 significant digits.

Exercise 2: Derive the result in the box on the previous page.

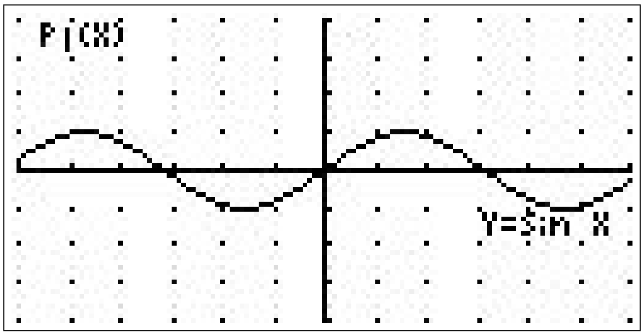
Hint: Repeat and extend the process begun in Section 12.2.2, but starting with Eq. (5) for P_n instead of Eq. (1) for P_2.

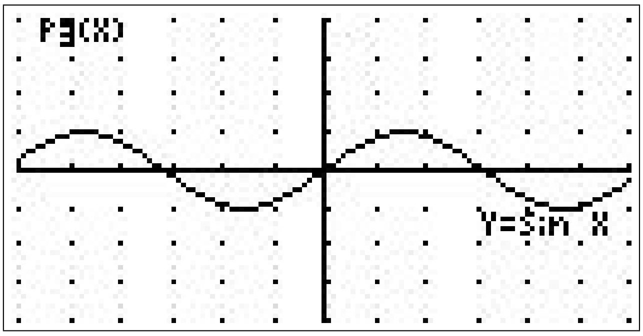
Once you have established a pattern in the coefficients, you may write $\vdots$, then the nth term.

Exercise 3: Find the Taylor polynomials centred at $x = 0$ for $f(x) = \sin(x)$, up to degree 11 (look for a pattern in the first few coefficients).

Plot P_1 and P_3 on the relevant graphs below; the window is $[-6, 6, 1] \times [-4, 4, 1]$.

Use each polynomial up to P_{11} to approximate $\sin(0.2)$, and compare with the calculator value. Underline the digits that are correct in each approximate value.





12.3.2 Centred at $x = c$

If instead of approximating a function $f(x)$ near $x = 0$, we want to approximate it near some large $x = c$, for the functions considered so far we would probably need a large number of terms in the Taylor polynomial about $x = 0$ to give an accurate approximation at $x = c$; the further c is from the origin, the more terms would be required. It makes more sense to use the Taylor polynomial centred at $x = c$.

Suppose we know the values of f and its derivatives at $x = c$. We want to find a polynomial $P_n(x)$ which is a good approximation to $f(x)$ for values of x close to c.

As the expression $(x - c)$ tells us how close x is to c, we use it to construct the polynomials approximating f at $x = c$. We write our general polynomial as

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n.$$

If we require the derivatives of the approximating polynomial and the original function to agree at $x = c$, we get the following result.

Taylor polynomial of degree n for $f(x)$ centred at $x = c$

$$f(x) \approx P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Example 3: Construct the Taylor polynomials centred at $x = 1$ for $f(x) = \ln(x)$ up to degree 5.

$$P_5(x) = f(1) + f'(1)\,(x-1) + \frac{f''(1)}{2!}\,(x-1)^2 + \frac{f'''(1)}{3!}\,(x-1)^3 + \frac{f^{(4)}(1)}{4!}\,(x-1)^4 + \frac{f^{(5)}(1)}{5!}\,(x-1)^5.$$

$f(x) = \ln(x) \qquad f(1) = 0.$

$f'(x) = \dfrac{1}{x} \qquad f'(1) = 1.$

$f''(x) = -\dfrac{1}{x^2} \qquad f''(1) = -1.$

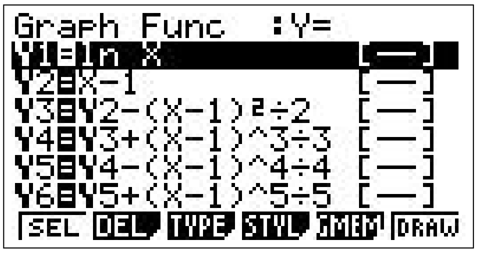
$f'''(x) = \dfrac{2}{x^3} \qquad f'''(1) = 2.$

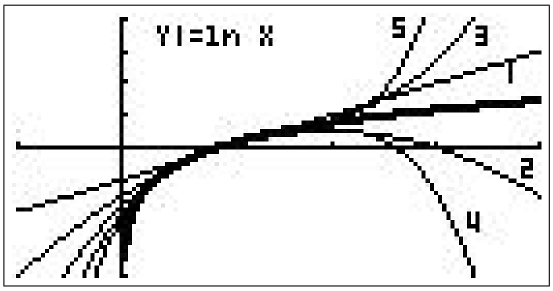
$f^{(4)}(x) = -\dfrac{6}{x^4} \qquad f^{(4)}(1) = -6.$

$f^{(5)}(x) = \dfrac{24}{x^5} \qquad f^{(5)}(1) = 24.$

$$\begin{aligned} \therefore\ P_5(x) &= (x-1) - \frac{1}{2!}\,(x-1)^2 + \frac{2}{3!}\,(x-1)^3 - \frac{6}{4!}\,(x-1)^4 + \frac{24}{5!}\,(x-1)^5 \\ &= x-1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}. \end{aligned}$$

Plot f and $P_1 - P_5$ on your calculator using a window $[-1, 4, 1] \times [-4, 4, 1]$.





Use the successive polynomials to approximate ln(1.2), and compare with the calculator value. Underline the digits that are correct in each approximate value.

From the calculator, $\ln(0.2) = 0.1823215568$ to 10 significant digits.

$\ln(1.2) \approx P_1(1.2) = 0.2$.

$\ln(1.2) \approx P_2(1.2) = \underline{0.18}$.

$\ln(1.2) \approx P_3(1.2) = \underline{0.182}\dot{6}$.

$\ln(1.2) \approx P_4(1.2) = \underline{0.182}2\dot{6}$.

$\ln(1.2) \approx P_5(1.2) = \underline{0.1823}30\dot{6}$.

Exercise 4: Find the fifth-degree Taylor polynomial for $f(x) = \dfrac{1}{x}$ centred at $x = 3$.

Plot f and $P_1 - P_5$ on your calculator using a window $[0, 10, 1] \times [-2, 2, 1]$.

12.4 Taylor series

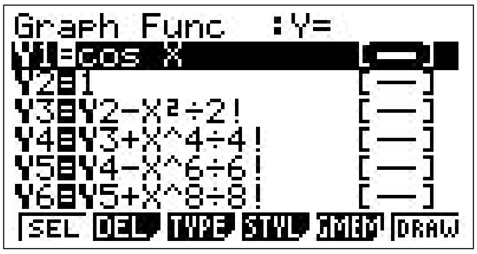
12.4.1 Centred at $x = 0$

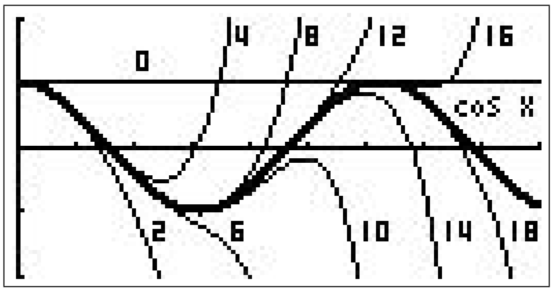
We have seen how to approximate a function near a point by Taylor polynomials.

Example 4: The first few Taylor polynomials for $f(x) = \cos(x)$ centred at $x = 0$ are

$$\begin{aligned}
\cos(x) &\approx P_0(x) = 1 \\
\cos(x) &\approx P_2(x) = 1 - \frac{x^2}{2!} \\
\cos(x) &\approx P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\
\cos(x) &\approx P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\
\cos(x) &\approx P_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.
\end{aligned}$$

Note the $\approx$ *sign.* Each polynomial is a better approximation to $\cos(x)$ than the previous one.





V-Window $[0, 9, 1] \times [-2, 2, 1]$

We are therefore tempted to write

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots,$$

indicating that the cosine function is **equal** to this "infinite-degree" polynomial. *Note the* $=$ *sign.* This infinite sum is called the **Taylor series** for $\cos(x)$, centred at $x = 0$.

Note that $\cos(x)$ and the $P_n(x)$ here are all even functions: they are symmetric about the y axis.

What do we mean by such an infinite sum? You may already have some intuition about what it means, for it can be interpreted in exactly the same way we interpret a more familiar statement like

$$\frac{1}{3} = 0.333333\cdots .$$

This means that 1/3 is the *limit* of successive finite decimals 0.3, 0.33, 0.333, 0.3333, ...

Similarly, for any x, the value of $\cos(x)$ is the *limit* of the values of successive finite Taylor polynomials. We say that the Taylor series *converges* to $\cos(x)$ for all x, because for all x, the values

$$P_0(x),\ P_1(x),\ P_2(x), \cdots , P_n(x), \cdots$$

converge (get closer and closer to) to the value $\cos(x)$ as $n \to \infty$.

For example, at $x = 1$,

$$\begin{aligned}
P_0(1) &= 1 \\
P_2(1) &= 0.\underline{5} \\
P_4(1) &= 0.\underline{54}16 \\
P_6(1) &= 0.\underline{540}27 \\
P_8(1) &= 0.\underline{540302}5794 \\
P_{10}(1) &= 0.\underline{540302302}8 \\
&\vdots \\
P_{18}(1) &= 0.\underline{5403023058681}5 \\
&\vdots \\
\cos(1) &= 0.54030230586184 \quad \text{(to 14 significant digits)},
\end{aligned}$$

and thus we see that $P_n(1)$ gets closer and closer to $\cos(1)$ as $n \to \infty$.

Some common Taylor series centred at $x = 0$

For all x,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^n \frac{x^{2n-1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$$

In the last two series, the first term corresponds to $n = 0$. Note that $0! = 1$ by definition.

12.4.2 Centred at $x = c$

Any function f, all of whose derivatives exist at $x = c$, has a Taylor series centred at $x = c$. Assuming that the series converges to $f(x)$, we have the following result,

Taylor series for $f(x)$ centred at $x = c$

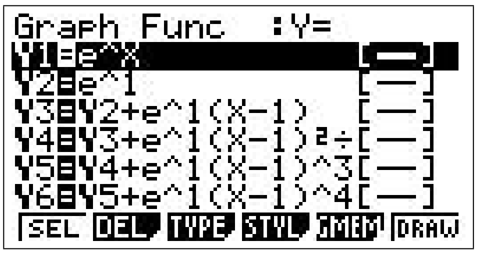
$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

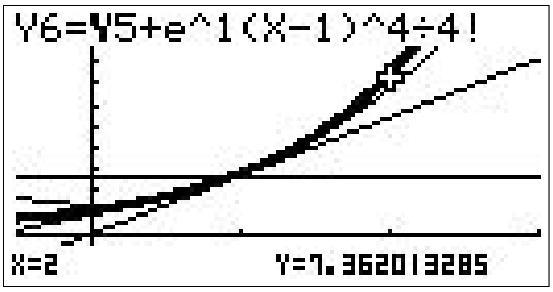
Example 5: Find the Taylor series centred at $x=1$ for $f(x)=e^x$.

As $f'(x)=f(x)$, we have $f(1)=f'(1)=f''(1)=f'''(1)=f^{(4)}(1)=\ldots=e$.

Therefore,

$$\begin{aligned}
e^x &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \ldots \\
&= e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots + \frac{e}{n!}(x-1)^n + \ldots \\
&= e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \frac{(x-1)^n}{n!} + \ldots\right) \\
&= e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}.
\end{aligned}$$





V-Window $[-0.5, 2.5, 1] \times [-1, 10, 1]$
The cursor is on P_4.

Exercise 5: Find the Taylor series centred at $x=\pi/2$ for $f(x)=\cos(x)$.

12.4.3 A neat method

Example 6: Find the Taylor series for $f(x)=e^{x^2}$ about $x=0$.

We have, from the box on page 83,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Therefore, replacing x with x^2,

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \cdots + \frac{x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Clearly, this method is quicker than the standard method (see Exercise 7). Keep an eye out for other cases, especially if the standard method looks complicated.

Exercise 6: Find the Taylor series for $f(t)=\dfrac{\sin(t)}{t}$ centred at $t=0$.

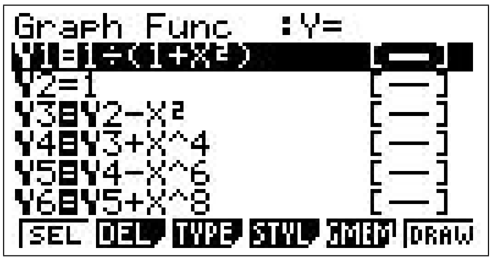
Exercise 7: Use the standard method to find the first three non-zero terms in the Taylor series for $f(x)=e^{x^2}$ centred at $x=0$. Plot together with f.

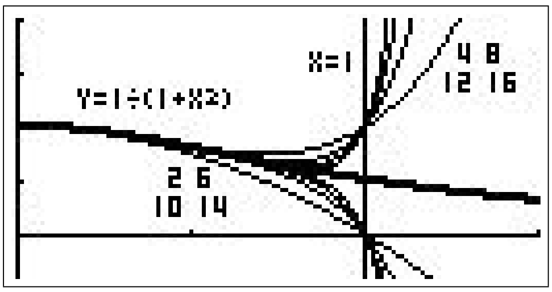
12.5 Interval of convergence

Example 7: Let's look at the graph of $f(x)=1/(1+x^2)$ and its successive Taylor polynomials centred at $x=0$ (below center),

$$P_n(x) = 1 - x^2 + x^4 - x^6 + \cdots \pm x^n.$$

with $n = 2, 4, 6, 8, 10, 12, 14, 16, 200, 202$. As the graphs of all the functions are symmetric about the y axis (*why is this?*), we only draw the them for positive x. There is also a vertical line drawn at $x=1$.





V-Window $[0, 1.5, 0.5] \times [-0.4, 2, 0.5]$

It appears that the graphs of the Taylor polynomials $P_n(x)$ approach the graph of $1/(1+x^2)$ (the bold curve) very nicely *as long as* $x<1$. If $x>1$, it looks like there is no convergence, no matter how many terms in the Taylor series we add. In fact, when $x>1$, the larger the n, the worse the approximation P_n is to the function.

We can therefore write

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \qquad \text{for } |x| < 1,$$

where the restriction on x is *essential* if we want to use the = sign.

We say that the interval $-1 < x < 1$ is the **interval of convergence** (IC) of the Taylor series for $1/(1+x^2)$ centred at $x=0$.

Some Taylor series, such as those for $\sin(x)$, $\cos(x)$ and e^x, converge for *all* values of x, that is their IC is $(-\infty, \infty)$. Other Taylor series, such as those for $1/(1+x^2)$ and $\ln(x)$, have *finite* ICs.

Exercise 8: From the Taylor polynomials for $f(x) = \ln(x)$ centred at $x = 1$ in Example 3, write down the Taylor series and determine its IC by graphing the function and its first few Taylor polynomials.

Check by evaluating $P_n(1.2)$ and $P_n(2.2)$ for the first few values of n and comparing them with the values of $\ln(1.2)$ and $\ln(2.2)$.

Exercise 9: Find the Taylor series for $f(x) = \dfrac{1}{1-x}$ centred at $x=0$ and determine its IC.

Exercise 10: Find the Taylor series for the function

$$f(x) \quad = \quad \begin{cases} \cos(x) & |x| \leqslant \pi \\ -1 & |x| > \pi \end{cases}$$

centred at $x=0$. What is its IC?

Plot the function as Y1 = cos(X)(abs(X) ⩽ π) − 1(abs(X) > π). The logical expressions in brackets evaluate to 1 if the expression is true and 0 if the expression is false.

Exercise 11

(a) Find the Taylor series and its IC for $\ln(1+x)$ centred at $x=0$.

(b) The Taylor series for $\ln(x)$ about $x=1$ has an IC of $0 < x < 2$ (see Example 3), the series for $f(x) = \ln(1+x)$ about $x = 0$ an IC of $-1 < x < 1$. Neither is therefore suitable for finding $\ln(x)$ over its whole domain $x > 0$.

Find the Taylor series centred at $x=0$ for $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$ and its IC.
Hint: use a log law and previous results.

(c) As x ranges across the IC you found in (b), what values does $g(x) = \dfrac{1+x}{1-x}$ take?

(d) Hence write out an algorithm to find the natural log of any number $x > 0$.

Plot the difference between $\ln(x)$ and $P_5(x)$, and the difference between $\ln(x)$ and $P_{10}(x)$, for $0 < x < 20$.

12.6 Problems

Solutions to the problems are on page 103.

When the function is given, check your answer by plotting the function and the Taylor polynomial(s).

1. Find the Taylor polynomials of degree $n=2,3,4$, centred at $x=0$, for $f(x)=\sqrt{1+x}$.

2. Find the Taylor polynomials of degree $n=3,4$, centred at $x=0$, for $f(x) = \arctan(x) \equiv \tan^{-1}(x)$. Note that $f'(x) = 1/(1+x^2)$.

3. Find the Taylor polynomials of degree $n=2,3,4$, centred at $x=0$, for $f(x)=\dfrac{1}{\sqrt{1+x}}$.

4. Suppose a function $f(x)$ is approximated by the sixth-degree Taylor polynomial centred at $x=0$: $P_6(x) = 3x - 4x^3 + 5x^6$.

Give the value of **(a)** $f(0)$; **(b)** $f'(0)$; **(c)** $f'''(0)$; **(d)** $f^{(5)}(0)$; **(e)** $f^{(6)}(0)$.

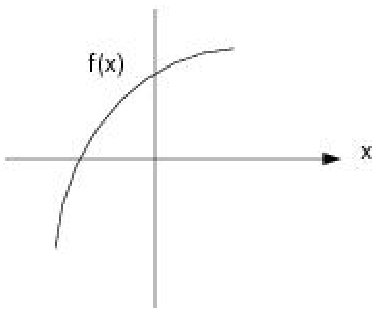
5. Suppose that g is a function which has continuous derivatives, and that $g(5)=3$, $g'(5)=-2$, $g''(5)=1$ and $g'''(5)=-3$.

(a) What is the Taylor polynomial of degree 2 centred at $x=5$ for g?
What is the Taylor polynomial of degree 3 centred at $x=5$ for g?

(b) Use the two polynomials that you found in (a) to approximate $g(4.9)$.

6. Suppose $P_2(x)=a+bx+cx^2$ is the second-degree Taylor polynomial, centred at $x=0$, for a function f.

What can you say about the signs of a, b and c if f has the graph given below?



PTO

7. **(a)** Find the Taylor polynomial of degree 3 centred at $x = 0$ for

$$\sinh^{-1}(x) \quad = \quad \int_0^x \frac{dt}{\sqrt{1+t^2}}.$$

Remember the Second Fundamental Theorem of Calculus?

(b) Hence estimate $\sinh^{-1}(0.25)$.

Check your estimate using the $\sinh^{-1}$ function in your calculator[13] and/or using numerical integration. How accurate is the estimate?

(c) For what values of x is the approximation accurate to within 0.1?

8. **(a)** Integrate the series for $\sin(t)/t$ (Exercise 6, Section 12.4.3) term by term, including the general term, to find the Taylor series centred at $x = 0$ for the sine-integral function

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\, dt.$$

(b) How many terms of the series do you need to take to give an estimate for Si(0.25) accurate to 8 significant digits?

Hint: Check the approximation from successive Taylor polynomials.

Check your value using numerical integration.

9. The Taylor series of $f(x) = x^2 e^{x^2}$, centred at $x = 0$, is (Section 12.4.3)

$$x^2 + x^4 + \frac{x^6}{2!} + \frac{x^8}{3!} + \frac{x^{10}}{4!} + \cdots .$$

Use the Taylor series to find $\left.\frac{d}{dx}\left(x^2 e^{x^2}\right)\right|_{x=0}$ and $\left.\frac{d^6}{dx^6}\left(x^2 e^{x^2}\right)\right|_{x=0}$.

10. Show how you can use the Taylor approximation for $\sin(x)/x$ centred at $x = 0$ (Exercise 6) to explain why

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

11. By graphing f and several of its Taylor polynomials, estimate the interval of convergence of the Taylor series, centred at $x = 0$, for $f(x) = \frac{1}{\sqrt{1+x}}$.

Hint: Question 3 might be useful here.

12. By looking at the Taylor series about $x = 0$ for each function, decide which of the functions

$$f(\theta) = 1 + \sin(\theta), \qquad g(\theta) = e^\theta, \qquad h(\theta) = \frac{1}{\sqrt{1-2\theta}}$$

is largest and which is smallest for small positive θ.

Plot the functions using a suitable [V-Window] to check your answer.

[13] $\sinh^{-1}$ is in the [OPTN] [HYP] menu.

13. *Force due to gravity*

When a body is near the surface of the Earth, we usually assume that the force due to gravity on the body is a constant $F = mg$, where m is the mass of the body and g is the acceleration due to gravity at sea level. For a body at a distance h above the surface of the Earth, a more accurate expression for the gravitational force on the body is

$$F(h) = \frac{mgR^2}{(R+h)^2},$$

where R is the radius of the Earth. We will consider the situation in which the body is not too far from the surface of the Earth, so that h is much smaller than R ($h << R$).

(a) Express F as mg multiplied by a series in $x = h/R$.

(b) Show that F reduces to the Newtonian form $F = mg$ when $x << 1$.

(c) The first-order correction to the approximation $F = mg$ is obtained by taking the linear term in the series, but no higher terms. How far can you go above the Earth's surface before the first-order correction changes the approximation $F = mg$ by more than 10%? Take $R = 6400\,\text{km}$.

Answer: 320 km.

14. *Special Relativity*

In Einstein's Theory of Special Relativity, the mass of an object moving with velocity v is

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where $m_0 = m(0)$ is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy at velocity v and its energy at rest:

$$K(v) = m(v)c^2 - m_0c^2.$$

(a) Use a Taylor series for K to investigate the behaviour of the kinetic energy at non-relativistic velocities. To begin, find the first three non-zero terms in the Taylor series for K in terms of $x = (v/c)^2$ about $x = 0$.

(b) Using your calculator, draw the graph of the exact K/m_0c^2 against $x = (v/c)^2$, showing asymptotes.

(c) Show that when $v << c$, the expression for K agrees with classical Newtonian Physics: $K(v) = \frac{1}{2}m_0v^2$.

(d) The second-order correction to the approximation $K \approx \frac{1}{2}m_0v^2$ is obtained by taking the next term in the series, but no higher terms. Use it to estimate the change in an object's kinetic energy for $v = 3440\,\text{m/s}$ (Mach 10). Take $c = 3 \times 10^8\,\text{m/s}$.

Answer: $9.9 \times 10^{-9}\,\%$

15. *Physical Chemistry*

The potential energy V of two gas molecules separated by a distance r is given by

$$V(r) = -V_0 \left(2\left(\frac{r_0}{r}\right)^6 - \left(\frac{r_0}{r}\right)^{12} \right),$$

where V_0 and r_0 are positive constants.

(a) Show that the global minimum of V, $V = -V_0$, occurs when $r = r_0$.

(b) Sketch V/V_0 versus r/r_0 for $0.8 < r/r_0 < 1.2$.

(c) The force F between the molecules is given by $F = -dV/dr$. Derive the expression for the force and draw a graph of $F/\left(\frac{V_0}{r_0}\right)$ against r/r_0.

(d) We note that a positive F represents a repulsive force; a negative F represents an attractive force. Is F repulsive or attractive when $r = r_0$? What does this mean physically?

(e) *Qualitatively*, what happens if the molecules start with $r = r_0$ and are pulled slightly further apart?

(f) *Qualitatively*, what happens if the molecules start with $r = r_0$ and are pushed slightly closer together?

(g) The above expression for the force between the gas molecules is rather complicated and therefore not very useful. A Taylor series can help us develop a quantitative understanding of the nature of the force between the gas molecules when they are not too far from the equilibrium configuation. To do this, first write V as a Taylor series in $x = r/r_0$ centred at $x = 1$, retaining up to the quadratic term.

(h) Differentiate this result to find a series representation for the force F.

(i) By discarding all except the first non-zero term in this series, describe how the force between the atoms depends on the displacement from the equilibrium when that displacement is small.

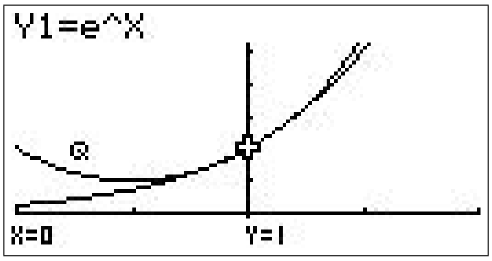
(j) Thus, for small displacements from the equilibrium, what sort of motion results?

12.7 Solutions

12.7.1 Solutions to the exercises

Exercise 1

(a) Use the general result for Taylor polynomials to find the quadratic approximation (graph Q in the figure below) to $f(x) = e^x$ near $x = 0$.



V-Window $[-2, 2, 1] \times [-0.5, 3, 1]$

The quadratic approximation to a function f near $x = 0$ is

$$P_2(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}\,x^2.$$

For $f(x) = e^x$, $f(0) = f'(0) = f''(0) = 1$, so that the quadratic approximation to $f(x) = e^x$ near $x = 0$ is

$$P_2(x) = 1 + x + \frac{1}{2}\,x^2.$$

(b) Use the approximation to estimate $e^{0.1}$. Compare with the calculator value and $P_1(0.1)$ (Example 1).

$e^{0.1} \approx P_2(0.1) = 1.105.$

$e^{0.1} \approx P_1(0.1) = 1.1.$

From the calculator, $e^{0.1} = 1.1052$ to 5 significant digits.

Exercise 2: Derive the general result for a Taylor polynomial of degree n centred at $x=0$.

Hint: Repeat and extend the process begun in Section 12.2.2, but starting with Eq. (5) for P_n instead of Eq. (1) for P_2.

Once you have established a pattern in the coefficients, you may write $\vdots$ and then the nth term.

We have from Eq. (5)

$$P_n(x) \;=\; a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_{n-1}x^{n-1} + a_nx^n. \tag{6}$$

From Eq. (6),

$$\begin{aligned} P_n(0) &= a_0 = f(0). \\ \therefore\; a_0 &= f(0). \end{aligned}$$

Differentiating Eq. (6),

$$\begin{aligned} P_n'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots. \\ \therefore\; P_n'(0) &= a_1 = f'(0). \\ \therefore\; a_1 &= f'(0). \\ P_n''(x) &= 2a_2 + 6a_3x + 12a_4x^2 + \cdots \\ \therefore\; P_n''(0) &= 2a_2 = f''(0). \\ \therefore\; a_2 &= \frac{f''(0)}{2} = \frac{f''(0)}{2!}. \\ P_n'''(x) &= 6a_3 + 24a_4x + \cdots \\ \therefore\; P_n'''(0) &= 6a_3 = f''(0). \\ \therefore\; a_3 &= \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}. \\ P_n^{(4)}(x) &= 24a_4 + \cdots \\ \therefore\; P_n^{(4)}(0) &= 24a_4 = f^{(4)}(0). \\ \therefore\; a_4 &= \frac{f^{(4)}(0)}{24} = \frac{f^{(4)}(0)}{4!}. \end{aligned}$$

The pattern is emerging: $a_n = \dfrac{f^{(n)}(0)}{n!}$.

Therefore,

$$P_n(x) \;=\; f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Exercise 3: Find the Taylor polynomials centred at $x = 0$ for $f(x) = \sin(x)$ up to degree 11 (look for a pattern in the first few coefficients).

The general formula for the 11th-degree Taylor polynomial centred at $x = 0$ for a function f is

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots + \frac{f^{(11)}(0)}{11!}\,x^{11}.$$

$f(x) = \sin(x) \qquad f(0) = 0.$

$f'(x) = \cos(x) \qquad f'(0) = 1.$

$f''(x) = -\sin(x) \qquad f''(0) = 0.$

$f'''(x) = -\cos(x) \qquad f'''(0) = -1.$

$f^{(4)}(x) = \sin(x) \qquad f^{(4)}(0) = 0.$

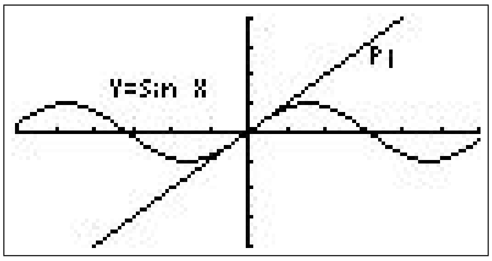
The pattern then repeats. Therefore, all the even powers of x have coefficient 0, all the odd powers coefficient ± 1.

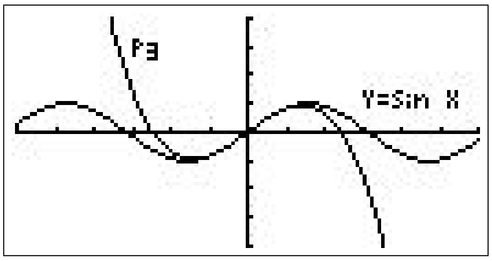
Therefore, the Taylor polynomial of degree 11 centred at $x = 0$ for $f(x) = \sin(x)$ is

$$P_{11}(x) \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}.$$

This contains all the other Taylor polynomials up to degree 11.

Plot P_1 and P_3 on the relevant graphs. The window for the graphs here is $[-6, 6, 1] \times [-4, 4, 1]$.





Use each polynomial to approximate sin(0.2), and compare with the calculator value. Underline the digits that are correct in each approximate value.

$P_0(x) = 0$ for all x. P_n for even n is the same as P_{n-1}.

From the calculator, $\sin(0.2) = 0.19866933080494$ to 14 significant digits.

$\sin(0.2) \approx P_1(0.2) = 0.2.$

$\sin(0.2) \approx P_3(0.2) = \underline{0.19866}\dot{6}.$

$\sin(0.2) \approx P_5(0.2) = \underline{0.19866933}\dot{3}.$

$\sin(0.2) \approx P_7(0.2) = \underline{0.1986693308}.$

$\sin(0.2) \approx P_9(0.2) = \underline{0.19866933080}635.$

$\sin(0.2) \approx P_{11}(0.2) = \underline{0.19866933080494}.$

Exercise 4: Find the fifth-degree Taylor polynomial for $f(x)=\dfrac{1}{x}$ centred at $x=3$.

$$P_5(x) = f(3)+f'(3)\,(x-3)+\frac{f''(3)}{2!}\,(x-3)^2+\frac{f'''(3)}{3!}\,(x-3)^3+\frac{f^{(4)}(3)}{4!}\,(x-3)^4-\frac{f^{(5)}(3)}{5!}\,(x-3)^5.$$

$$f(x)=\frac{1}{x} \qquad f(3)=\frac{1}{3}.$$

$$f'(x)=-\frac{1}{x^2} \qquad f'(3)=-\frac{1}{3^2}.$$

$$f''(x)=\frac{2}{x^3} \qquad f''(3)=\frac{2}{3^3}=\frac{2!}{3^3}.$$

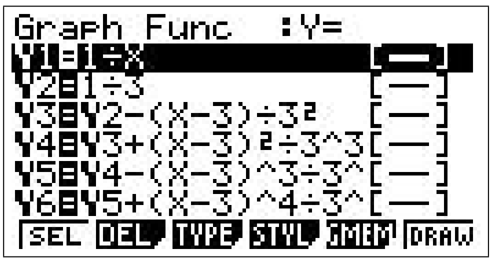
$$f'''(x)=-\frac{6}{x^4} \qquad f'''(3)=-\frac{6}{3^4}=-\frac{3!}{3^4}.$$

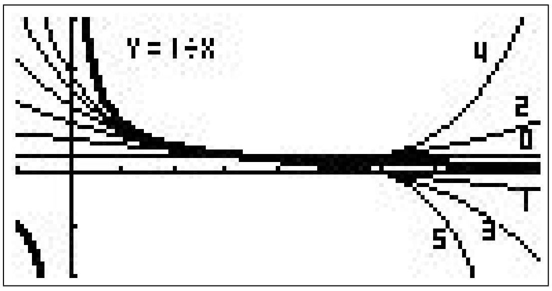
$$f^{(4)}(x)=\frac{24}{x^5} \qquad f^{(4)}(3)=\frac{24}{3^5}=\frac{4!}{3^5}.$$

$$f^{(5)}(x)=-\frac{120}{x^6} \qquad f^{(5)}(3)=-\frac{120}{3^6}=-\frac{5!}{3^6}.$$

$$\therefore\ P_5(x)=\frac{1}{3}-\frac{x-3}{3^2}+\frac{(x-3)^2}{3^3}-\frac{(x-3)^3}{3^4}+\frac{(x-3)^4}{3^5}-\frac{(x-3)^5}{3^6}.$$

Plot f and $P_1 - P_5$ on your calculator using a window $[-1,9,1]\times[-2,2,1]$.





Exercise 5: Find the Taylor series centred at $x=\pi/2$ for $f(x)=\cos(x)$.

$$f(x) = f(\pi/2) + f'(\pi/2)\,(x-\pi/2) + \frac{f''(\pi/2)}{2!}\,(x-\pi/2)^2 + \frac{f'''(\pi/2)}{3!}\,(x-\pi/2)^3 + \cdots .$$

$f(x) = \cos(x) \qquad f(\pi/2) = 0.$

$f'(x) = -\sin(x) \qquad f'(\pi/2) = -1.$

$f''(x) = -\cos(x) \qquad f''(\pi/2) = 0.$

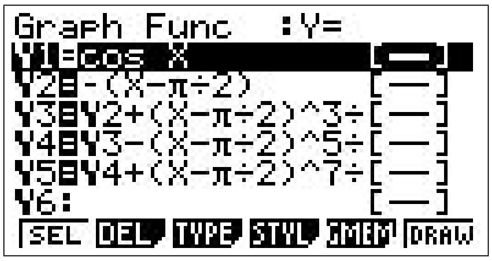
$f'''(x) = \sin(x) \qquad f'''(\pi/2) = 1.$

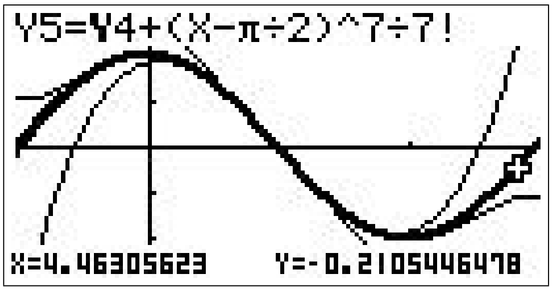
$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(\pi/2) = 0.$

The pattern then repeats. Therefore, all the even powers of $x-\pi/2$ have coefficient 0, all the odd powers coefficient ± 1.

Therefore, the Taylor series centred at $x=\pi/2$ for $f(x)=\cos(x)$ is

$$\begin{aligned}\cos(x) &= -(x-\pi/2) + \frac{(x-\pi/2)^3}{3!} - \frac{(x-\pi/2)^5}{5!} + \frac{(x-\pi/2)^7}{7!} - \cdots \\ &= \sum_{n=1}^{\infty}(-1)^n\frac{(x-\pi/2)^{2n-1}}{(2n-1)!}.\end{aligned}$$





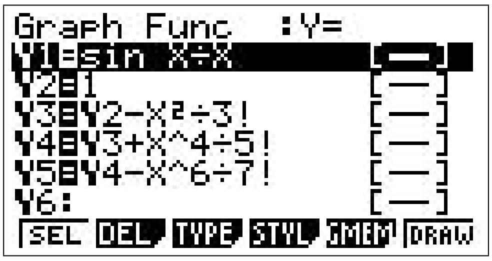
V-Window $[-\pi/2, 3\pi/2, \pi/2]\times[-1.4, 1.4, 0.5]$

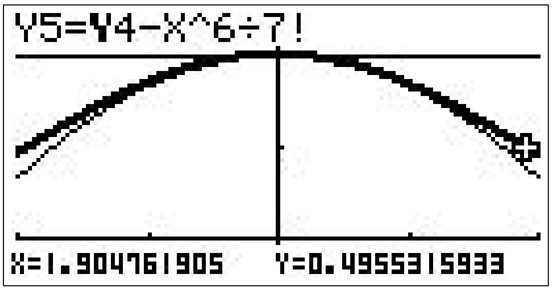
Exercise 6: Find the Taylor series for $f(t)=\dfrac{\sin(t)}{t}$ centred at $t=0$.

From the box on page 83,

$$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \cdots .$$

$$\therefore \quad \frac{\sin(t)}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots + (-1)^n \frac{t^{2n}}{(2n+1)!} + \cdots .$$





V-Window $[-2, 2, 1] \times [0, 1.2, 0.5]$

Exercise 7: Use the standard method to find the first three non-zero terms in the Taylor series centred at $x=0$ for $f(x)=e^{x^2}$.

$$P_4(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \frac{f^{(4)}(0)}{4!}\,x^4 + \dots .$$

$f(x) = e^{x^2} \qquad f(0) = 1.$

$f'(x) = 2xe^{x^2} \qquad f'(0) = 0.$

$f''(x) = 2e^{x^2} + 4x^2e^{x^2} \qquad f''(0) = 2.$

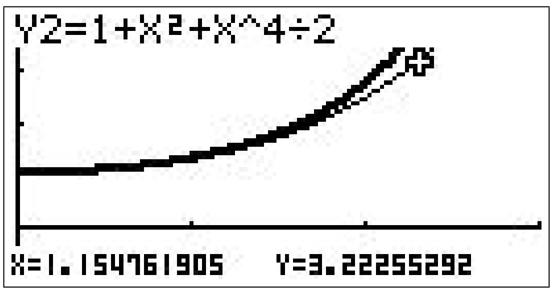
$f'''(x) = 8xe^{x^2} + (2{+}4x^2)\cdot 2xe^{x^2} \qquad f'''(0) = 0.$

$f^{(4)}(x) = (12{+}24x^2)e^{x^2} + (10x{+}8x^3)\cdot 2xe^{x^2} \qquad f^{(4)}(0) = 12.$

$$\therefore \quad e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \cdots .$$

Note that $f(x)=e^{x^2}$ and $P_4(x)$ are even functions: Taylor polynomials/series centred at $x=0$ of even (odd) functions only contain even (odd) powers of x.

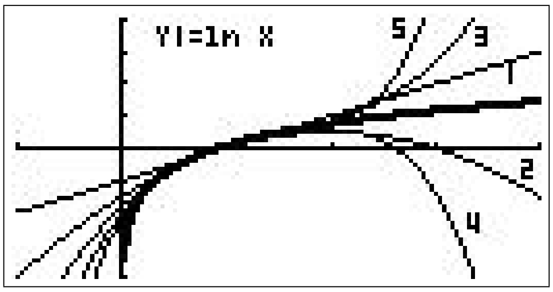
Plot together with f.



V-Window $[0, 1.5, 0.5] \times [-1, 4, 1]$

Exercise 8: From the Taylor polynomials for $f(x) = \ln(x)$ centred at $x = 1$ in Example 3, write down the Taylor series and determine its interval of convergence by graphing the function and its first few Taylor polynomials.

$$\begin{aligned}\ln(x) &= x-1-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\frac{(x-1)^5}{5}+\cdots\\ &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}.\end{aligned}$$



V-Window $[-1, 4, 1] \times [-4, 4, 1]$

The interval of convergence, from the graph, is $0 < x < 2$.

Check by evaluating $P_n(1.2)$ and $P_n(2.2)$ for the first few values of n and comparing them with the values of $\ln(1.2)$ and $\ln(2.2)$.

$\ln(1.2) = 0.1823216$ and $\ln(2.2) = 0.7884574$

n	$P_n(1.2)$	$P_n(2.2)$
1	0.2	1.2
2	0.18	0.48
3	$0.182\dot{6}$	1.056
4	$0.1822\dot{6}$	0.5376
5	$0.182330\dot{6}$	1.035264
	converges	*diverges*

Exercise 9: Find the Taylor series centred at $x=0$ for $f(x)=\dfrac{1}{1-x}$ and determine its interval of convergence.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots .$$

$f(x) = \dfrac{1}{1-x}$ $\quad f(0) = 1.$

$f'(x) = \dfrac{1}{(1-x)^2}$ $\quad f'(0) = 1.$

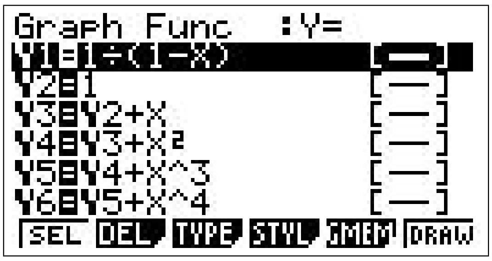
$f''(x) = \dfrac{2}{(1-x)^3}$ $\quad f''(0) = 2 = 2!.$

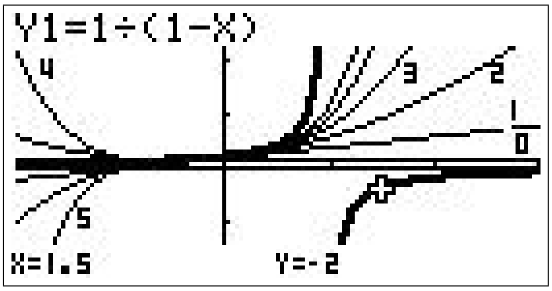
$f'''(x) = \dfrac{6}{(1-x)^4}$ $\quad f'''(0) = 6 = 3!.$

$f^{(4)}(x) = \dfrac{24}{(1-x)^5}$ $\quad f^{(4)}(0) = 24 = 4!.$

$f^{(5)}(x) = \dfrac{120}{(1-x)^6}$ $\quad f^{(5)}(0) = 120 = 5!.$

$$\therefore\ f(x) \;=\; 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \;=\; \sum_{n=0}^{\infty} x^n.$$





V-Window $[-2,3,1]\times[-10,14,5]$

$f(x) = 1/(1-x)$ has an infinite discontinuity at $x = 1$. The graphs of the first few Taylor polynomials show that the fit to f is good for $-1 < x < 1$, and improves as the degree of the polynomial increases. However, the polynomials clearly diverge from f outside that interval, particularly obvious for the second branch of f at $x > 1$. This is the interval of convergence of the Taylor series centred at $x=0$ for $f(x)=1/(1-x)$.

Exercise 10: Find the Taylor series for the function

$$f(x) \;=\; \begin{cases} \cos(x) & |x| \leqslant \pi \\ -1 & |x| > \pi \end{cases}$$

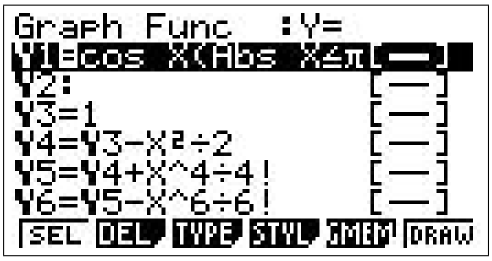
centred at $x=0$.

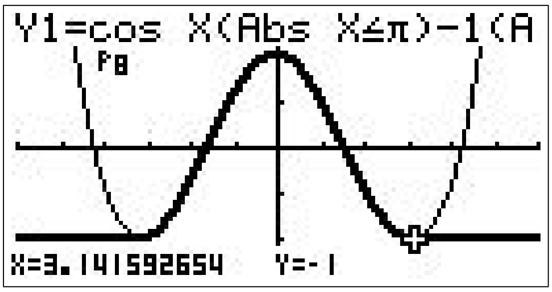
The Taylor series for f is the same as the Taylor series for $\cos(x)$, because the series is centred on $x=0$, at which $f(x)=\cos(x)$.

$$\cos(x) \;=\; 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}\;+\;\cdots\;=\;\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}.$$

What is its interval of convergence?

The interval of convergence for the function f here is $-\pi\leqslant x\leqslant\pi$, whereas the series for $\cos(x)$ converges for all x.





V-Window $[-6,6,1]\times[-1.4,1.4,0.5]$

Exercise 11

(a) Find the Taylor series and its interval of convergence for $\ln(1+x)$ about $x = 0$.

$f(x) = \ln(1+x) \qquad f(0) = 0$

$f'(x) = \dfrac{1}{1+x} \qquad f(0) = 1$

$f''(x) = -\dfrac{1}{(1+x)^2} \qquad f''(0) = -1$

$f'''(x) = \dfrac{2}{(1+x)^3} \qquad f''(0) = 2 = 2!$

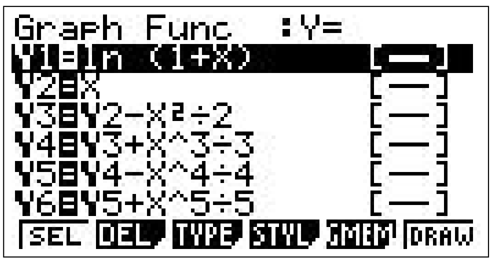
$f^{(4)}(x) = -\dfrac{6}{(1+x)^4} \qquad f'''(0) = -6 = -3!$

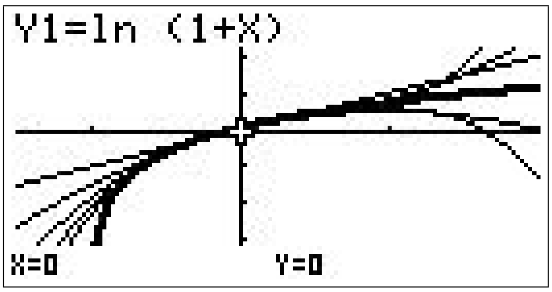
$f^{(5)}(x) = \dfrac{24}{(1+x)^5} \qquad f^{(4)}(0) = 24 = 4!$

Therefore,

$$\begin{aligned}
\ln(1+x) &= f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \frac{f^{(4)}(0)}{4!}\,x^4 + \frac{f^{(5)}(0)}{5!}\,x^5 + \ldots \\
&= x - \frac{1}{2!}\,x^2 + \frac{2!}{3!}\,x^3 - \frac{3!}{4!}\,x^4 + \frac{4!}{5!}\,x^5 + \ldots \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots \\
&= \sum_{n=1}^{\infty} (-1)^{n-1}\,\frac{x^n}{n}.
\end{aligned}$$

The graphs of $f(x) = \ln(1+x)$ and $P_1 - P_5$ are plotted in the figure below.





V-Window $[-1.5, 2, 1] \times [-4, 3, 1]$

The Taylor series for $\ln(1+x)$ about $x = 0$ is therefore $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\,\frac{x^n}{n}$.

From the graph, the interval of convergence is $-1 < x < 1$.

(b) The Taylor series for $\ln(x)$ about $x=1$ has an interval of convergence of $0<x<2$ (see Example 3). The Taylor series for $f(x)=\ln(1+x)$ about $x=0$ in (a) has an interval of convergence of $-1<x<1$. Neither is therefore suitable for finding the natural log of all x in the domain of $\ln(x)$, $x>0$.

Find the Taylor series for the function

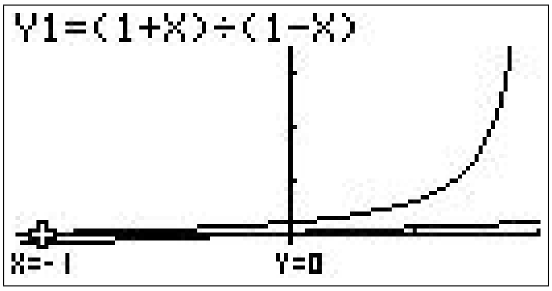
$$f(x) \;=\; \ln\left(\frac{1+x}{1-x}\right)$$

and its interval of convergence. *Hint:* use a log law and previous results.

From (a),

$$\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots \qquad -1<x<1 \\
&= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \qquad -1<x<1. \\
\therefore\ \ln(1-x) &= (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} + \cdots \\
&= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots \qquad -1<x<1 \\
&= -\sum_{n=1}^{\infty}\frac{x^n}{n} \qquad -1<x<1. \\
\therefore\ \ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \qquad \text{log law} \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots \qquad -1<x<1 \\
&\quad - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots\right) \qquad -1<x<1 \\
&= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots \qquad -1<x<1 \\
&= 2\sum_{n=1,3,5,\ldots}^{\infty}\frac{x^n}{n} \qquad -1<x<1 \\
&= 2\sum_{k=1}^{\infty}\frac{x^{2k-1}}{2k-1} \qquad -1<x<1 \qquad (n=2k-1).
\end{aligned}$$

(c) As x ranges across the interval of convergence you found in (a), what range of values does $g(x)=\dfrac{1+x}{1-x}$ take?



V-Window $[-1.1,1,0.5]\times[-4,20,5]$

The graph $y=\dfrac{1+x}{1-x}$ above shows that for $-1<x<1$, $0<g(x)=\dfrac{1+x}{1-x}<\infty$, that is g takes all values greater than 0.

Hence, given some number $a>0$, we can find an x, $-1<x<1$, such that $g(x)=a$. Then, using the series above we can find $\ln(a)$ for all $a>0$.

(d) Hence write out an algorithm to find the natural log of any number $x>0$.

If $\dfrac{1+x}{1-x}=a$, then $x=\dfrac{a-1}{a+1}$. Hence, from our series above,

$$\ln(a) \;=\; 2\sum_{k=1}^{\infty}\frac{\left(\dfrac{a-1}{a+1}\right)^{2k-1}}{2k-1} \qquad a>0.$$

Replacing a by a general variable x in the series above, we have our final result, a series for the natural log of all positive numbers,

$$\ln(x) \;=\; 2\sum_{k=1}^{\infty}\frac{\left(\dfrac{x-1}{x+1}\right)^{2k-1}}{2k-1} \qquad x>0.$$

12.7.2 Solutions to the problems

1. Find the Taylor polynomials of degree $n=2,3,4$, centred at $x=0$, for $f(x)=\sqrt{1+x}$.

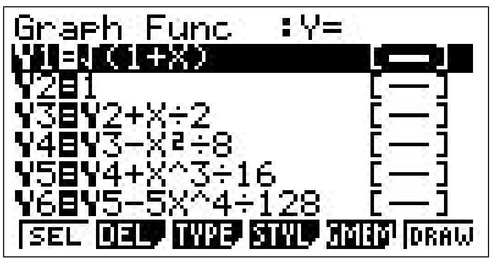
$$f(x) = \sqrt{1+x} = (1+x)^{1/2} \qquad f(0) = 1.$$
$$f'(x) = \frac{1}{2(1+x)^{1/2}} \qquad f'(0) = \frac{1}{2}.$$
$$f''(x) = -\frac{1}{4(1+x)^{3/2}} \qquad f''(0) = -\frac{1}{4}.$$
$$f'''(x) = \frac{3}{8(1+x)^{5/2}} \qquad f'''(0) = \frac{3}{8}.$$
$$f^{(4)}(x) = -\frac{15}{16(1+x)^{7/2}} \qquad f^{(4)}(0) = -\frac{15}{16}.$$

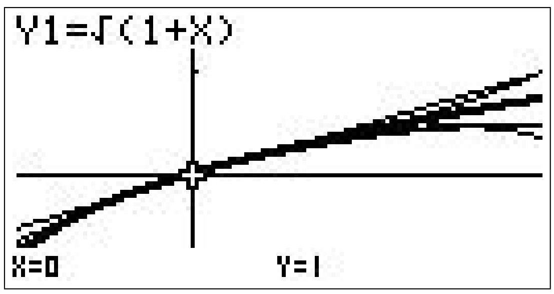
Therefore, using the general formula,

$$\begin{aligned} P_2(x) &= f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 \\ &= 1 + \frac{1}{2}\,x - \frac{1}{4}\,\frac{x^2}{2!} \\ &= 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2. \end{aligned}$$

$$\begin{aligned} P_3(x) &= P_2(x) + \frac{f'''(0)}{3!}\,x^3 \\ &= P_2(x) + \frac{3}{8}\,\frac{x^3}{3!} \\ &= 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3. \end{aligned}$$

$$\begin{aligned} P_4(x) &= P_3(x) + \frac{f^{(4)}(0)}{4!}\,x^4 \\ &= P_3(x) - \frac{15}{16}\,\frac{x^4}{4!} \\ &= 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3 - \frac{5}{128}\,x^4. \end{aligned}$$





V-Window $[-1,2,1]\times[0,2.5,1]$

2. Find the Taylor polynomials of degree $n=3,4$, centred at $x=0$, for $f(x) = \arctan(x) \equiv \tan^{-1}(x)$.

$$f(x) = \arctan(x) \qquad f(0) = 0.$$

$$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$$

$$f''(x) = -\frac{2x}{(1+x^2)^2} \qquad f''(0) = 0.$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3} \qquad f'''(0) = -2.$$

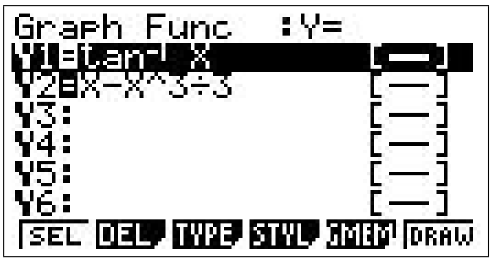
$$f^{(4)}(x) = \frac{8x}{(1+x^2)^3} + \frac{16x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4} \qquad f^{(4)}(0) = 0.$$

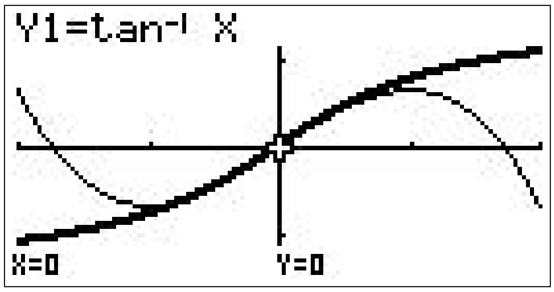
Therefore, using the general formula,

$$P_3(x) = P_4(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3$$

$$= x - 2\,\frac{x^3}{3!}$$

$$= x - \frac{1}{3}\,x^3.$$





V-Window $[-2, 2, 1] \times [-1.5, 1.5, 1]$

3. Find the Taylor polynomials of degree $n = 2, 3, 4$, centred at $x = 0$, for $f(x) = \dfrac{1}{\sqrt{1+x}}$.

$$f(x) = \frac{1}{\sqrt{1+x}} \equiv \frac{1}{(1+x)^{1/2}} \equiv (1+x)^{-1/2} \qquad f(0) = 1.$$

$$f'(x) = -\frac{1}{2(1+x)^{3/2}} \qquad f'(0) = -\frac{1}{2}.$$

$$f''(x) = \frac{3}{4(1+x)^{5/2}} \qquad f''(0) = \frac{3}{4}.$$

$$f'''(x) = -\frac{15}{8(1+x)^{7/2}} \qquad f'''(0) = -\frac{15}{8}.$$

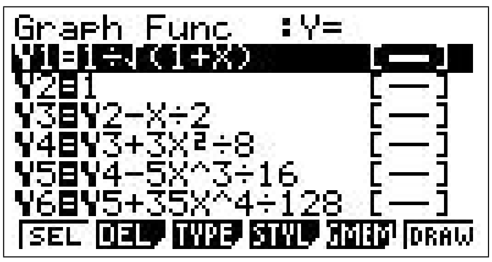
$$f^{(4)}(x) = \frac{105}{16(1+x)^{9/2}} \qquad f^{(4)}(0) = \frac{105}{16}.$$

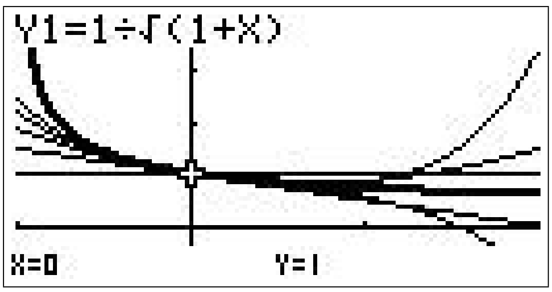
Therefore, using the general formula,

$$\begin{aligned} P_2(x) &= f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 \\ &= 1 - \frac{1}{2}\,x + \frac{3}{4 \times 2!}\,x^2 \\ &= 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2. \end{aligned}$$

$$\begin{aligned} P_3(x) &= P_2(x) + \frac{f'''(0)}{3!}\,x^3 \\ &= P_2(x) - \frac{15}{8 \times 3!}\,x^3 \\ &= 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2 - \frac{5}{16}\,x^3. \end{aligned}$$

$$\begin{aligned} P_4(x) &= P_3(x) + \frac{f^{(4)}(0)}{4!}\,x^4 \\ &= P_3(x) + \frac{105}{16 \times 4!}\,x^4 \\ &= 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2 - \frac{5}{16}\,x^3 + \frac{35}{128}\,x^4. \end{aligned}$$





V-Window $[-1, 2, 1] \times [-1, 4, 1]$

4. Suppose a function $f(x)$ is approximated by the sixth-degree Taylor polynomial centred at $x=0$

$$P_6(x) \;=\; 3x \;-\; 4x^3 \;+\; 5x^6. \qquad (7)$$

Give the value of **(a)** $f(0)$; **(b)** $f'(0)$; **(c)** $f'''(0)$; **(d)** $f^{(5)}(0)$; **(e)** $f^{(6)}(0)$.

The general formula for P_6 is

$$P_6(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \frac{f^{(4)}(0)}{4!}\,x^4 + \frac{f^{(5)}(0)}{5!}\,x^5 + \frac{f^{(6)}(0)}{6!}\,x^6. \quad (8)$$

(a) Setting $x=0$ in Eqs. (7) and (8) gives $P_6(0)=f(0)=0$.

(b) Differentiating Eq. (7) and setting $x=0$ gives $P_6'(0)=3$.
Differentiating Eq. (8) and setting $x=0$ gives $P_6'(0)=f'(0)$.
Therefore, $f'(0)=3$.

(c) Differentiating Eq. (7) twice more and setting $x=0$ gives $P_6'''(0)=-24$.
Differentiating Eq. (8) twice more and setting $x=0$ gives
$P_6'''(0)=f'''(0)$.
Therefore, $f'''(0)=-24$.

Continuing this process, we have

(d) $f^{(5)}(0)=P_6^{(5)}(0)=0$, and

(e) $f^{(6)}(0)=P_6^{(5)}(0)=3600$.

5. Suppose g is a function which has continuous derivatives, and that $g(5)=3$, $g'(5)=-2$, $g''(5)=1$, $g'''(5)=-3$.

(a) What is the Taylor polynomial of degree 2 centred at $x=5$ for g?
What is the Taylor polynomial of degree 3 centred at $x=5$ for g?

$$\begin{aligned} P_2(x) &= g(5) \;+\; g'(5)(x-5) \;+\; \frac{g''(5)}{2!}(x-5)^2 \\ &= 3 \;-\; 2(x-5) \;+\; \frac{1}{2}(x-5)^2. \end{aligned}$$

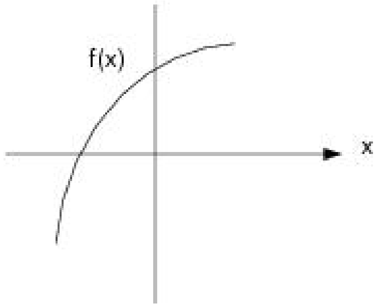
$$\begin{aligned} P_3(x) &= P_2(x) \;+\; \frac{g'''(5)}{3!}(x-5)^3 \\ &= 3 \;-\; 2(x-5) \;+\; \frac{1}{2}(x-5)^2 \;-\; \frac{1}{2}(x-5)^3. \end{aligned}$$

(b) Use the two polynomials that you found in (a) to approximate $g(4.9)$.

$g(4.9) \approx P_2(4.9) = 3.205$.
$g(4.9) \approx P_3(4.9) = 3.2055$.

6. Suppose $P_2(x) = a + bx + x^2$ is the second-degree Taylor polynomial centred at $x=0$ for a function f.

What can you say about the signs of a, b and c if f has the graph given below?



We know $f(0)=P_2(0)=a$. From the graph, $f(0)>0$, so that a is positive.

Similarly, $f'(0) = P_2'(0) = b$. The slope of the graph at $x = 0$ is positive, so that b is positive.

$f''(0)=P_2''(0)=2c$. The graph of f is concave down at $x=0$, so that f'' is negative and therefore c is negative.

7. (a) Find the Taylor polynomial of degree 3 centred at $x=0$ for

$$\sinh^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1+t^2}}.$$

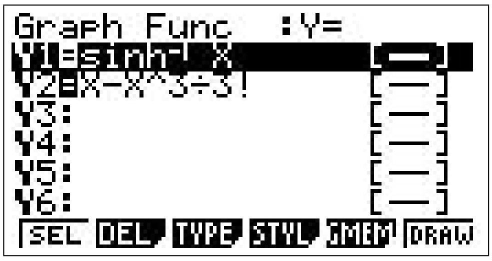
$$f(x) = \int_0^x \frac{dt}{\sqrt{1+t^2}} \qquad f(0) = 0.$$

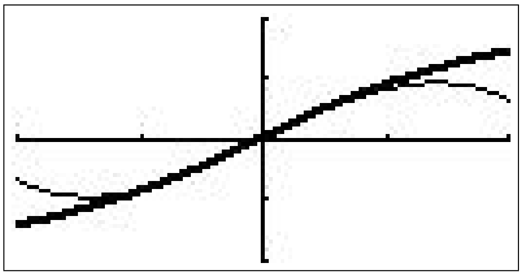
$$f'(x) = \frac{1}{\sqrt{1+x^2}} \qquad f'(0) = 1 \quad \text{Second Fundamental Theorem of Calculus.}$$

$$f''(x) = -\frac{x}{(1+x^2)^{3/2}} \qquad f''(0) = 0.$$

$$f'''(x) = \frac{2x^2 - 1}{(1+x^2)^{5/2}} \qquad f'''(0) = -1.$$

$$\begin{aligned} \therefore\ P_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= 0 + x + 0 - \frac{x^3}{3!} \\ &= x - \frac{x^3}{3!}. \end{aligned}$$





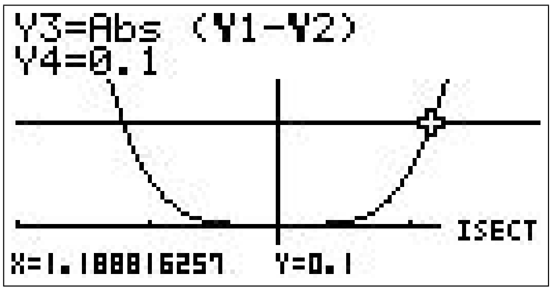
V-Window $[-2,2,1]\times[-2,2,1]$

(b) Hence estimate $\sinh^{-1}(0.25)$. Check your estimate using the $\sinh^{-1}$ function in your calculator and/or using numerical integration. How accurate is the estimate?

$P_3(0.25)=0.247396$ (6 SD).
$\sinh^{-1}(0.25)=0.247466$ (6 SD), using either the calculator $\sinh^{-1}$ function or numerical integration.
The error in P_3 is 7×10^{-5}.

(c) For what values of x is the approximation accurate to within 0.1?

Plot the graph of $\left|\sinh^{-1}(x)-P_3(x)\right|$ and use *intersect* to find where this function is equal to 0.1.



When $-1.189<x<1.189$ (4 SD), the error in P_3 is less than 0.1.

8. (a) Integrate the series for $\sin(t)/t$ (Exercise 5) term by term, including the general term, to find the Taylor series centred at $x=0$ for the sine-integral function

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt.$$

$$\begin{aligned}
\mathrm{Si}(x) &= \int_0^x \frac{\sin(t)}{t}\,dt \\
&= \int_0^x \left(1-\frac{t^2}{3!}+\frac{t^4}{5!}-\frac{t^6}{7!}+\cdots+(-1)^n\frac{t^{2n}}{(2n+1)!}+\ldots\right)dt \\
&= x-\frac{x^3}{3\cdot3!}+\frac{x^5}{5\cdot5!}-\frac{x^7}{7\cdot7!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)\cdot(2n+1)!}+\cdots.
\end{aligned}$$

(b) How many terms of the series do you need to take to give an estimate for Si(0.25) accurate to 8 significant digits?

Evaluating the successive Taylor-polynomial approximations, $P_n(0.25)$, to Si(0.25), we obtain

n	$P_n(0.25)$
1	0.25000000
3	0.24913194
5	0.24913357
7	0.24913357

Therefore, because $P_5(0.25) = P_7(0.25)$ to 8 significant digits, $P_5(0.25) = 0.24913357$ is accurate to 8 significant digits.

Check your value using numerical integration.

The numerical integrator on a 9860 gives $\mathrm{Si}(0.25) = 0.2491335703$ (all displayed digits), confirming the answer above.

9. The Taylor series centred at $x = 0$ for $f(x) = x^2 e^{x^2}$ is

$$x^2 + x^4 + \frac{x^6}{2!} + \frac{x^8}{3!} + \frac{x^{10}}{4!} + \cdots .$$

Use the Taylor series to find $\left.\frac{d}{dx}\left(x^2 e^{x^2}\right)\right|_{x=0}$.

Differentiating the series term by term, we have

$$\frac{d}{dx}\left(x^2 e^{x^2}\right) = 2x + 4x^3 + \cdots ,$$

giving

$$\left.\frac{d}{dx}\left(x^2 e^{x^2}\right)\right|_{x=0} = 0.$$

Find $\left.\frac{d^6}{dx^6}\left(x^2 e^{x^2}\right)\right|_{x=0}$.

Differentiating the series term by term to find the sixth derivative, we have

$$\frac{d^6}{dx^6}\left(x^2 e^{x^2}\right) = \frac{6!}{2!} + \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{3!} x^2 + \cdots ,$$

giving

$$\left.\frac{d^6}{dx^6}\left(x^2 e^{x^2}\right)\right|_{x=0} = \frac{6!}{2!} = 360.$$

10. Show how you can use the Taylor approximation for $\sin(x)/x$ centred at $x = 0$ (Exercise 6) to explain why

$$\lim_{x\to 0} \frac{\sin(x)}{x} \;=\; 1.$$

From Exercise 6, for x near 0,

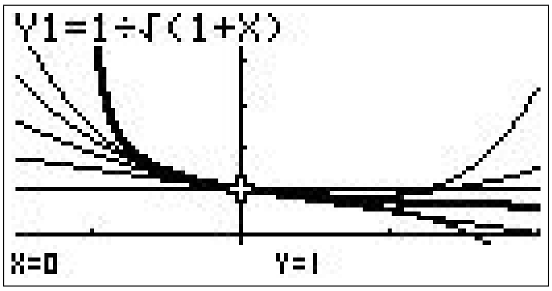
$$\frac{\sin(x)}{x} \;\approx\; 1 - \frac{x^3}{3!}.$$

Letting $x \to 0$, we obtain the result.

11. By graphing f and several of its Taylor polynomials, estimate the interval of convergence of the Taylor series centred at $x = 0$ for $f(x) = \dfrac{1}{\sqrt{1+x}}$.

From Question 3,

$$P_4(x) \;=\; 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4.$$



V-Window $[-1.5, 2, 1] \times [-1, 5, 1]$

The divergence of the polynomials for x greater than about 1 is reasonably obvious, but it is not so clear at $x = -1$. However, the interval of convergence is always symmetrical about the point at which the polynomials are centred ($x = 0$ here).

We could also argue in this case that, as f is undefined for $x \leqslant -1$, any polynomial (all of which are defined for $x \leqslant -1$) cannot converge to f for $x \leqslant -1$.

The interval of convergence is $-1 < x < 1$.

12. By looking at the Taylor series about $x = 0$ for each function, decide which of the functions

$$f(\theta) = 1 + \sin(\theta), \qquad g(\theta) = e^{\theta}, \qquad h(\theta) = \frac{1}{\sqrt{1-2\theta}}$$

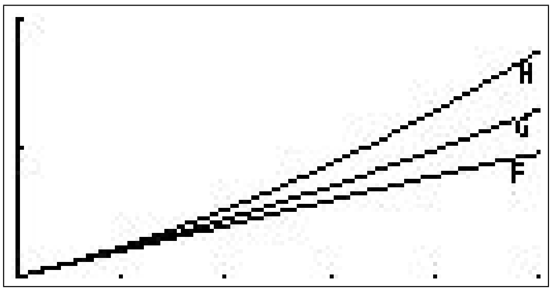
is largest and which is smallest for small positive θ.

Using the given series for $\sin(\theta)$ and e^{θ}, and the standard Taylor-series method for $\dfrac{1}{\sqrt{1-2\theta}}$:

$$\begin{aligned}
f(\theta) &= 1 + \theta - \frac{\theta^3}{6} + \cdots . \\
g(\theta) &= 1 + \theta + \frac{\theta^2}{2} + \frac{\theta^3}{6} + \cdots \\
h(\theta) &= (1-2\theta)^{-1/2}. \\
&= 1 + \theta + \frac{3\theta^2}{2} + \cdots .
\end{aligned}$$

Comparing the third term in each series, we see that $f(\theta) < g(\theta) < h(\theta)$ for small positive θ.

Plot the functions using a suitable V-Window to check your answer.



V-Window $[0, 0.5, 0.1] \times [1, 2, 0.5]$

The plot verifies the algebraic result.

13. *Force due to gravity*

When a body is near the surface of the Earth, we usually assume that the force due to gravity on the body is a constant mg, where m is the mass of the body and g is the acceleration due to gravity at sea level. For a body at a distance h above the surface of the Earth, a more accurate expression for the gravitational force on the body is

$$F(h) \;=\; \frac{mgR^2}{(R{+}h)^2},$$

where R is the radius of the Earth. We will consider the situation in which the body is not too far from the surface of the Earth, so that h is much smaller than R ($h{<<}R$).

(a) Express F as mg multiplied by a series in $x{=}h/R$.

Let $x{=}h/R$. Then,

$$\begin{aligned} F(h) &= \frac{mgR^2}{(R{+}h)^2} \\ &= mg\,\frac{R^2}{(R{+}h)^2} \\ &= mg\,\frac{R^2}{R^2\left(1+\dfrac{h}{R}\right)^2} \\ &= mg\,\frac{1}{\left(1+\dfrac{h}{R}\right)^2}. \\ \therefore\;\; F(x) &= mg\,\frac{1}{(1{+}x)^2} \qquad x=\frac{h}{R} \\ &= mg\,(1{+}x)^{-2}. \end{aligned}$$

Find the Taylor series for $g(x){=}(1{+}x)^{-2}$ about $x{=}0$.

$g(x) = (1{+}x)^{-2} \qquad g(0) = 1.$

$g'(x) = -2(1{+}x)^{-3} \qquad g'(0) = -2.$

$g''(x) = 6(1{+}x)^{-4} \qquad g''(0) = 6.$

Then $(1{+}x)^{-2} = 1 - 2x + \dfrac{6x^2}{2!} + \cdots = 1 - 2x + 3x^2 + \ldots.$

Therefore,

$$F \;=\; mg\big(1 - 2x + 3x^2 + \cdots\big) \;=\; mg\left(1 - \frac{2h}{R} + 3\left(\frac{h}{R}\right)^2 + \cdots\right).$$

(b) Show that F reduces to the Newtonian form $F = mg$ when $x << 1$.

When $x = h/R << 1$, we can neglect all but the first term of the series above. This gives directly $F = mg$.

(c) The first-order correction to the approximation $F = mg$ is obtained by taking the linear term in the series, but no higher terms. How far can you go above the Earth's surface before the first-order correction changes the approximation $F = mg$ by more than 10%? Take $R = 6400$ km.

The ratio of the first-order term to the zeroth-order term is $\dfrac{2h/R}{1} = \dfrac{2h}{R}$.

For this correction to be 10%, we require $\dfrac{2h}{R} = 0.1$, giving $h = 0.05R = 320$.

Therefore, we can go up to 320 km above the Earth's surface before the first-order correction is 10%.

PTO

14. *Special Relativity*

In Einstein's Theory of Special Relativity, the mass of an object moving with velocity v is

$$m(v) \;=\; \frac{m_0}{\sqrt{1-v^2/c^2}},$$

where $m_0 = m(0)$ is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy at velocity v and its energy at rest:

$$K(v) \;=\; m(v)c^2 \;-\; m_0c^2.$$

(a) Use a Taylor series for K to investigate the behaviour of the kinetic energy at non-relativistic velocities. To begin, find the first three non-zero terms in the Taylor series for K in terms of $x = (v/c)^2$ about $x = 0$.

$$\begin{aligned} K(m) &= mc^2 \;-\; m_0c^2 \\ &= \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 \\ &= m_0c^2\left(\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right). \\ \therefore\; K(x) &= m_0c^2\big((1-x)^{-1/2} - 1\big) \qquad x = (v/c)^2. \end{aligned}$$

Find the Taylor series for $g(x) = (1-x)^{-1/2} - 1$ about $x = 0$.

$g(x) = (1-x)^{-1/2} - 1 \qquad g(0) = 0.$

$g'(x) = \dfrac{(1-x)^{-3/2}}{2} \qquad g'(0) = \dfrac{1}{2}.$

$g''(x) = \dfrac{3(1-x)^{-5/2}}{4} \qquad g''(0) = \dfrac{3}{4}.$

$g'''(x) = \dfrac{15(1-x)^{-7/2}}{8} \qquad g'''(0) = \dfrac{15}{8}.$

Therefore,

$$g(x) \;=\; \frac{1}{2}x \;+\; \frac{3}{4}\frac{x^2}{2!} \;+\; \frac{15}{8}\frac{x^3}{3!} \;+\cdots \;=\; \frac{1}{2}x \;+\; \frac{3}{8}x^2 \;+\; \frac{5}{16}x^3 \;+\; \cdots.$$

Therefore,

$$K(x) \;=\; m_0c^2\left(\frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots\right).$$

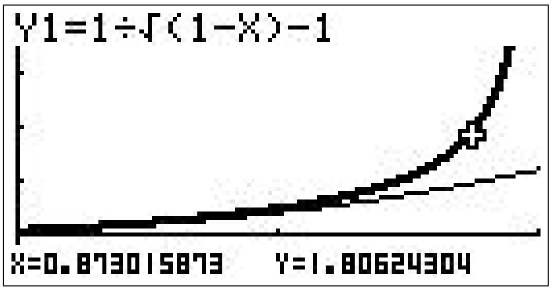
$$\therefore\; K(v) \;=\; m_0c^2\left(\frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \frac{5}{16}\left(\frac{v}{c}\right)^6 + \ldots\right).$$

(b) Using your calculator, draw the graph of the exact K/m_0c^2 against $x = (v/c)^2$, showing asymptotes.

$$K(v) = mc^2 - m_0c^2.$$

$$\therefore \frac{K}{m_0c^2} = \frac{m}{m_0} - 1 = \frac{1}{\sqrt{1-v^2/c^2}} - 1 = \frac{1}{\sqrt{1-x}} - 1.$$

Graphing $K(x)/m_0c^2$ against x, we get



V-Window $[0, 1, 0.5] \times [-0.8, 4, 1]$

This function has a vertical asymptote at $x=1$.
Also shown is the Taylor-series approximation up to the term in $x^3 = (v/c)^6$.

(c) Show that when $v << c$, the expression for K agrees with classical Newtonian Physics: $K(v) = \frac{1}{2}m_0v^2$.

If $v<<c$, we can take just the first term in the series, giving

$$K(v) = m_0c^2 \cdot \frac{1}{2}\left(\frac{v}{c}\right)^2 = \frac{1}{2}m_0v^2.$$

(d) The second-order correction to the approximation $K \approx \frac{1}{2}m_0v^2$ is obtained by taking the next term in the series, but no higher terms. Use it to estimate the change in an object's kinetic energy for $v=3440$ m/s (Mach 10). Take $c=3\times10^8$ m/s.

$$K \approx m_0c^2\left(\frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4\right) = \frac{1}{2}m_0v^2\left(1 + \frac{3}{4}\left(\frac{v}{c}\right)^2\right).$$

When $v=3440$, the second-order term, $\frac{3}{4}\left(\frac{v}{c}\right)^2 \approx 9.9\times10^{-11}$.

In percentage terms, the (relativistic) correction for Mach 10, due to the second-order term, is 9.9×10^{-9} %, a very small number.

15. *Physical Chemistry*

The potential energy V of two gas molecules separated by a distance r is given by

$$V(r) \;=\; -V_0\left(2\left(\frac{r_0}{r}\right)^6 - \left(\frac{r_0}{r}\right)^{12}\right),$$

where V_0 and r_0 are positive constants.

(a) Show that the global minimum of V, $V=-V_0$, occurs when $r=r_0$.

$$V(r) \;=\; -V_0\left(2\left(\frac{r_0}{r}\right)^6 - \left(\frac{r_0}{r}\right)^{12}\right),$$

with domain $r>0$. The global minimum, if it exists, lies at a critical point.

$$\frac{dV}{dr} \;=\; -V_0\left(12\left(\frac{r_0}{r}\right)^5\cdot\frac{-1}{r^2} - 12\left(\frac{r_0}{r}\right)^{11}\cdot\frac{-1}{r^2}\right)$$

$$=\; \frac{12V_0r_0}{r^2}\cdot\left(\frac{r_0}{r}\right)^5\left(1-\left(\frac{r_0}{r}\right)^6\right).$$

Therefore, dV/dr is defined for all $r>0$ and is 0 when $r=r_0$.

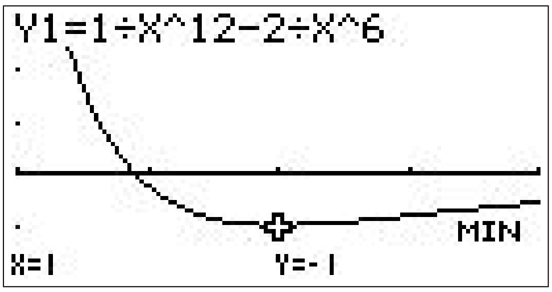
Critical point: $V(r_0) = -V_0 < 0$.
'Endpoint': As $r\to 0$, $V\to\infty$.
'Endpoint': As $r\to\infty$, $V\to 0$.

Therefore, the global minimum $V=-V_0$ occurs when $r=r_0$.

(b) Sketch V/V_0 versus r/r_0 for $0.8<r/r_0<1.2$.

$$\frac{V}{V_0} \;=\; \left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^6.$$

Set $Y_1 = 1/X^{12} - 2/X^6$, where $X = r/r_0$.

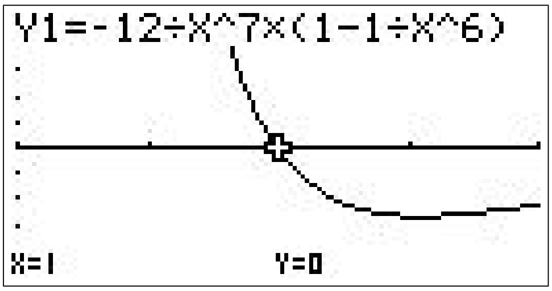


V-Window $[0.8, 1.2, 0.1]\times[-2, 3, 1]$

(c) The force, F, between the molecules is given by $F = -dV/dr$. Derive the expression for the force and draw a graph of $F/\left(\frac{V_0}{r_0}\right)$ against r/r_0.

$$\begin{aligned} F(r) &= -\frac{dV}{dr} \\ &= -\frac{12V_0 r_0}{r^2} \cdot \left(\frac{r_0}{r}\right)^5 \left(1 - \left(\frac{r_0}{r}\right)^6\right), \quad \text{using the result from (a).} \\ \therefore \quad \frac{F}{(V_0/r_0)} &= -12\left(\frac{r_0}{r}\right)^7 \left(1 - \left(\frac{r_0}{r}\right)^6\right). \end{aligned}$$

Set Y2 $= -12/\mathrm{X}\wedge 7\left(1-1/\mathrm{X}\wedge 6\right)$: $\mathrm{X} = r/r_0$.



V-Window $[0.8, 1.2, 0.1] \times [-5, 5, 1]$

(d) We note that a positive F represents a repulsive force; a negative F represents an attractive force. Is F repulsive or attractive when $r = r_0$? What does this mean physically?

At $r = r_0$, $F = 0$. The force is neither attractive nor repulsive: the molecules are in equilibrium.

(e) *Qualitatively*, what happens if the molecules start with $r = r_0$ and are pulled slightly further apart?

If the molecules are pulled apart, r becomes slightly greater than r_0 and so the force is negative (attractive): the molecules are pulled back together, towards equilibrium.

(f) *Qualitatively*, what happens if the molecules start with $r = r_0$ and are pushed slightly closer together?

If the molecules are pushed together, r becomes slightly less than r_0 and so the force is positive (repulsive): the molecules are forced apart, back towards equilibrium.

(g) The above expression for the force between the gas molecules is rather complicated and therefore not very useful. A Taylor series can help us develop a quantitative understanding of the nature of the force between the gas molecules when they are not too far from the equilibrium configuation. To do this, first write V as a Taylor series in $x=r/r_0$ centred at $x=1$, retaining up to the quadratic term.

Write $V=-V_0(2x^{-6}-x^{-12})$, where $x=r/r_0$.

Find the Taylor series about $x=1$ ($r=r_0$) for $g(x)=2x^{-6}-x^{-12}$.

$g(x)=2x^{-6}-x^{-12}$ $\quad g(1)=1.$

$g'(x)=-12x^{-7}+12x^{-13}$ $\quad g'(1)=0.$

$g''(x)=84x^{-8}-156x^{-14}$ $\quad g''(1)=-72.$

Therefore,

$$g(x) \;=\; 1 \;-\; 72\,\frac{(x-1)^2}{2} \;+\; \cdots \;=\; 1 \;-\; 36(x-1)^2 \;+\; \cdots .$$

Therefore,

$$V \;=\; -V_0\big(1-36(x-1)^2+\cdots\big) \;=\; -V_0\left(1-36\,\frac{(r-r_0)^2}{r_0^2}+\cdots\right).$$

(h) Differentiate this result to find a series representation for the force F.

$$F \;=\; -\frac{dV}{dr} \;=\; V_0\left(-\frac{72(r-r_0)}{r_0^2}+\cdots\right) \;=\; -\frac{72V_0(r-r_0)}{r_0^2} \;+\; \cdots .$$

(i) By discarding all except the first non-zero term in this series, describe how the force between the atoms depends on the displacement from the equilibrium when that displacement is small.

If $r>r_0$, F is negative, i.e. attractive. If $r<r_0$, F is positive, i.e. repulsive. This is what we found graphically in (e) and (f).

(j) Thus, for small displacements from the equilibrium, what sort of motion results?

The motion is always back towards the equilibrium position $r=r_0$, which is therefore a (locally) stable equilibrium.

13 Differential Equations Graphical and Numerical Methods

13.1 Introduction

The material here is based on my lecture notes for a first-year university Calculus course at UNSW Canberra. It goes beyond the 'how do I do this?' of some of the other material in the book in that it provides more context and modelling examples.

13.1.1 Why solve differential equations?

> Perhaps the most important of all the applications of Calculus is to differential equations. When physical scientists or engineers use Calculus, more often than not it is to analyze a differential equation that has arisen in the process of modelling some phenomenon that they are studying.
>
> Although it is often impossible to find an explicit formula for the solution of a differential equation, we will see that graphical and numerical approaches provide the needed information.
>
> James Stewart
> *Calculus: Concepts and Contexts*

13.1.2 Why graphics calculators?

Graphics calculators are a relatively cheap and easy-to-use way of implementing the *graphical and numerical approaches* referred to by James Stewart above. The programs for slope fields and Euler's Method are relatively simple and easy to relate to hand calculations. Both are also a good introduction to simple coding. Once you have drawn one or two slope fields or done several Euler calculations by hand, the advantages of a program become obvious. Any meaningful form of modelling without programs is impossible.

Euler's Method is a good introduction to numerical differential-equation solvers, their geometric basis and the important idea of accuracy. More-sophisticated methods are just elaborations of Euler's Method; an understanding of the basis of Euler's Method gives one intuition for the other methods and the general approach.

13.1.3 Modelling with differential equations

Scientists and engineers (and demographers, epidemiologists, economists, financial modellers, etc) use differential equations (DEs) in their modelling. That involves capturing the most important aspects of the system being modelled in a set of equations, solving the equations somehow and relating the findings back to the system. Here are some common models resulting in first-order DEs.

Population dynamics

In the absence of limiting factors, such as lack of food, biological populations tend to grow at a rate proportional to the total population, that is

$$\frac{dP(t)}{dt} = kP(t),$$

where $P(t)$ is the population at time t and k, the growth rate, is the constant of proportionality. Limiting factors, the presence of predators and other complexities can be added to the model to make it more realistic.

Other processes leading to a similar DE are radioactive decay, change of air pressure with height, compound interest, absorption of light and electrical circuits without capacitors.

Heating and cooling

Newton's Law of Cooling (and Heating)

The rate of change of the temperature T of a cooling body with time t is proportional to the difference between the temperature of the body and that of its surroundings, T_s. Mathematically, this is written as

$$\frac{dT(t)}{dt} = -k\big(T(t) - T_s\big),$$

where k is the proportionality constant.

Other processes leading to a similar DE are mixing of substances in tanks, concentration of drugs in the body and simple learning models.

Projectile motion

Newton's Second Law of Motion:

$$\text{mass} \times \text{acceleration} = \text{sum of forces acting on the body.}$$

Acceleration is the time rate of change of velocity $\frac{dv}{dt}$. The DE which describes a simple model of a body falling under gravity with air resistance (drag) is

$$m\frac{dv}{dt} = mg - kv^n,$$

where m is the mass and v the velocity of the body, k is the coefficient of air resistance and n is a constant that depends on the speed and the size of the body. mg is the gravitational force and kv^n the drag or air-resistance force.

13.2 First-order DEs

13.2.1 Solving DEs: A graphical method — slope fields

The slope field of a first-order DE is a grid of short line segments on the Cartesian (xy) plane, with the slope of the line segment at a particular point being the slope of the (unique) solution to the DE that passes through that point; this slope is given by the DE. Slope fields are useful in showing the global behaviour of solutions to a DE and require no knowledge of methods of solving DEs.

Before the ready availability of graphics, slope fields had to be drawn by hand, an easy but time-consuming process. The use of isoclines, lines of equal slope, helped a little but no-one was going to draw many slope fields. With the use of a relatively simple program, graphics calculators can draw slope fields quickly and accurately. By changing the calculator V-Window, you can examine particular regions of the solutions in greater detail or 'zoom out' for the big picture.

The calculator program SLPFIELD draws a slope field for a DE in the form

$$\frac{d\mathrm{Y}}{d\mathrm{X}} \;=\; f(\mathrm{X},\mathrm{Y}).$$

Use: Press [MENU] [5] and enter the function $f(\mathrm{X},\mathrm{Y})$, the RHS of the DE, into Y1. This function will not be graphed (and cannot if it contains Y) but is evaluated by the program at each grid point to find the derivative/slope there.
Choose a suitable [V-Window], press [MENU] [B], select and run the SLPFIELD program.

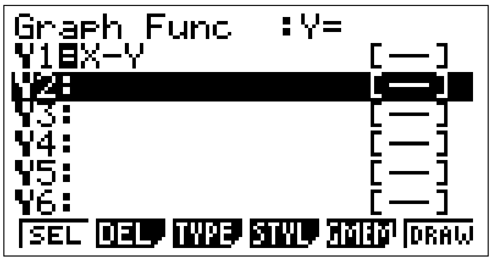
Example: Plot the slope field for the differential equation

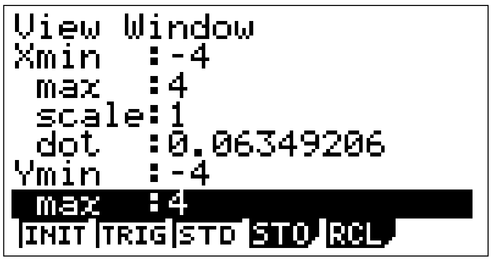
$$\frac{dy}{dx} = x - y$$

over the region $-4 \leqslant x \leqslant 4$, $-4 \leqslant y \leqslant 4$.

Sketch the curve of the solution of the DE that passes through the point $(0,0)$, that through $(1,0)$ and that through $(2,0)$.

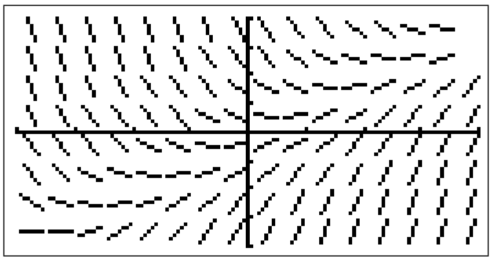
Set Y1 = X−Y and a [V-Window] of $[-4,4,1]\times[-4,4,1]$.

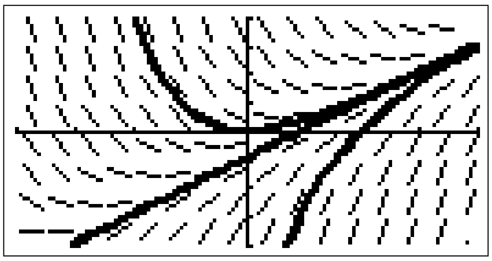




PTO

Run the program to give the slope field below left. This shows three different types of solutions: concave-up solutions in most of three of the four quadrants; concave-down solutions in most of the fourth quadrant; and what looks like a straight-line solution running from bottom left to top right and separating the other two types.





Following the slope lines, draw the solution curves that pass through $(0,0)$, through $(1,0)$ and through $(2,0)$ (above right).

One way to do this is generate the slope field on your calculator, download the screenshot to your computer using FA-124 or ScreenReceiver (see Section 9.6) and print it out so you can draw in the solutions.[14]

You can plot solutions on the slope field using one of the DE solvers in the next section (as in the figure above right); just delete the *ClrGraph*[15] command near the start of the relevant program. If you also change *S–L–Normal* to *S–L–Thick* (SET UP F6 S/L F2), the solution curve will be easier to see on the slope field.

To plot the exact (algebraic) solution on the slope field, save the slope field as a PICT[16]and set this as the background in SET UP. Plot the exact solution as usual.

Exercises

Solutions in Section 13.4.

1. Sketch a slope field for the differential equation $\dfrac{dy}{dx} = y - 2x$.

Then use it to sketch the solution curve that passes through the point $(1,0)$.

Use a V-Window $[-3,3,1]\times[-2,2,1]$.

2. Sketch a slope field for the differential equation $\dfrac{dy}{dx} = x - xy$.

Then use it to sketch the solution curve that passes through the point $(1,0)$.

3. Sketch a slope field for the differential equation $\dfrac{dy}{dx} = 2 - y$.

Then use it to sketch three representative solution curves, i.e. curves showing the different types of behaviour of the solutions.

Use a V-Window $[0,3,1]\times[0,4.5,1]$.

[14] From the FA-124 *File* menu: Print Properties Page Setup Scaled 300 OK OK.
[15] To put it back in, PRGM F6 CLR Grph.
[16] Press OPTN F1 after the slope field has been plotted.

13.2.2 Solving DEs: Numerical methods

Euler's Method

Rather than trying to follow the slope lines in a slope field in drawing an approximate solution curve, it would seem to make sense to draw in the necessary slope lines as we go. This simple idea is the basis of Euler's Method.

The method

Example *Using Euler's Method — a simple case*

Use Euler's Method to plot an approximation to the solution $y(x)$ of the initial-value problem

$$\frac{dy}{dx} = 2x \qquad y(-2) = 4.$$

Take $-2 < x < 2$.

This is a simple case that can be done easily algebraically; we use it here to understand Euler's Method. The exact (algebraic) answer can be used as a check.

We are going to construct an approximate solution to this initial-value problem. The solution curve will consist of a number of slope lines joined end to end.

In drawing the slope lines, we need to specify how long they are to be. The shorter they are, the smoother the curve and the closer the approximate curve will be to the exact one, but the more calculations we shall have to do.

It turns out to be easier to specify not the length of the slope line, but the change in x from one end of the slope line to the other.

In drawing a slope line, we start at a given point (x_0, y_0), and draw in the slope line of slope m, calculated at the point (x_0, y_0) from the DE. By definition, $m = \Delta y/\Delta x$.

Along the slope line, if x changes by a given amount called the step length $h = \Delta x$, the change in y is given by $\Delta y = m\Delta x = mh$.

The next point on our approximate curve is then $(x_1, y_1) = (x_0 + h,\ y_0 + mh)$.

In effect, we are approximating the solution curve on the interval $x_0 \leqslant x \leqslant x_0 + h$ by a linear function, the tangent to the curve at (x_0, y_0).

To continue, we repeat the whole process starting now at the new point (x_1, y_1).

For our example, we start with a step length $h = 0.5$. The starting point for our example is the initial condition $(x_0, y_0) = (-2, 4)$.

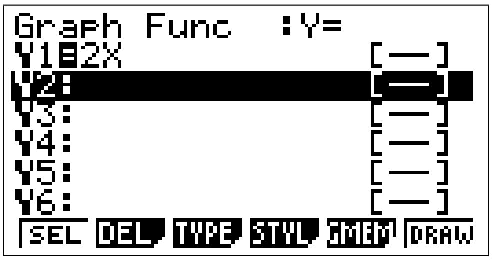
PTO

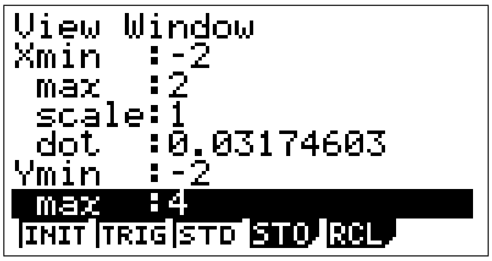
The EULER1 program

As with slope fields, this calculation is tedious by hand, but is easily programmed, as in the EULER1 program. To use this program, first write the DE in terms of the calculator variables X (independent variable) and Y (dependent variable), as we did for SLPFIELD:

$$\frac{dY}{dX} = f(X, Y).$$

- Press [y=] and set Y1 = the RHS of the DE: in this example Y1 = 2X.





- Set the [V-Window] variables to suitable values (so we see the whole graph): $[-2, 2, 1] \times [-2, 4, 1]$ is about right here.

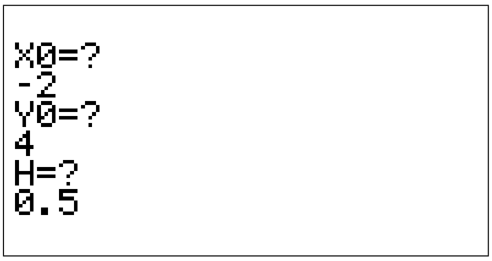
  The initial X value gives Xmin.

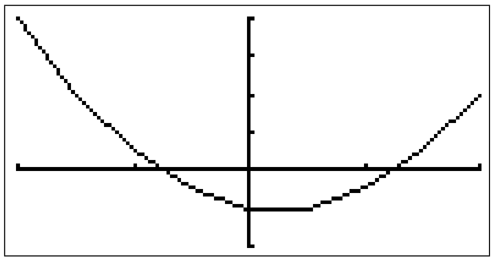
  As with plotting graphs, you have to experiment a bit with the Y V-Window, but the initial Y value helps.

  The two steps above are the same as for SLPFIELD.

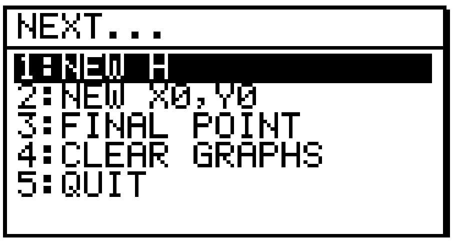
- Run the EULER1 program.

  Enter the initial point, $(X0, Y0) = (-2, 4)$ here; the step H, here 0.5; and press [EXE]. The calculator should then draw in the whole curve from Xmin to Xmax.





- Press [SHIFT] [F6] to redisplay the graph if you wish.
- Press [EXE] to go to the menu where you can choose to enter a new H value, a new initial point or one of several other options.



- Select [5] (QUIT) to stop the program.

Exercise *Changing the step length*

Try some smaller H values, say 0.1, 0.01. The value 0.5 in the example is relatively large. What do you observe? *Solutions in Section 13.4.*

Notes

- Intuitively, as h gets smaller, the approximation to the solution $y(x)$ gets better.

  If you magnify a differentiable function sufficiently at any point, it looks like a straight line, the tangent line at that point. Therefore approximating the function y **on each subinterval** $[x, x+h]$ by a straight line should become more and more accurate as h (the width of a subinterval) gets smaller.

- In the limit $h \to 0$, the *approximate* solution generated by Euler's Method tends to the *exact* solution y.

- We can get closer and closer to the exact solution by taking h smaller and smaller, but as you have observed, it takes the calculator longer and longer to do the calculations.

- Ultimately we are limited in our accuracy by round-off error. If h is too small, the number of digits carried by the calculator (14) will not be sufficient to generate an accurate answer. Therefore *all* numerical solutions are approximations to the exact solution.

Exercise *Changing the initial point*

Now experiment with different initial Y values. Take H = 0.1. What do you observe? *Solutions in Section 13.4.*

It's a good idea to clear the graphs (menu item) or stop the program and restart it at this point, otherwise the screen gets cluttered.

Notes

- In the example above, we know the general solution of the DE is $y = x^2 + C$, where C is a constant. Different initial points will give different C values, hence generating a family of parallel curves displaced from one another along the y axis, as you should have found in the exercise above.

- The value of C is determined by specifying a point on the curve, here the initial point $(-2, 4)$: $y(-2) = 4 \;\Rightarrow\; 4 = (-2)^2 + C = 4 + C \;\Rightarrow\; C = 0$.

  The particular solution is then $y = x^2$. Compare with the Euler approximations.

Example *Numerical calculations using Euler's Method*

Find $y(2)$ if

$$\frac{dy}{dx} = \frac{2y}{x} \quad \text{and} \quad y(1) = 3.$$

Set Y1 = 2Y/X; set the [V-Window] to $[1, 2, 0.5] \times [-2, 12, 2]$.

Note: Xmin should be set to the initial x value X0, so Xmin = 1 here.

If we set up the [V-Window] so that Xmax is the X value of the point we want, pressing [EXE] [3] after the graph is plotted will display the required Y value.

Note: There has to be an integral number of step lengths H between Xmin/X0 and Xmax for the Y value at Xmax to be displayed.

Run EULER1 with X0 = 1 and Y0 = 3 (the initial condition), and step length[17] H = 0.1 to see what the solution looks like.

Press EXE 3 to display the last values of X and Y calculated (menu option 3).

Exercise: Calculate $y(2)$ in this manner, reducing H in powers of 10. Round values to 4 decimal places and enter them in the table below. *Solutions in Section 13.4.*

H	$y(2)$
0.1	11.4545
0.01	
0.001	

What's the best estimate for $y(2)$ from EULER1?

Round off the last two y values (whose H values differ by a factor of 10) until they are the same. That tells you the exact y value (in most cases), **accurate to that number of digits**.

Here the two y values are 11.9406 and 11.9940. We have to round these back to 2 significant digits before they are the same.

Hence, from these calculations, $y(2) = 12$, accurate to 2 significant digits.

Euler's Method is equivalent to the Left-Endpoint Rule for numerical integration.

Modified Euler's Method

The program MODEULR1 uses the modified Euler or Heun Method. The slope of the slope line it draws at each step is the average of the slopes at each end of the line that would be drawn using Euler's Method. You run MODEULR1 the same way you run EULER1.

The ME1CALC version of the program just calculates the final values (no plots); you have to specify the final X value XF.

MODEULR1 is more accurate than EULER1 for a given H value, and therefore produces an answer accurate to a given number of significant digits faster than EULER1 does. The modified Euler's Method is equivalent to the Trapezoidal Rule for numerical integration.

Exercise: Use MODEULR1 (or ME1CALC) to complete the table below, finding $y(2)$ if

$$\frac{dy}{dx} = \frac{2y}{x} \quad \text{and} \quad y(1) = 3.$$

H	$y(2)$
0.1	11.9600
0.01	
0.001	

What's your best estimate for $y(2)$ from MODEULR1? How many significant digits?

Solutions in Section 13.4.

[17] H = 0.1 gives 10 steps from Xmin to Xmax, a reasonable number of steps to start with.

How accurate are our answers?

If two consecutive values in a column in the table (with H values differing by a factor of 10) are the same when rounded to n significant digits, we can be *reasonably* sure that this is the exact answer, accurate to n significant digits.

Therefore, comparing the last two answers in each table:

EULER1: $y(2)=12$, accurate to 2 significant digits.

MODEULR1: $y(2)=12.000$, accurate to 5 significant digits.

The numbers generated by Euler's Method appear to converge, albeit slowly, as H becomes smaller. When $\text{H}=0.001$, the calculation takes a long time (it has to do 1000 steps), and the result is still only accurate to 2 significant digits. If we wanted a value of y for x larger than 2, we would have to take still smaller step sizes to achieve the same accuracy.

Therefore, although in principle we can achieve any given accuracy using Euler's Method by taking a small enough step size (providing round-off error does not become significant), the time taken to do the calculations can become large.

We then look to methods that converge faster, such as the modified Euler's Method (MODEULR1). For graphing and numerical purposes over not-too-large ranges of x, however, EULER1 is OK.

Computer packages such as Matlab use fancier methods, such as the Runge-Kutta methods, but the basic idea is the same. In fact, most numerical DE solvers use fancier versions of Euler's Method;[18] if you understand it, you will have a good grasp of the others.

Exercises

Solutions in Section 13.4.

1. Consider the differential equation $\dfrac{dy}{dx} = x - y$, with initial condition $y(0) = 1$.

(a) Use the EULER1 program to find $y(0.4)$ *accurate* to 2 significant digits.[19] Set up a table of values of $y(0.4)$ versus H, like in the exercises above.

(b) How do you *know* each answer in (a) is *accurate* to 2 significant digits?

(c) Use MODEULR1 or ME1CALC to find $y(0.4)$ *accurate* to 4 significant digits.

(d) The exact solution to the DE for which $y(0)=1$ is

$$y(x) \;=\; x \;-\; 1 \;+\; 2e^{-x}.$$

How does the error in using EULER1 to calculate $y(0.4)$ vary with H in this case? You may need to calculate $y(0.4)$ for a few more H values and keep more digits. Add a column to your table for the error in using Euler's Method.

What return in improved accuracy does Euler's Method give for the increased work caused by reducing the step size H by a factor of 10?

What about the modified Euler's Method?

[18] essentially, they use a weighted average of values of the derivative/slope at various x values as the slope of the line to the next point

[19] Rounded to 2 significant digits is not the same as accurate to 2 significant digits.

2. (a) Find $P(3)$ and $P(12)$, accurate to 3 significant digits, if $P(0) = 1$ and

$$\frac{dP}{dt} = 0.3e^{-0.1t^2}P.$$

Hint: Use MODEULR1 or ME1CALC; write the DE in calculator variables; set up a table of $P(3)$ and $P(12)$ vs H to determine when you have reached the required accuracy.

(b) What is the approximate value of $P(t)$ as $t \to \infty$?

3. (a) Use Euler's Method to find $y(2,)$ accurate to 3 significant digits, if $\frac{dy}{dx} = \frac{1}{x}$ and $y(1)=0$.

(b) What step length H do you need in MODEULR1 or ME1CALC to obtain an answer accurate to 5 significant digits?

4. (a) Find $y(1.5)$ accurate to 3 significant digits for the initial-value problem

$$y' + \tan(x)\,y = \cos^2(x) \qquad y(0) = -1$$

using MODEULR1 or ME1CALC. *Is your calculator in RADIAN mode?*

(b) The algebraic (exact) solution to the initial-value problem here (on the interval $0<x<\pi/2$) is $y(x) = \sin(x)\cos(x) - \cos(x)$. Use this to check that your answer to (a) is indeed accurate to 3 significant digits.

13.2.3 Systems of DEs

Two coupled first-order DEs

Both Euler's and modified Euler's Methods are easily adapted to systems of differential equations. The modified Euler's Method in this case is related to Euler's Method exactly as it was for a single first-order DE. Again, the modified Euler's Method is the better method to use because it gives a better accuracy for a given step size.

We illustrate the modified Euler's Method using the MODEULR2 program with the following simple example.

Example: Solve the following system of coupled differential equations numerically, subject to the given initial conditions.

$$\frac{dx_1}{dt} = -x_1 + x_2 \qquad (1)$$

$$\frac{dx_2}{dt} = 2x_1 \qquad (2)$$

$$x_1(0) = 0 \qquad x_2(0) = 1.$$

Plot the solution curves $x_1(t)$ and $x_2(t)$ for $0<t<2$.

Find $x_1(2)$ and $x_2(2)$ accurate to 3 significant digits.

Using the MODEULR2 program[20]

MODEULR2 does for a system of two first-order differential equations what MODEULR1 did for a single first-order differential equation. It too works for a system in which the right-hand sides of the DEs are **non-linear**.

MODEULR2 assumes a system of first-order DEs of the form

$$\frac{d\mathrm{U}}{d\mathrm{X}} = \mathrm{Y1}\,(\mathrm{X},\mathrm{U},\mathrm{V}) \qquad \frac{d\mathrm{V}}{d\mathrm{X}} = \mathrm{Y}_2\,(\mathrm{X},\mathrm{U},\mathrm{V}), \qquad (3)$$

that is independent variable X, dependent variables U and V.

We therefore have to translate the differential equations in our problem to **calculator variables** X, U, V.

The right-hand side of the first DE, Eq. (1), is stored in Y1 on the calculator in terms of U and V; the right-hand side of the second DE, Eq. (2), in Y2.

U and V are just generated by [ALPHA] [U] and [ALPHA] [V].

PTO

[20]ME2CALC just calculates the final values (no plots).

Step 1: Convert from problem variables to calculator variables

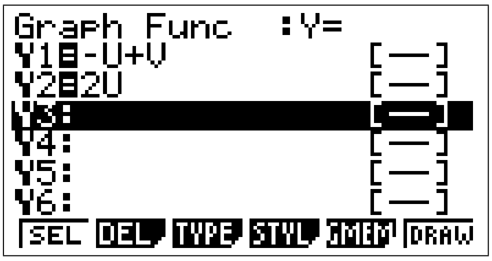
Comparing Eqs. (1) and (2) with the calculator equations, Eq. (3), we see that, in this case,

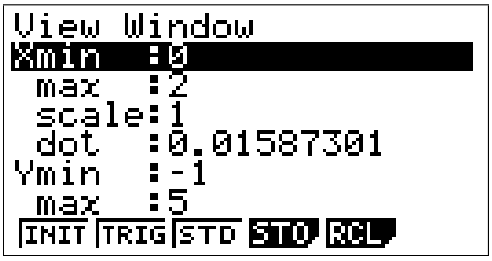
$$t \to \mathrm{X} \qquad x_1 \to \mathrm{U} \qquad x_2 \to \mathrm{V}.$$

Therefore (using the correct minus on your calculator),

$$\frac{dx_1}{dt} = -x_1 + x_2 \quad \to \quad \frac{d\mathrm{U}}{d\mathrm{X}} = -\mathrm{U} + \mathrm{V}. \qquad \text{Set Y1} = -\mathrm{U} + \mathrm{V}.$$

$$\frac{dx_2}{dt} = 2x_1 \quad \to \quad \frac{d\mathrm{V}}{d\mathrm{X}} = 2\mathrm{U}. \qquad \text{Set Y2} = 2\mathrm{U}.$$





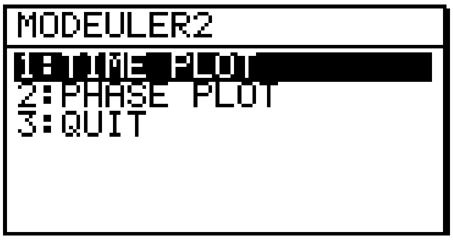
Step 2: Set the V-Window for the solution

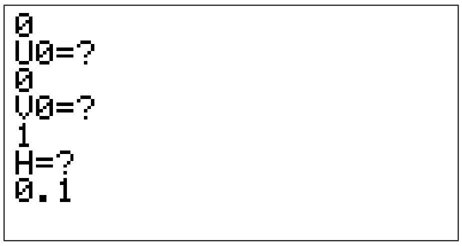
We need to tell the program the range of X (t) values for which we want the solution. We do this by setting the V-Window variables.

For this problem, set the V-Window to $[0, 2, 1] \times [-1, 5, 1]$. Thus, we are solving the DEs for time $t = 0$ to time $t = 2$, as stated in the problem. The Y (U and V) range is found by experimenting and taking into account the initial conditions.

Step 3: Running the program

Now run MODEULR2.

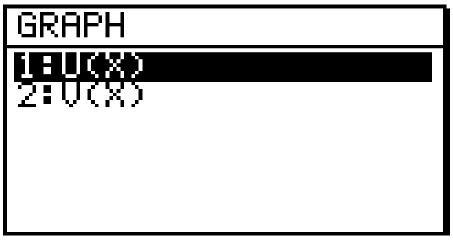




- Choose TIME PLOT.[21]
- Set X0 = 0.
- Set U0 = 0 U0 = $x_1(0) = 0$.
- Set V0 = 1 V0 = $x_2(0) = 1$.
- Set STEP H = 0.1 initially to give a reasonable number of steps between $t = 0$ and $t = 2$.
- Graph U(X), that is $x_1(t)$ in this case (figure over the page).

[21] A phase plot is V vs U.

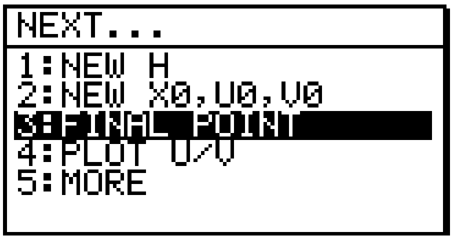
The program then plots an (approximate) modified Euler solution for the system.

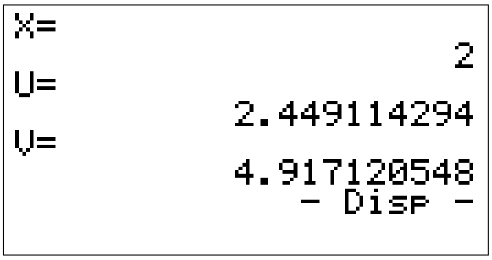


Step 4: The results

Pressing EXE after the graph has been plotted gives a range of options: new H; new X0, U0 and V0; displaying the final point; plotting the other solution V/U; clearing the graphs before continuing; changing to a phase plot; and quitting (below left).

Press 3 to see the final (approximate) values calculated (below right).[22]





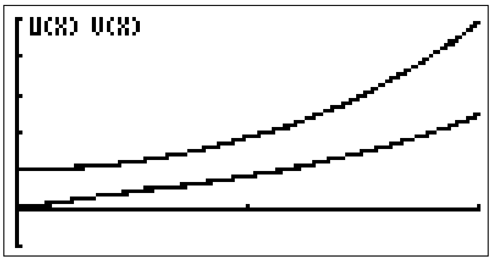
How accurate is the solution?

We proceed exactly as we did with MODEULR1.

We write down the (approximate) values, given by the program, for $x_1(2)$ (U) and $x_2(2)$ (V) with a step size of H = 0.1 in the table below. Then repeat with H = 0.01 and, in this case to achieve the desired accuracy, H = 0.001 (have a think while this one is running).

The figure below shows the graphs of both U (bottom curve) and V for the three H values. The graphs for all three H values are close, the graphs for H = 0.01 and H = 0.001 almost indistinguishable.

H	$\boldsymbol{x_1(2)}$	$\boldsymbol{x_2(2)}$
0.1	2.45	4.92
0.01	2.46	4.93
0.001	2.46	4.93



We conclude from the values in the table that $x_1(2) = 2.46$ and $x_2(2) = 4.93$, both accurate to 3 significant digits.

[22]You only have to plot one of U or V to see all the calculated values.

Exercises

1. Plot the solutions to the following system on $0 \leqslant x \leqslant 2$, with initial conditions $y_1(0) = 8$ and $y_2(0) = 2$:

$$\begin{aligned} \frac{dy_1}{dx} &= -2y_1 + 3y_2 + 12e^x \\ \frac{dy_2}{dx} &= y_1 - 4y_2. \end{aligned}$$

Find $y_1(2)$ and $y_2(2)$ accurate to 4 significant digits.

2. Let $u_P(t)$ be the amount (mass) of a drug in the blood plasma at time t days, and let $u_T(t)$ be the amount in the tissue (organs in the body) for certain positive rate constants k_a, k_b and k_e. The following equations model the process in which the drug enters the plasma at rate $g(t)$, moves between the plasma and the tissue, and is excreted from the plasma:

$$\begin{aligned} \frac{du_P}{dt} &= -k_b u_P - k_e u_P + k_a u_T + g(t) \\ \frac{du_T}{dt} &= k_b u_P - k_a u_T. \end{aligned}$$

Consider the case in which $k_a = 4$, $k_b = 2$, $k_e = 3$ and $g(t) = 0$, giving equations

$$\begin{aligned} \frac{du_P}{dt} &= -5u_P + 4u_T \\ \frac{du_T}{dt} &= 2u_P - 4au_T. \end{aligned}$$

Suppose the initial conditions are $u_P(0) = 1$ and $u_T(0) = 0$: a dose of 1 unit of a drug is injected into the plasma or bloodstream at time $t = 0$, with none in the tissue. Plot $u_P(t)$ and $u_T(t)$ on the interval $0 \leqslant t \leqslant 2$ and comment: do these make sense for the amount of drug in the plasma and tissue?

From Section 6.1 of *Differential Equations: A Toolbox for Modeling the World* by Kurt Bryan, SIMIODE, Cornwall, New York.

13.3 Second-order DEs

13.3.1 Theory

We convert a second-order DE to two coupled first-order DEs by defining a new dependent variable. If the second-order DE is

$$\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu = f(x),$$

let

$$v = \frac{du}{dx}. \tag{4}$$

Then, $\frac{dv}{dx} = \frac{d^2u}{dx^2}$, and the second-order DE becomes

$$\frac{dv}{dx} + av + bu = f(x), \tag{5}$$

a first-order system of two coupled DEs in dependent variables u and v that we can solve numerically with, for example, MODEULR2. *This is a general method.*

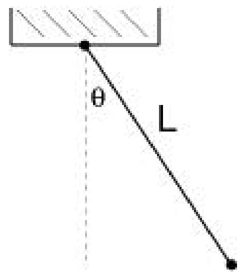
13.3.2 Example: Pendulum motion

The DE describing pendulum motion is[23]

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta), \tag{6}$$

where θ gives the angle from the vertical, positive in the anticlockwise direction, g is the gravitational acceleration constant and L the length of the pendulum.

This DE is non-linear in the dependent variable θ because of the $\sin(\theta)$ term, and cannot be solved algebraically.



We convert our *second-order*, non-linear differential equation into a pair of coupled, *first-order*, non-linear differential equations.

Let the pendulum angular velocity be $v = \frac{d\theta}{dt}$. This defines a new variable $v = v(t)$.

Differentiating both sides of this equation with respect to t gives the angular acceleration

$$\frac{dv}{dt} = \frac{d^2\theta}{dt^2}. \tag{7}$$

Substitute Eq. (7) into Eq. (6). We then have the system of two **two coupled first-order** (non-linear) DEs

$$\frac{d\theta}{dt} = v, \tag{8}$$

$$\frac{dv}{dt} = -\frac{g}{L}\sin(\theta). \tag{9}$$

[23]This equation follows directly from Newton's Second Law for a body moving in a circular arc with $u = L\theta$, or from energy conservation.

We can now use MODEULR2 on these two DEs, as described in Section 13.2.3, page 129. This program assumes coupled first-order DEs of the form

$$\frac{d\text{U}}{d\text{X}} = \text{Y1}\,(\text{X},\text{U},\text{V}) \qquad \frac{d\text{V}}{d\text{X}} = \text{Y2}\,(\text{X},\text{U},\text{V}).$$

We translate the DEs in the problem to **calculator variables** X, U, V. Remember that the right-hand side of the first DE, Eq. (8), is stored (in terms of X, U, V) in Y1 on the calculator, the right-hand side of the second DE, Eq. (9), in Y2.

Example: Plot the graph of $\theta(t)$, the solution of the pendulum equation

$$\frac{d^2\theta}{dt^2} = -10\sin(\theta)$$

or, as a first-order system,

$$\frac{d\theta}{dt} = v \qquad \frac{dv}{dt} = -10\sin(\theta),$$

on the interval $0 \le t \le 2.5$ seconds, given $\theta(0) = \pi/12$ rad and $v(0) = \theta'(0) = 0$ (rad/s).

Find $\theta(2.5)$ and $v(2.5)$ accurate to 3 decimal places.

In practical terms, we are pulling the pendulum out to an angle of $\pi/12$ radians to the right (θ positive), and letting it go from rest: $v(0) = 0$.

Step 1: Convert from problem variables to calculator variables

Comparing the pendulum equations with the calculator equations, we see that, in this case,

$$t \to \text{X} \qquad \theta \to \text{U} \qquad v = \frac{d\theta}{dt} \to \text{V}.$$

Thus,

$$\frac{d\theta}{dt} = v \;\to\; \frac{d\text{U}}{d\text{X}} = \text{V}. \qquad \text{Set Y1} = \text{V}.$$

$$\frac{dv}{dt} = -10\sin(\theta) \;\to\; \frac{d\text{V}}{d\text{X}} = -10\sin(\text{U}). \qquad \text{Set Y2} = -10\sin\text{U}.$$

Note: Y1 = V for all second-order DEs that we solve using MODEULR2. Y2 will be different for different second-order DEs.

Step 2: Set the V-Window for the solution

We need to tell the program the range of times for which we want the solution. We do this by setting the V-Window variables.

For this problem, set the V-Window to $[0, 2.5, 1] \times [-0.5, 0.5, 0.1]$. Note that X0 = Xmin = 0.

Thus, we are solving the DEs for time $t = 0$ s to time $t = 2.5$ s.

The Y range is suggested by the initial condition $\theta(0) = \pi/12 \approx 0.26$, and from experimenting. Of course, you should always be in Radians.

Step 3: Running the program

Now run MODEULR2.

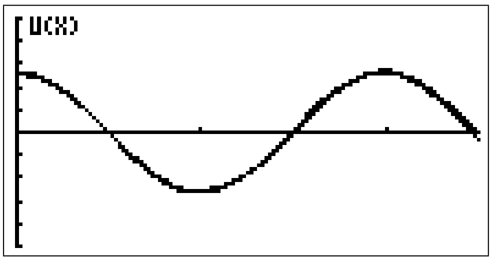
- Choose TIME PLOT.
- Set $X0 = 0$.
- Set $U0 = \pi/12$ $\quad$ $U0 = \theta(0) = \pi/12$.
- Set $V0 = 0$ $\quad$ $V0 = v(0) = \theta'(0) = 0$.
- Start with a step size $H = 0.1$.
- Choose the graph of U(X), that is $\theta(t)$.

Step 4: Converging to the solution

The program then generates an approximate solution to the pendulum equations using the modified Euler's Method. As always, we determine accuracy by reducing the step size H in powers of 10.

The table below shows the MODEULR2 values for $\theta(2.5)$ and $v(2.5) = \theta'(2.5)$ for step sizes H of 0.1, 0.01 and 0.001.

The graphs for $H = 0.1$ and 0.01 in the figure below are almost indistinguishable. However, for an accuracy of 3 decimal places in the numerical values, we need the results for $H = 0.001$.



V-Window $[0, 2.5, 1] \times [-0.5, 0.5, 0.1]$

H	$\boldsymbol{\theta(2.5)}$	$\boldsymbol{v(2.5)}$
0.1	−0.038	−0.842
0.01	−0.005	−0.825
0.001	−0.005	−0.825

Conclusion: On the interval $0 < t < 2.5$, a step size of 0.01 gives a good approximation to the graph of the exact answer, which looks periodic.

From the table, $\theta(2.5) = -0.005$ rad, that is slightly to the left of the equilibrium (vertical) position, and $v(2.5) = -0.825$ rad/s, that is moving to the left, **both values accurate to 3 decimal places**.

13.3.3 Example: Damped oscillations

Example: Use MODEULR2 to sketch a reasonably accurate solution $\theta(t)$ to the initial-value problem for a pendulum,

$$\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} + 10\sin(\theta) = 0 \qquad \theta(0) = \pi/12 \quad \theta'(0) = 0,$$

on the interval $0 \leqslant t \leqslant 2.5$ seconds by first converting the second-order DE to two coupled first-order DEs.

What appears to be happening to the oscillations as t increases? What is different about this DE compared to the one in the previous example that might account for this?

What is the position of the pendulum (radians), and how fast and in what direction is it swinging, at $t = 2.5$ s? Find values accurate to 2 decimal places.

Let

$$\frac{d\theta}{dt} = v. \tag{10}$$

Therefore,

$$\frac{d^2\theta}{dt^2} = \frac{dv}{dt}.$$

The second-order DE then becomes

$$\frac{dv}{dt} + v + 10\sin(\theta) = 0 \qquad \text{or}$$

$$\frac{dv}{dt} = -v - 10\sin(\theta). \tag{11}$$

Equations (10) and (11) are coupled first-order non-linear DEs, non-linear because of the $\sin(\theta)$.

The corresponding calculator equations are $\big(t \to \mathrm{X};\ \theta \to \mathrm{U};\ v \to \mathrm{V}\big)$

$$\frac{d\mathrm{U}}{d\mathrm{X}} = \mathrm{V} \qquad \text{and} \qquad \frac{d\mathrm{V}}{d\mathrm{X}} = -\mathrm{V} - 10\sin\mathrm{U}.$$

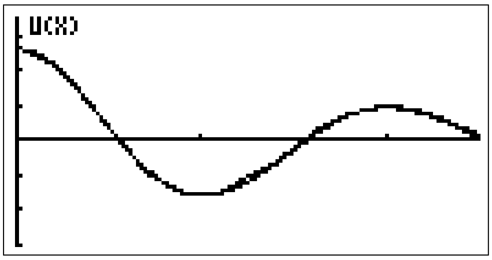
Set Y1 = V and Y2 = −V−10 sin(U).

Careful with the different − signs on your calculator.

The initial conditions are U0 = $\pi/12$ and V0 = 0.

A suitable V-Window is $[0, 2.5, 0.5] \times [-0.3, 0.35, 0.1]$.

Run MODEULR2, TIME PLOT, with H = 0.1 and 0.01, from which we conclude that H = 0.1 gives a reasonably accurate graph. The two graphs (over the page) are almost indistinguishable.



V-Window $[0, 2.5, 0.5] \times [-0.3, 0.35, 0.1]$

H	$\boldsymbol{\theta(2.5)}$	$\boldsymbol{v(2.5)}$
0.1	0.007	−0.232
0.01	0.017	−0.239
0.001	0.017	−0.239

The amplitude of the oscillations is decreasing with time because the DE now contains a damping term $d\theta/dt$.

At $t = 2.5$ s, the pendulum is 0.02 rad to the right of its equilibrium position (positive θ value), swinging to the left (negative v value) at a rate of 0.24 rad/s; all values are accurate to 2 decimal places.

13.3.4 Exercises

Convert each second-order equation into an equivalent system of first-order DEs with initial conditions. Plot the solutions to each DE on $0 \leqslant x \leqslant 2$ and find $u(2)$ accurate to 4 significant digits.

1. $\dfrac{d^2u}{dt^2} + 5\dfrac{du}{dt} + 4u = 0, \quad u(0) = 7,\ u'(0) = 5.$

2. $2u'' + 2\cos(u') + u = \sin(t), \quad u(0) = 3,\ u'(0) = -1.$

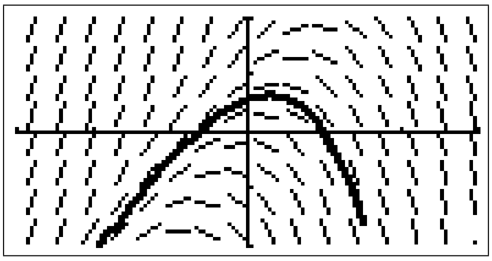
13.4 Solutions to exercises

Exercises page 122

1. Sketch a slopefield for the differential equation $\dfrac{dy}{dx} = y - 2x$.

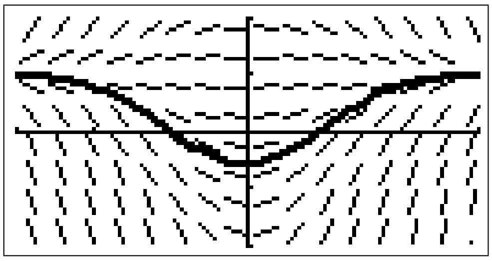
Then use it to sketch the solution curve that passes through the point $(1, 0)$.

Use a V-Window of $[-3, 3, 1] \times [-2, 2, 1]$.



2. Sketch a slopefield for the differential equation $\dfrac{dy}{dx} = x - xy$.

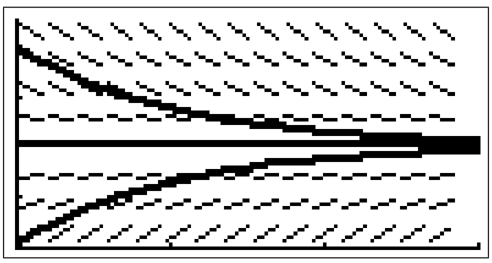
Then use it to sketch the solution curve that passes through the point $(1, 0)$.



V-Window $[-3, 3, 1] \times [-2, 2, 1]$

3. Sketch a slopefield for the differential equation $\dfrac{dy}{dx} = 2 - y$.

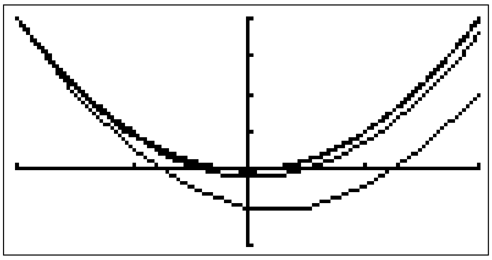
Then use it to sketch three representative solution curves, i.e. curves showing the different types of behaviour of the solutions. Use a V-Window $[0, 3, 1] \times [0, 4.5, 1]$.



The horizontal line $y = 2$ is a solution of the DE

Exercise *Changing the step length* page 125

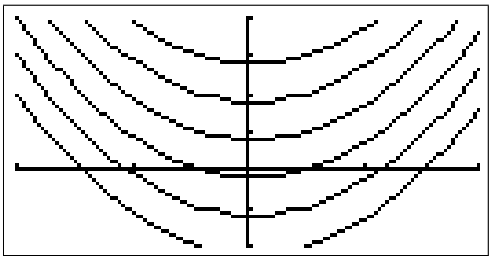
Try some smaller H values, say 0.1, 0.01. The value 0.5 in the example is relatively large. What do you observe?



The curves appear to be converging to some final curve as H is decreased. In fact, the curve for H = 0.01 is almost indistinguishable from the exact curve, $y = x^2$. However, the smaller H, the longer the program takes to run.

Exercise *Changing the initial point* page 125

Now experiment with different initial Y values. Take H = 0.1. What do you observe?



A whole series of parallel curves, one for each initial Y value.

Exercise page 126

Use EULER1 to calculate $y(2)$, reducing H in powers of 10. Round values to 4 decimal places and enter them in the table.

H	$\boldsymbol{y(2)}$
0.1	11.4545
0.01	11.9406
0.001	11.9940

Exercise page 126

Use MODEULR1 or ME1CALC to complete the table below. What's your best estimate for $y(2)$ from MODEULR1? How many significant digits?

H	$\boldsymbol{y(2)}$
0.1	11.9600
0.01	11.9996
0.001	12.0000

Round the last two values until they agree: $y(2) = 12.000$, accurate to 5 significant digits.

Exercises page 127

1. Consider the differential equation $\dfrac{dy}{dx} = x - y$, with initial condition $y(0) = 1$.

(a) Use EULER1 to find $y(0.4)$ *accurate* to 2 significant digits.

H	$\boldsymbol{y(0.4)}$
0.1	0.7122
0.01	0.7379
0.001	0.7404

The values in the table for H = 0.01 and H = 0.001 are both 0.74 when rounded to 2 significant digits. Therefore, accurate to 2 significant digits, $y(0.4) = 0.74$.

(b) How do you *know* each answer in (a) is *accurate* to 2 significant digits?

We continue reducing the step length h in powers of 10 until two successive answers are the same to the required accuracy, that is when rounded to 2 significant digits.

We know that the approximate answers tend to the exact answer as $h \to 0$, so we can be reasonably confident that, if successive answers obtained with h reduced by a factor of 10 are the same to 2 significant digits, this is the exact answer accurate to 2 significant digits.

(c) Use MODEULR1 or ME1CALC to find $y(0.4)$ *accurate* to 4 significant digits.

H	$\boldsymbol{y(0.4)}$
0.1	0.7416
0.01	0.7406
0.001	0.7406

The values in the table for H = 0.01 and H = 0.001 are both 0.7406. Therefore, accurate to 4 significant digits, $y(0.4) = 0.7406$.

(d) The exact solution to the DE for which $y(0) = 1$ is

$$y(x) = x - 1 + 2e^{-x}.$$

How does the error in using EULER1 to calculate $y(0.4)$ vary with H in this case? You may need to calculate $y(0.4)$ for a few more H values and keep more digits.

What return in improved accuracy does Euler's Method give for the increased work caused by reducing the step size H by a factor of 10? What about the modified Euler's Method?

Calculating $y(0.4)$ from EULER1 and MODEULR1 or ME1CALC, and the corresponding errors (error = exact − approx) gives, all rounded to 5 decimal places,

H	**EULER1**	**Error**	**MODEULR1**	**Error**
0.1	0.71220	0.02844	0.74160	−0.00096
0.01	0.73794	0.00270	0.74065	−0.00001
0.001	0.74037	0.00027	0.74064	0
0.0001	0.74061	0.00003		

We can see that for each reduction of H by a factor of 10 (meaning the calculator has to do 10× more calculations), we achieve a reduction in the error by a factor of about 10 using EULER1, and by a factor of about 100 using MODEULR1.

For this reason, Euler's Method is a first-order method (error reduction 10^1), whereas the modified Euler's Method is a second-order method (error reduction 10^2).

2. (a) Find $P(3)$ and $P(12)$, accurate to 3 significant digits, if $P(0) = 1$ and

$$\frac{dP}{dt} = 0.3e^{-0.1t^2}P.$$

In calculator variables, the DE is

$$\frac{d\text{Y}}{d\text{X}} = 0.3e^{-0.1\text{X}^2}\text{Y}.$$

Set Y1 = 0.3e^(−0.1X^2)Y, V-Windows of $[0, 3, 1]\times[-0.3, 3, 1]$ and $[0, 12, 2]\times[-0.3, 3, 1]$, respectively. Run MODEULR1 or ME1CALC with X0 = 0, Y0 = 1 and reducing step lengths H. The table below shows the results.

H	$\boldsymbol{P(3)}$	$\boldsymbol{P(12)}$
1	1.97	2.30
0.1	1.99	2.32
0.01	1.99	2.32

Therefore, accurate to 3 significant digits, $P(3) = 1.99$ and $P(12) = 2.32$.

(b) What is the approximate value of $P(t)$ as $t \to \infty$?
The graph of $P(t)$ looks to have levelled out at $P \approx 2.32$: this is the approximate limiting value as $t \to \infty$.

3. (a) Use Euler's Method to approximate $y(2)$ if $\dfrac{dy}{dx} = \dfrac{1}{x}$ and $y(1) = 0$ accurate to 3 significant digits.

H	$\boldsymbol{y(2)}$
0.1	0.7188
0.01	0.6956
0.001	0.6934
0.0001	0.6932

Therefore, $y(2) = 0.693$, accurate to 3 significant digits.

(b) What step length do you need in MODEULR1 or ME1CALC to obtain an answer accurate to 5 significant digits?

H	$\boldsymbol{y(2)}$
0.1	0.69377
0.01	0.69315
0.001	0.69315

A step length of 0.01 gives an answer accurate to 5 significant digits, verified by using a step length of 0.001.

4. (a) Find $y(1.5)$ accurate to 3 significant digits for the initial-value problem

$$y' + \tan(x)\,y = \cos^2(x) \qquad y(0) = -1$$

using MODEULR1 or ME1CALC. In calculator variables, the DE is

$$\frac{d\mathrm{Y}}{d\mathrm{X}} = -\tan(\mathrm{X})\mathrm{Y} + \cos(\mathrm{X})^2.$$

Set $\mathrm{Y1} = -\tan(\mathrm{X})\mathrm{Y} + \cos(\mathrm{X})^2$ and a V-Window of $[0, 1.5, 0.5] \times [-1, 0.15, 0.2]$. Run MODEULR1 with $\mathrm{X0} = 0$, $\mathrm{Y0} = -1$ and reducing H.

H	$\boldsymbol{y(1.5)}$
0.1	-2.37×10^{-3}
0.01	-1.88×10^{-4}
0.001	-1.77×10^{-4}
0.0001	-1.77×10^{-4}

Therefore, accurate to 3 significant digits, $y(1.5) = -1.77 \times 10^{-4}$.

(b) The solution to the initial-value problem above is $y(x) = \sin(x)\cos(x) - \cos(x)$ (on the interval $0 < x < \pi/2$). Use this to check that your answer to (a) is indeed accurate to 3 significant digits.
From the exact solution,

$$y(1.5) = \sin(1.5)\cos(1.5) - \cos(1.5) = -1.77 \times 10^{-4},$$

rounded to 3 significant digits, in agreement with MODEULR1.

Exercises page 132

1. Plot the solutions to the following system on $0 \leqslant x \leqslant 2$, with initial conditions $y_1(0) = 8$ and $y_2(0) = 2$:

$$\frac{dy_1}{dx} = -2y_1 + 3y_2 + 12e^x$$
$$\frac{dy_2}{dx} = y_1 - 4y_2.$$

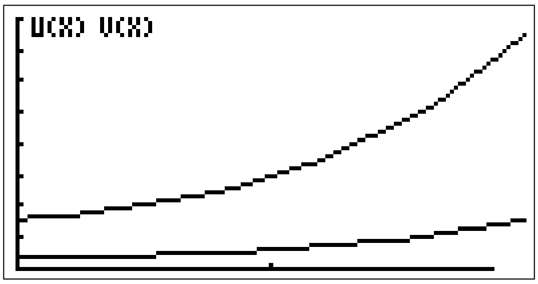
Find $y_1(2)$ and $y_2(2)$ accurate to 4 significant digits.

Here, $x \to$ X; $y_1 \to$U; $y_2 \to$V. The equations in calculator variables are then

$$\frac{d\text{U}}{d\text{X}} = -2\text{U} + 3\text{V} + 12e\text{\^{}X}$$
$$\frac{d\text{V}}{d\text{X}} = \text{U} - 4\text{V}.$$

The initial conditions are U0 = 8, V0 = 2.

Running MODEULR2 gives the following graphs of $y_1(t)$ (top) and $y_2(t)$.



V-Window $[0, 2, 1] \times [0, 40, 5]$

The values for $y_1(2)$ are given below.

H	$y_1(2)$
0.1	37.420
0.01	37.351
0.001	37.351

Therefore, $y_1(2) = 37.35$, accurate to 4 significant digits. From the same calculations, $y_2(2) = 7.524$, accurate to 4 significant digits.

2. Let $u_P(t)$ be the amount (mass) of a drug in the blood plasma at time t days, and let $u_T(t)$ be the amount in the tissue (organs in the body) for certain positive rate constants k_a, k_b and k_e. The following equations model the process in which the drug enters the plasma at rate $g(t)$, moves between the plasma and the tissue, and is excreted from the plasma:

$$\begin{aligned}\frac{du_P}{dt} &= -k_b u_P - k_e u_P + k_a u_T + g(t)\\ \frac{du_T}{dt} &= k_b u_P - k_a u_T.\end{aligned}$$

Consider the case in which $k_a = 4$, $k_b = 2$, $k_e = 3$ and $g(t) = 0$, giving equations

$$\begin{aligned}\frac{du_P}{dt} &= -5u_P + 4u_T\\ \frac{du_T}{dt} &= 2u_P - 4u_T.\end{aligned}$$

Suppose the initial conditions are $u_P(0) = 1$ and $u_T(0) = 0$: a dose of 1 unit of a drug is injected into the plasma or bloodstream at time $t = 0$, with none in the tissue. Plot $u_P(t)$ and $u_T(t)$ on the interval $0 \leqslant t \leqslant 2$ and comment: do these make sense for the amount of drug in the plasma and tissue?

Here, $t \to$ X; $u_P \to$U; $u_T \to$V. The equations in calculator variables are then

$$\begin{aligned}\frac{d\text{U}}{d\text{X}} &= -5\text{U} + 4\text{V}\\ \frac{d\text{V}}{d\text{X}} &= 2\text{U} - 4\text{V}.\end{aligned}$$

The initial conditions are U = 1, V(0 = 0.

Running MODEULR2 gives the following graphs of $u_P(t)$ (top) and $u_T(t)$.

V-Window $[0, 2, 1] \times [0, 1, 0.5]$

The graphs make sense. The intial drug in the plasma is lost to the tissue or excreted. Although some comes back from the tissue, it is always less than the amount lost, so the amount of the drug in the plasma always decreases.

The drug in the tissue increases from zero as it receives drug from the plasma. However, this amount decreases as the drug is excreted, and eventually the amount of drug in the tissue reaches a maximum, then decreases.

Exercises page 137

Convert each second-order equation into an equivalent system of first-order DEs with initial conditions. Plot the solutions to each DE on $0 \leqslant x \leqslant 2$ and find $u(2)$ accurate to 4 significant digits.

1. $\dfrac{d^2u}{dt^2} + 5\dfrac{du}{dt} + 4u = 0, \quad u(0) = 7,\ u'(0) = 5.$

Let $\dfrac{du}{dt} = v$. Then, the second-order DE becomes

$$\frac{dv}{dt} + 5v + 4u = 0, \quad u(0) = 7,\ v(0) = 5,$$

or

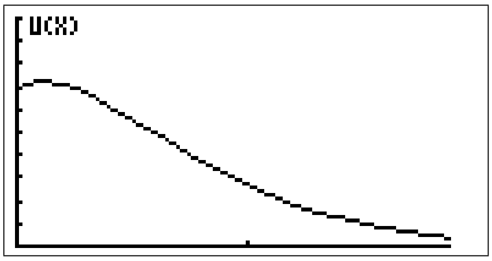
$$\frac{dv}{dt} = -5v - 4u \quad u(0) = 7,\ v(0) = 5.$$

Here, $t \to$ X; $u \to$U; $v \to$V. The equations in calculator variables are then

$$\begin{aligned} \frac{d\text{U}}{d\text{X}} &= \text{V} \\ \frac{d\text{V}}{d\text{X}} &= -5\text{U} - 4\text{V}. \end{aligned}$$

The initial conditions are U(0) = 7, V(0) = 5.

Running MODEULR2 gives the following graph of $u(t)$.



V-Window $[0, 2, 1] \times [0, 10, 1]$

The values for $u(2)$ are given below.

H	$\boldsymbol{u(2)}$
0.1	0.27534
0.01	0.26319
0.001	0.26308
0.0001	0.26308

Therefore, $u(2) = 0.2631$, accurate to 4 significant digits.

2. $2u'' + 2\cos(u') + u = \sin(t), \quad u(0) = 3, \ u'(0) = -1.$

Let $\dfrac{du}{dt} = v$. Then, the second-order DE becomes

$$2\frac{dv}{dt} + 2\cos(v) + u = \sin(t), \quad u(0) = 3, \ v(0) = -1,$$

or

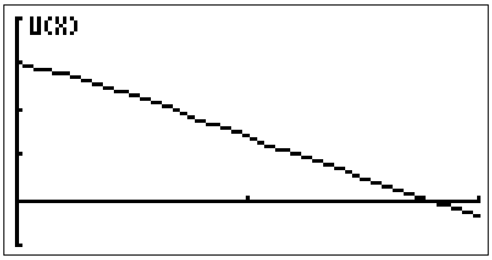
$$\frac{dv}{dt} = -\cos(v) - 0.5u + 0.5\sin(t) \quad u(0) = 3, \ v(0) = -1.$$

Here, $t \to$ X; $u \to$U; $v \to$V. The equations in calculator variables are then

$$\begin{aligned} \frac{d\text{U}}{d\text{X}} &= \text{V} \\ \frac{d\text{V}}{d\text{X}} &= -0.5\text{U} - \cos(\text{V}) + 0.5\sin(\text{X}). \end{aligned}$$

The initial conditions are U0 = 3, V0 = −1.

Running MODEULR2 gives the following graph of $u(t)$.



V-Window $[0, 2, 1] \times [-1, 4, 1]$

The values for $u(2)$ are given below.

H	$\boldsymbol{u(2)}$
0.1	−0.32823
0.01	−0.32714
0.001	−0.32714

Therefore, $u(2) = -0.3271$, accurate to 4 significant digits.

13.5 Appendix: Calculator programs

These are available at *canberramaths.org.au* under *Resources*.

13.5.1 First-order DE solvers

SLPFIELD

Plots slope fields. Instructions for use on page 121.

EULER1

Uses Euler's Method to plot approximate solutions to a DE; also gives final values. Instructions for use on page 124.

MODEULR1/ME1CALC

Uses the modified Euler's Method to plot approximate solutions to a DE; also gives final values. Instructions for use on page 126.

ME1CALC just calculates the final values (no plots); you have to specify the final X value XF.

13.5.2 Coupled first-order/second-order DE solvers

MODEULR2/ME2CALC

Uses the modified Euler's Method to plot approximate solutions to two coupled first-order DEs or a second-order DE; also gives final values. Instructions for use on page 129.

ME2CALC just calculates the final values (no plots); you have to specify the final X value XF.

14 Population Modelling 2 Logistic and Epidemic Models

14.1 Logistic model

The logistic model arose from an attempt by Verhulst to come up with a more realistic population model than Malthus' exponential model (see Population Modelling 1 in Volume 1 of this book). He reasoned that no organism grows without bound, otherwise the Earth would be covered in this organism. Restrictions to growth are imposed by the need for food and space, so that the growth rate k, assumed constant in the exponential model, must decrease as the population increases. Verhulst chose the simplest such form for the growth rate, a linear decrease with population P: $k = a - bP$, where a and b are constants. This led to the so-called logistic curves (see Section 14.1.1 below), which start out like exponentials but eventually saturate or tend asymptotically to a constant value.

14.1.1 Mathematical background

This section briefly outlines the steps to the logistic model in terms of differential equations. You can skip down to *Solution* if you haven't covered/encountered differential equations.

The assumption that the (instantaneous) rate of change (increase or decrease) in a population $P(t)$ at time t is proportional to the population at that time gives rise to the exponential differential equation

$$\frac{dP}{dt} = kP,$$

where the constant k is the growth (or decay) rate. Solutions are of the form $P(t) = P_0 e^{kt}$, where $P_0 = P(0)$ is the initial population.

Putting the Verhulst growth rate $k = a - bP$, where a and b are positive constants, into the exponential differential equation above, re-arranging and renaming some constants gives the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right),$$

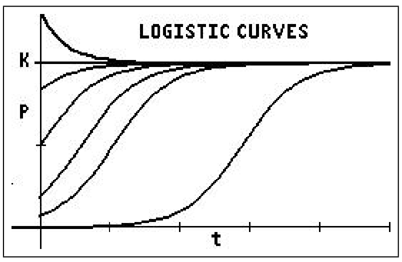
where it turns out that K is the maximum sustainable population or carrying capacity.

Solution: The solution to the logistic differential equation is

$$P(t) = \frac{KEe^{kt}}{Ee^{kt} - 1}, \tag{1}$$

where $E = P_0/(P_0 - K)$, with $P_0 = P(0)$ the initial population ($P_0 \neq K$).

If $P_0 = K$, the solution is $P(t) = K$, a constant or equilibrium solution. $P(t) = 0$ is the other equilibrium solution but not of great interest in population modelling other than acknowledging that it makes sense. Some logistic curves $P(t)$ are shown below.



Logistic curves are close to exponential for small t but, instead of increasing indefinitely like the exponentials, they level out and approach the constant value K asymptotically. Curves starting above the carrying capacity K decrease down to it asymptotically.

14.1.2 Logistic problem

Exercise: A population $P(t)$ of fish in a pond is modelled by Eq. (1) with $k = 2.5$ and carrying capacity $K = 100$. Time t is measured in weeks. Initially there are 10 fish in the pond.

1. Write down the formula for $P(t)$ and graph it for $0 < t < 5$.
2. How many fish are there after 1 week?
3. How long does it take the fish population to reach 80?
4. How many fish are there after a long time?

Solutions to this problem are in Section 14.4.1.

14.2 Discrete logistic model

Here we use a discrete version of the Verhulst or logistic model, called the discrete logistic model, to predict the growth of a population of bacteria or kangaroos. In a discrete model, time goes in steps rather than being continuous, as it was in previous models. The model is in the form of a difference equation which tells you how to calculate the population after the next time step, P_{n+1}, if you know the population P_n now:

$$P_{n+1} = AP_n(1 - P_n), \tag{2}$$

where A is a constant

14.2.1 Exercise: Bacteria

In this model, P_n is a measure of the population (in millions of bacteria) at the end of the nth hour and A is a number that depends on how fast the bacteria reproduce. For our calculations, we take $A = 2$ and the starting population $P_0 = 0.1$.

To calculate P_1, the population after 1 hour, put $n = 0$ and $P_0 = 0.1$ in Eq. (2):

$$P_1 = 2P_0(1 - P_0) = 2 \times 0.1 \times (1 - 0.1) = 0.18.$$

To calculate P_2, the population after the second hour, put $n = 1$ in Eq. (2):

$$P_2 = 2P_1(1 - P_1) = 2 \times 0.18 \times (1 - 0.18) = 0.2952,$$

and so on. After a few more steps (hours), you should find the population stabilises at a particular number. *What is the number?*

Calculator Hint

To speed up this process, on a calculator, type:

0.1 → [ALPHA] [P] [EXE] store the starting population in memory P;

2P(1−P) → [P] [EXE] evaluate Eq. (2) and store the result back in P.

This will give you the next value for P. If you now keep pressing [EXE], the calculator will repeatedly execute the last line to give successive values for P.

Next let $A = 3.2$ and keep $P_0 = 0.1$:

0.1 → ALPHA P EXE ;

3.2P(1–P) → P EXE ;

and keep pressing EXE .

You'll need to run the population for about 18 (model) hours this time before it settles down. What happens here? *Draw a plot of population versus time.*

Now try $A = 3.8$ and $P_0 = 0.1$. This one is weird! The population varies wildly between 0.1 and 1, with no hope of prediction. *Plot this one too.* You've discovered chaos (the mathematical version). *What happens with other values of A and P_0?*

Solutions to the above are in Section 14.4.2.

Sequence grapher: Can be used to graph values of P_n vs n. The LOGISTIC program (Section 14.2.3) sets this up for you for the bacteria here and for the kangaroos in the next section.

14.2.2 Exercise: Kangaroo management

Part of an UNSW Canberra Maths Lab adapted from *Stimulating Mathematical Interest with Dynamical Systems* by M.B. Durkin, The Maths Teacher 89, 242–24 (1996).

You are hired by the State Forestry Department, with your main task to assist in the management of the kangaroo population in a remote forest called Hamt Reserve. The possibility of culling of kangaroos in the reserve is under consideration.

The kangaroo population in the reserve is given by the discrete logistic model, a difference equation,

$$P_{n+1} = 1.8P_n - 0.8(P_n)^2, \tag{3}$$

where P_n is the number of kangaroos in the reserve at the end of year n in tens of thousands, i.e. *one unit of P equals 10, 000 kangaroos.* At the end of 2005, there were 8000 kangaroos in the reserve ($P_0 = 0.8$).

The first task

As a training exercise, management asks you to model and report on a scenario containing several events that would affect the kangaroo population.

Write a short report on the outcome of the following scenario. The report should include a mathematical analysis with calculations, tables and/or graphs to substantiate your conclusions.

The scenario

- If there are no natural disasters in 2006, what is the kangaroo population at the end of 2006? Do this and the following calculations manually (without a program) using Eq. (3).[24]
- Unfortunately, at the end of 2006, there is a short but fatal outbreak of the dreaded rootoxis which kills around 4000 kangaroos. What is the population of kangaroos at the end of 2007? When does the kangaroo population recover to more than 9000 kangaroos if are no more natural disasters?

[24] *Calculator hint*: As in Section 14.2.1, store the initial population in memory P and repeatedly execute the calculation $1.8\text{P} - 0.8\text{P}^2 \to \text{P}$ by pressing EXE the required number of times.

- Following the rootoxis epidemic, on Christmas Day 2008 there is a forest fire in a nearby forest which results in 2000 kangaroos from that forest migrating into Hamt Reserve. What is the population of kangaroos in Hamt Reserve at the end of 2009?
- After these two events, there are no more natural disasters. What is the kangaroo population after a long time? The number here is the limiting capacity or maximum sustainable population of the reserve.

Effect of culling

Impressed by your previous report, management has now put you in charge of undertaking a feasibility study into whether culling of kangaroos is necessary/desirable in Hamt Reserve. Your analysis will be a crucial factor in the decision-making process.

Write a report addressing the following questions. Again, a mathematical analysis including calculations, tables and/or graphs is required to substantiate your conclusions. Add an executive summary for your boss, summarising your findings and making suitable recommendations.

1. What is the modified form of Eq. (3) if H kangaroo units are culled each year?

We assume here, for simplicity, that all the kangaroos are killed close to the end of the year, otherwise the killing of the female kangaroos in particular would affect the number of births and deaths, and consequently the growth rate.

2. What would happen if 720 kangaroos were culled each year ($H = 0.072$), a value used in a nearby reserve? Assume the initial population is that given above for the year 2005, $P_0 = 0.8$. What is the long-term population?

What if the initial population were $P_0 = 0.3$? $P_0 = 0.095$?

3. What would happen if 2400 kangaroos were culled each year ($H = 0.24$)? Assume again that $P_0 = 0.8$. What is the long-term population?

What if the initial population were $P_0 = 1$? $P_0 = 1.5$?

4. What about $H = 0.2$? It turns out[25] that this is the largest number of kangaroos which could be culled annually without the kangaroos dying out in Hamt Reserve. Note that the initial population must be larger than 0.5. What is the long-term population in this case?

Solutions to the above are in Section 14.4.2.

[25]Experiment and see — the LOGISTIC program, next section, might help.

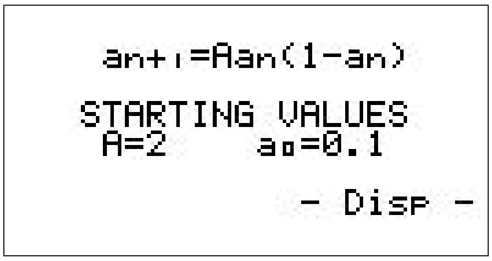
14.2.3 LOGISTIC program: Bacteria and kangaroos

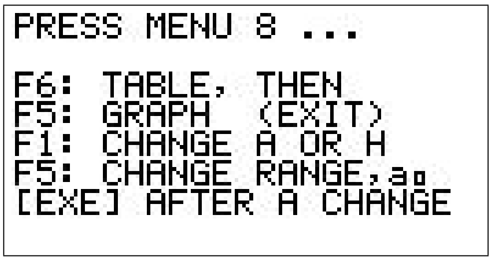
Available at *canberramaths.org.au* under *Resources*.

Sets up the graphics for a population of bacteria obeying the discrete logistic equation (Section 14.2.1) or for a population of kangaroos obeying the discrete logistic equation with culling (Section 14.2.2).

Use: Run the program. Select BACTERIA or KANGAROOS by pressing the appropriate number and EXE.

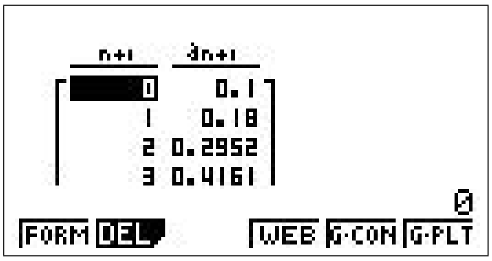
For BACTERIA, the program sets up the sequence $a_{n+1} = Aa_n(1-a_n)$, with starting values $A=2$ and $a_0=0.1$ (below left).



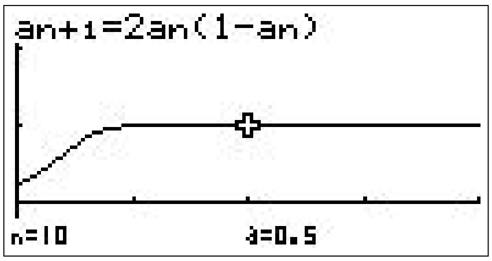


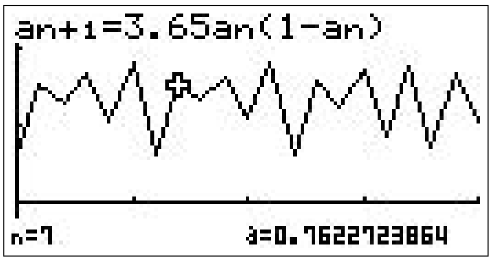
Pressing EXE gives the screen above right, essentially the commands for generating tables and graphs in RECUR mode (MENU 8). The program has now finished.

Pressing MENU 8 as directed gives the standard RECUR screen below



Pressing TABL gives a table of values of the sequence (above right). From this screen, pressing G-CON gives a line graph of population versus cycle number (time) (below left). F1 (Trace) has been pressed and the cursor moved to $n=10$.

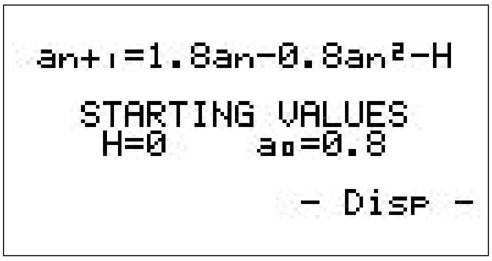




Press EXE twice to return to the main Recursion screen. Here you can change the value of A. Press F5 (SET) to change the value of a_0 or the table range.

$A=3.65$ in the figure above right.

For KANGAROOS, the program sets up the sequence $a_{n+1} = 1.8a_n - 0.8a_n^2 - H$, with the starting value for the harvesting rate $H = 0$ and $a_0 = 0.8$ (below left).



You then proceed as for BACTERIA to generate a table or graph, and to change the values of H in the formula or a_0 in SET.

14.3 SIR epidemic model

14.3.1 Introduction to epidemic modelling

Based on www.stat.columbia.edu/∼regina/research/notes123.pdf.

The modelling of infectious diseases is a tool that is used to study the mechanisms by which diseases spread, to predict the future course of an outbreak and to evaluate strategies to control an epidemic.

Infectious agents have had decisive influences on the history of mankind. Fourteenth-century Black Death took the lives of about a third of Europe's population at the time. Thucydides described the Plague of Athens (430 – 428 BC): 1,050 of 4,000 soldiers on an expedition died of a disease. He gave a detailed account of symptoms, some so horrendous that the last one – amnesia – seems a blessing (Bailey, 1975). An interesting feature of this account is that there was no mention of person-to-person contagion, which we now suspect with most new diseases; it was not until the 19th century that this was beginning to be discussed.

The practical use of epidemic models relies heavily on the assumptions underlying the models. A reasonable model does not have to include all possible effects but should incorporate the main mechanisms that influence disease propagation in the simplest possible fashion. Great care should be taken before epidemic models are used for prediction of real phenomena. However, even simple models should, and often do, pose important questions about the underlying mechanisms of infection spread and possible means of control of the disease or epidemic.

The classical papers by Kermack and McKendrick (1927, 1932, 1933) have had a major influence on the development of mathematical models for disease spread, and are still relevant in many epidemic situations. The first of these papers laid out a foundation for modelling infections which confer complete immunity after recovery (or, in the case of lethal diseases, death). If infected individuals are introduced into a large population, a basic problem is to describe the spread of the infection within the population as a function of time. One of the most important questions is whether the epidemic comes to an end only when all the initially susceptible individuals have contracted the disease or if some interplay of infectivity, recovery and mortality factors may result in epidemic "die out", with many susceptibles still present in the unaffected population.

In their first paper, Kermack and McKendrick started with the assumption that all members of the community were initially equally susceptible to the disease, and that complete immunity was conferred after the infection. The population was divided into three distinct classes: susceptibles S, healthy individuals who can catch the disease; infecteds I, those who have the disease and can transmit it; and removed R, individuals who have had the disease and are now immune to the infection (or removed from further propagation of the disease by some other means). Schematically, the individual goes through consecutive states S→I→R; such models are often called SIR models.

Bailey, NTJ (1975). *The Mathematical Theory of Infectious Diseases and its Applications.* 2nd ed., Hafner Press, NY.

Ethier, SN & Kurtz, TG (1986). *Markov Processes. Characterization and Convergence.* Wiley, NY.

Kermack, WO & McKendrick, AG (1927). Contributions to the mathematical theory of epidemics, i. *Proc. R. Soc. Edinb.: A Mathematics* **115**, 700–721.

Kermack, WO & McKendrick, AG (1932). Contributions to the mathematical theory of epidemics ii — the problem of endemicity. *Proc. R. Soc. Edinb.: A Mathematics* **138**, 55–83.

Kermack, WO & McKendrick, AG (1933). Contributions to the mathematical theory of epidemics iii — further studies of the problem of endemicity. *Proc. R. Soc. Edinb.: A Mathematics* **141**, 94–122.

14.3.2 Discrete SIR model

In the SIR model, the population is divided into three categories, **susceptible**, **infected** and **removed**, with S, I and R being the respective numbers in each, functions of time in days. Susceptible persons can catch the disease; infected persons are infectious and can therefore transmit the disease to susceptibles; removed persons have recovered, are then assumed immune to the disease and cannot spread it.

The discrete SIR model is the difference equations[26]

$$S_{n+1} = S_n - \beta S_n I_n \quad (1)$$

$$I_{n+1} = I_n + \beta S_n I_n - \alpha I_n \quad (2)$$

$$R_{n+1} = R_n + \alpha I_n, \quad (3)$$

where S_n, I_n and R_n are, respectively, the numbers of susceptible persons, infected persons and recovered persons after n time intervals; α (the recovery rate) and β (the transmission rate) are positive constants. The time interval here is 1 day.

$\beta S_n I_n$ in Eq. (1) is the number of new infections per day. These people move to the infected category, thereby decreasing S in Eq. (1) and increasing I in Eq. (2).

$\beta S_n I_n$ models the transmission of the disease. The expression here is one of several possible choices. It is called the *Law of Mass Action*, which states that the rate of new cases is proportional to the product of S_n and I_n. Transmission occurs when an infected meets a susceptible, so intuitively, transmission should depend somehow on the numbers of each. In this form, the number of contacts each individual makes per day is assumed to be large in a large population, and small in a small population.

People recover from the disease according to the third term on the RHS of Eq. (2), and move to the removed category. Assuming all in R are alive, no equation term represents births or deaths in this simplified SIR model. Infectious diseases such as influenza spread rapidly, then 'burn out' over a period of weeks or months. Natural births and deaths in the population can therefore be ignored on this timescale.

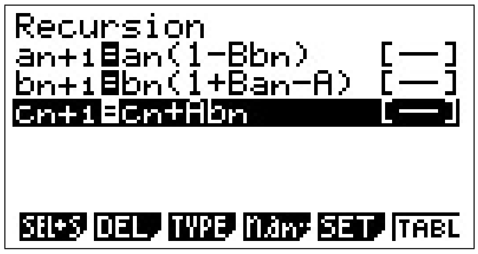
[26]See Section 14.2 for other examples.

14.3.3 Sequence graphing of the discrete SIR model

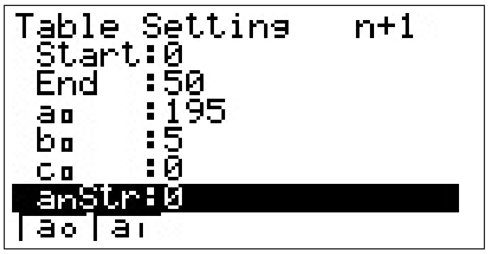
The 9860/CG20/CG50 has three built-in sequences a, b and c . To access them, press MENU 8, then F3 (TYPE) and F2. The left-hand sides of the three sequences should then be the $n+1$ terms. The right-hand sides will vary from those shown.

In calculator terms, the SIR model is
($S \to a$; $I \to b$; $R \to c$; $\alpha \to$ A; $\beta \to$ B)

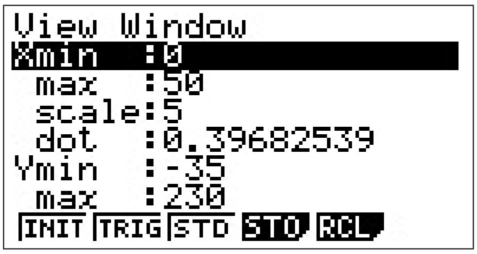
$$\begin{aligned} a_{n+1} &= a_n\big(1-Bb_n\big) \\ b_{n+1} &= b_n\big(1+Ba_n-A\big) \\ c_{n+1} &= c_n + Ab_n \end{aligned}$$



Press EXIT and F5 (SET). Start = 0 means we start with S_0 and I_0; S_1 is the S value after 1 day, etc. Start by plotting 50 points, End = 50, running the system through 50 days. We must also set Xmax to 50 in V-Window for the plots. This value may need to be changed once we know the duration of the epidemic. The initial conditions are contained in a_0, b_0 and c_0.

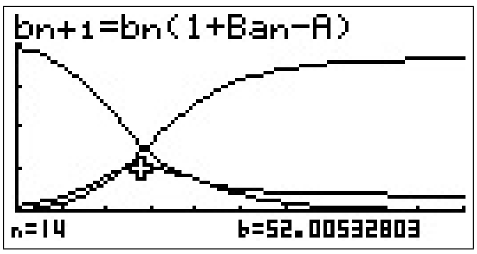


The next step is to choose a V-Window. Press SHIFT F3 and set the values as shown below. $N = 200$ is the maximum value for S, so Ymax must be at least 200. The values shown for Ymin amd Ymax allow the sequence name at the top and sequence point at the bottom in Trace to be displayed without obscuring the graphs. Yscl = 50.



Finally, store the values of α and β in memories A and B respectively. You will need to press MENU 1 to do this.

Press MENU 8 to return to the Recursion screen, TABL to display the table of values and F4 (G-CON) to plot the graphs. Use Trace to explore the values.



Adjust End in SET and Xmax in V-Window so that the disease runs its full course ($I < 1$).

14.3.4 Case study: On a Pacific island

The population on an idyllic Pacific island is stable at $N = 200$. The birth and death rates are small, and can be ignored. The island's post office employs five people,[27] who are all clustered around when a Christmas hamper is opened for customs inspection. Unfortunately the white powder in the hamper is not artificial snow. Within a day, during which the workers mix as usual with other persons on the island, all five workers are infected with a severe and mysterious virus.

You are the only person on the island with some knowledge of epidemiology, gained from Maths in Year 12. The island's Chief Medical Officer needs to know how many people might need to be treated for the virus and whether to call for emergency hospital facilities. The island's hospital can cope with at most 50 patients at any one time. The CMO asks for your help in predicting the course of the epidemic.

Based on the post-office experience, you assume that the incubation period is less than a day, and can be ignored. The mean infectious period, $1/\alpha$ in the SIR model, needs to be estimated, as does β.

1. Based on other viruses, you take the mean infectious period, $1/\alpha$, to be 5 days and the transmission parameter $\beta = 0.0025$. Draw graphs (time plots) of S_n and I_n vs n (time in days). Details of how to do this are in the Appendix, Section 14.3.3.

- What is the approximate duration of the epidemic ($I < 1$)? How many persons don't catch the disease?
- When will the peak of infection occur according to the discrete model? Will the hospital cope?
- Show, from the equation for I above, that the theoretical maximum value[28] of I in the discrete model occurs when $S = \alpha/\beta$.

  Compare this value with the value you obtained from your graph. Use [Trace] to do this. *Why might not these values be exactly the same?*

2. After a week you have some actual data. You find that the mean infectious period is actually 4 days, not 5, and that $I(8) = 30$ and $S(8) = 140$. Experiment with different β to find the value that gives $I_8 = 30$.

Hint: For each β value, use either a graph and [trace] or [table][29] (easier) to find I_8 until you obtain the actual value (or as close as possible).

When you have found the right β value, use the model to predict how many days it will take before the disease dies out and when the peak of infection will occur. *Will the hospital be able to cope according to this model? How many people will avoid catching the disease?*

Solutions in Section 14.4.3.

[27] The island's economy is based around issuing stamps, banking and processing asylum seekers.

[28] At a maximum or minimum, there is no change in I, so that $I_{n+1} = I_n$. This corresponds to the condition $dI/dt = 0$ in the continuous case.

[29] see the end of the Appendix

14.3.5 Exercise: Age distribution of trees in a forest

This is a similar problem to the discrete SIR model, and can be solved in a similar manner using difference equations and the Sequence grapher.

The population of trees in a forest is split into four age groups: b_n is the number of baby trees ($0-15$ years old) at time-point n; y_n the number of young trees ($16-30$ years); m_n the number of middle-aged trees ($31-45$ years old); and o_n the number of old trees (more than 45 years old).

The time step for our difference equations is 15 years.

In order to simplify the model we make the following assumptions:

A. a certain percentage of trees in each age group dies in each time interval;

B. surviving trees age into the next age group each time step. Old trees remain old trees (or die);

C. dead trees are replaced by an equal number of baby trees.

Define α, β, γ, δ as the fraction of dead trees in the respective age groups in each time interval. Then, the difference-equation model is

$$b_{n+1} = \alpha b_n + \beta y_n + \gamma m_n + \delta o_n \qquad \text{(Assumption C)} \tag{4}$$

$$y_{n+1} = (1-\alpha)b_n \qquad \text{(Assumptions A, B)} \tag{5}$$

$$m_{n+1} = (1-\beta)y_n \qquad \text{(Assumptions A, B)} \tag{6}$$

$$o_{n+1} = (1-\gamma)m_n + (1-\delta)o_n \qquad \text{(Assumptions A, B).} \tag{7}$$

1. If the population of trees in time interval n is $N = b_n + y_n + m_n + o_n$, show that the population stays the same size after one more time step, and so by induction the population of trees is a constant N.

This means that we only need solve Eqs. (1) – (3).

2. Three initial conditions are needed in order to fully solve these difference equations. Assume all baby trees initially.

Take $\alpha=0.2$, $\beta=0.5$, $\gamma=0.3$, $\delta=0.2$ and $N=1000$.

Set up Eqs. (1) – (3) with the given parameter values and initial conditions on your calculator. See Section 14.3.3 if you don't know how to do this.

Run the difference-equation model through 10 cycles (150 years) and plot the results. Do the individual populations appear to be stabilising?

Solutions in Section 14.4.4.

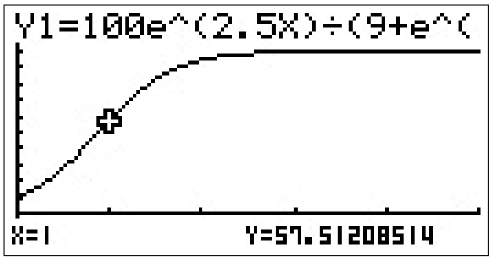
14.4 Solutions

14.4.1 Logistic problem

1. Write down the formula for $P(t)$ and graph it for $0<t<5$.

$P_0=10$ and $K=100$, so $E=P_0/(P_0-K)=-\frac{1}{9}$. After a little algebra,

$$P(t) = \frac{100e^{2.5t}}{9+e^{2.5t}}. \qquad (8)$$



V-Window $[0, 5, 1]\times[-20, 120, 10]$

2. How many fish are there after 1 week?

Read directly from your graph using trace (as in the figure above).

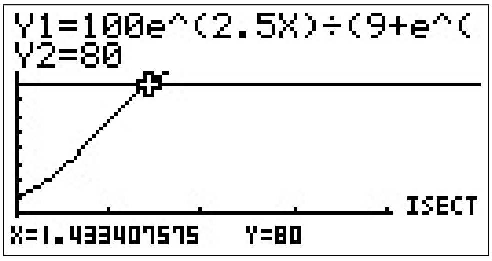
Algebraically, the number of fish after 1 week is, from Eq. (8),

$$P(1) = \frac{100e^{2.5}}{9+e^{2.5}} = 57.5 \quad \text{to 3 digits}$$

There are 57 or 58 fish in the pond after 1 week.

3. How long does it take the fish population to reach 80?

You can do this from your graph of $P(t)$ by graphing $y=80$, and finding its intersection (ISCT in G-Solv) with $P(t)$.



V-Window $[0, 5, 1]\times[-20, 120, 10]$

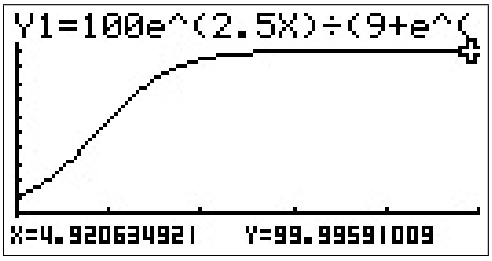
Algebraically, we need to solve $P(t)=80$. Therefore, from Eq. (8),

$$\begin{aligned}
\frac{100e^{2.5t}}{9+e^{2.5t}} &= 80. \\
\therefore \frac{e^{2.5t}}{9+e^{2.5t}} &= 0.8. \\
\therefore e^{2.5t} &= 0.8(9+e^{2.5t}). \\
\therefore 0.2e^{2.5t} &= 7.2. \\
\therefore e^{2.5t} &= 36. \\
\therefore t &= \frac{\ln(36)}{2.5} = 1.43 \quad \text{to 3 digits.}
\end{aligned}$$

It takes about 1.43 weeks or about 10 days for the fish population to reach 80.

4. How many fish are there after a long time?

You can do this from your graph of $P(t)$ by tracing along it and remembering what K means.



Algebraically, multiplying numerator and denominator of Eq. (8) by $e^{-2.5t}$,

$$P(t) = \frac{100e^{2.5t}}{9 + e^{2.5t}} = \frac{100}{9e^{-2.5t} + 1}.$$

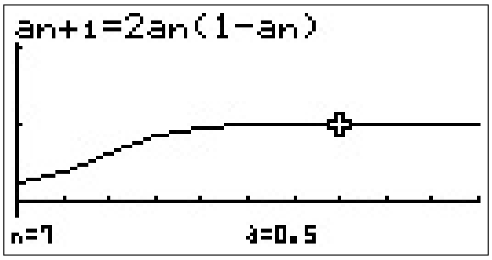
Now $e^{-2.5t}$ goes to 0 as $t \to \infty$, so $P(t) \to 100$. There are 100 fish in the pond after a long time. This is the carrying capacity or maximum sustainable population of the pond according to the logistic model, Eq. (8).

14.4.2 Discrete logistic model

Exercise: Bacteria

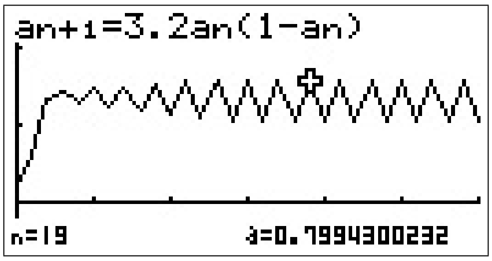
For the logistic difference equation $P_{n+1} = AP_n(1-P_n)$:

with $A=2$ and starting population $P_0=0.1$, the population stabilises at 0.5;



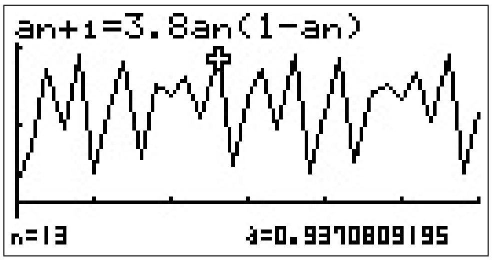
V-Window $[0, 10, 1] \times [-0.3, 1.2, 0.5]$

with $A=3.2$ and $P_0=0.1$, eventually the population alternates between 0.513 and 0.799;



V-Window $[0, 30, 5] \times [-0.3, 1.2, 0.5]$

with $A = 3.8$ and $P_0 = 0.1$, the population varies wildly between 0.1 and 1, with no hope of prediction. This one is weird! You've discovered mathematical chaos.



V-Window $[0, 30, 5] \times [-0.3, 1.2, 0.5]$

What happens with other values of A and P_0?

Exercise: Kangaroo management

The calculation of the population P_n can be done manually on a calculator (following the calculator hint in the question) or using the sequence grapher (MENU 8). The LOGISTIC program (Section 14.2.3) sets up the sequence grapher for the problem here.

We have the logistic difference equation for the kangaroo population

$$P_{n+1} = 1.8P_n - 0.8(P_n)^2,$$

with $P_0 = 0.8$ corresponding to the (end of) year 2005.

Using this and incorporating the rootoxis outbreak in 2006 by subtracting 0.4 (4000 kangaroos) from the 2006 population, we have the following number of kangaroos in subsequent years.

Year	n	P_n	**Number of kangaroos**
2005	0	0.8	8000
2006	1	$0.928 - 0.4 = 0.528$	5280
2007	2	0.7274	7274
2008	3	0.8860	8860
2009	4	0.9668	9668

The number of kangaroos has recovered to 9668 by the end of the year 2009. If we include the migration of 2000 kangaroos at the end of 2008, we have the following numbers.

Year	n	P_n	**Number of kangaroos**
2008	3	$0.8860 + 0.2 = 1.0860$	10, 860
2009	4	1.0113	10, 113
2010	5	1.0022	10, 022
2011	6	1.0004	10, 004
2012	7	1.0001	10, 001
2013	8	1.0000	10, 000

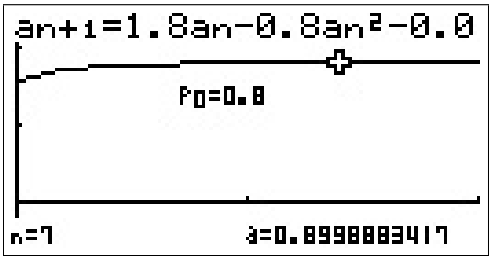
The population in the reserve at the end of 2009 would be 10, 113. In subsequent years, the population declines to the carrying capacity or maximum sustainable population of 10, 000, the population after a long time.

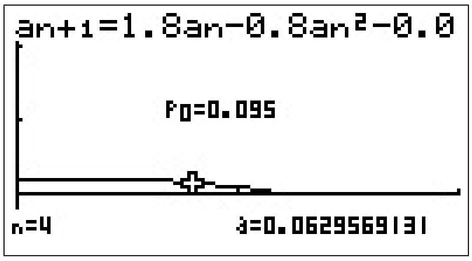
Effect of culling

1. If H kangaroo units are killed each year, this number is subtracted from the value for P_{n+1} that we calculated above, giving the difference equation

$$P_{n+1} = 1.8P_n - 0.8(P_n)^2 - H.$$

2. With $H = 0.072$ and an initial population of 0.8 units, the long-term population will be 0.9 units or 9000 kangaroos.

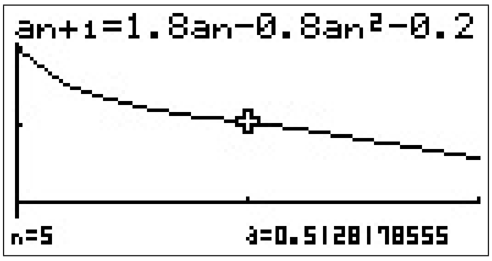




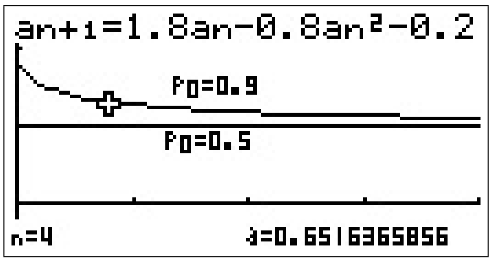
V-Window $[0, 10, 5] \times [-0.3, 1.2, 0.5]$

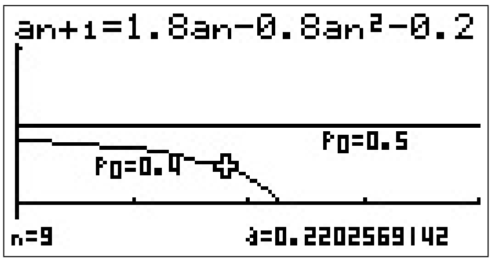
We find that[30] if the initial population is greater than 0.1 kangaroo units, the population will tend toward a stable value of 0.9. If the initial population is less than 0.1 kangaroo units, the population will tend to 0.

3. With $H = 0.24$, the population will die out, no matter what the initial population.



4. With $H = 0.2$, the long-term population will be 0.5 units or 5000 kangaroos, the maximum sustainable population with this level of hunting, provided that the initial population is greater than 5000. If the initial population is less than 5000, the population will die out.



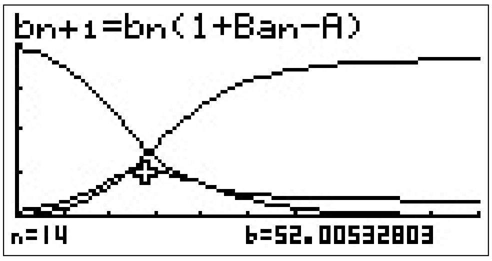


If this level of hunting were chosen, any natural disaster that killed more than a few kangaroos after the population had levelled off at 5000 would bring the population to below 5000, and it would therefore die out. There is no margin for error with this level of hunting. In practice, a value for H smaller than 2000 would be chosen for the number of kangaroos killed annually, thereby leaving a margin to allow for natural disasters.

[30] Theory helps a lot here, but you can reach the same conclusions by experimenting with numbers on your calculator. Using the LOGISTIC program may help with this.

14.4.3 Case study: On a Pacific island

1. $\alpha = 0.2$, $\beta = 0.0025$. The resulting time graph S, I and R vs n (time). The cursor is set on the maximum I value.



For these values of α and β (using trace on the calculator graph):

- the maximum I, the peak of infection, of 52 occurs after 14 days. The number of susceptibles then is 76.9;
- the epidemic lasts 44 days, with 18 persons not catching the disease;
- setting $I_{n+1} = I_n = I^*$ for maximum I_n, we have from Eq. (2),

$$\begin{aligned} I^* &= I^* + \beta S_n I^* - \alpha I^*. \\ \therefore\ I^*\big(\beta S_n - \alpha\big) &= 0. \\ \therefore\ \beta S_n - \alpha &= 0 \qquad \text{as } I^* \neq 0 \text{ at the maximum.} \end{aligned}$$

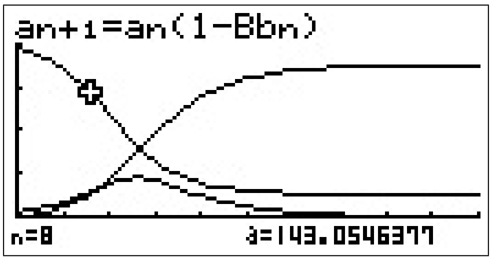
Therefore, at the peak of infection (maximum I_n), we find the same theoretical result as in the continuous case,

$$S_n = \frac{\alpha}{\beta} = \frac{0.2}{0.0025} = 80.$$

From the graph, the value of S at $I_{\max}$, Day 14, is 76.9, close to the theoretical value of 80. The error in the theoretical value is ± 1 day, as S_n is treated as a continuous variable here. Note that $S(13) = 88.2$.

2. A mean infectious period $1/\alpha = 4$ gives $\alpha = 0.25$. We are given $S_8 = 140$, $I_8 = 30$.

The value of $\beta = 0.00281$ gives $S_8 = 143.1$, $I_8 = 30.0$.



For these values of α and β (again using trace):

- the maximum I of 44 occurs after 13 days, with the corresponding value of S, 81.5, again close to the theoretical value $\alpha/\beta = 89$ (the S value on Day 14 is 93);
- the disease runs for 37 days ($I_{37} < 1$);
- 25 persons (S_{37}) avoid catching the disease.

The hospital therefore just copes.

14.4.4 Age distribution of trees in a forest

1. If the population of trees in time interval n is $N = b_n + y_n + m_n + o_n$, show that the population stays the same size after one more time step, and so by induction the population of trees is a constant N.

Adding the four equations gives

$$\begin{aligned} b_{n+1} + y_{n+1} + m_{n+1} + o_{n+1} &= \alpha b_n + \beta y_n + \gamma m_n + \delta o_n + (1-\alpha)b_n \\ &\qquad + (1-\beta)y_n + (1-\gamma)m_n + (1-\delta)o_n \\ &= b_n + y_n + m_n + o_n \\ &= N. \end{aligned}$$

Therefore, the population of trees stays the same size after one more time step, and so by induction the population of trees is a constant N.

Therefore, $o_n = N - \big(b_n+y_n+m_n\big)$, and we need only solve three equations.

2. Three initial conditions are needed in order to fully solve these difference equations. Assume all baby trees initially.

Take $\alpha=0.2$, $\beta=0.5$, $\gamma=0.3$, $\delta=0.2$ and $N=1000$.

Set up Eqs. (1) – (3) with the given parameter values and initial conditions on your calculator.

Run the difference-equation model through 10 cycles (150 years) and plot the results. Do the individual populations appear to be stabilising?

Substituting for o_n in Eqs. (1) – (3) gives

$$\begin{aligned} b_{n+1} &= \alpha b_n + \beta y_n + \gamma m_n + \delta\big(N - (b_n+y_n+m_n)\big) \\ &= (\alpha-\delta)b_n + (\beta-\delta)y_n + (\gamma-\delta)m_n + \delta N \\ y_{n+1} &= (1-\alpha)b_n \\ m_{n+1} &= (1-\beta)y_n. \end{aligned}$$

With $A = \alpha$, $B = \beta$, $C = \gamma$ and $D = \delta$, and using the sequences a, b and c on the calculator, the three equations become

$$\begin{aligned} a_{n+1} &= Aa_n + Bb_n + Cc_n + D\big(N - (a_n+b_n+c_n)\big) \\ &= (A-D)a_n + (B-D)b_n + (C-D)c_n + DN \\ b_{n+1} &= (1-A)a_n \\ c_{n+1} &= (1-B)b_n. \end{aligned}$$

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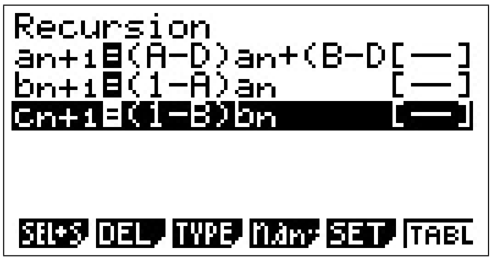
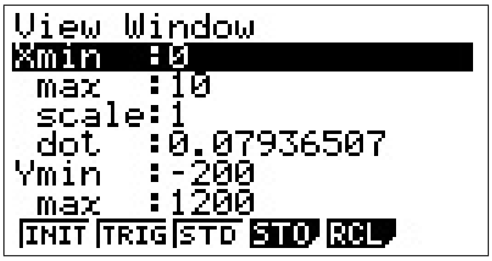
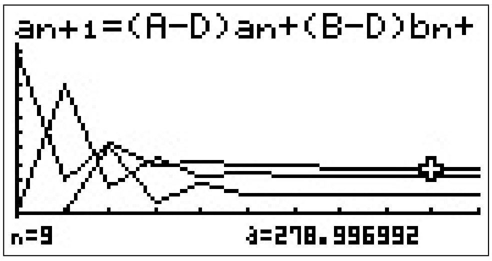
The three equations are shown in the figure below left. We leave the parameters in as A, B, C, D and N, as this makes it easy to change their values in further exploration of the model.

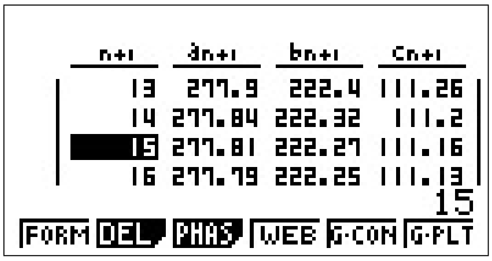
The initial conditions (all baby trees) are $a0 = 10000$, $b0 = 0$ and $c0 = 0$.





The window has n (X) from 0 to 10 cycles and Y from 0 to 1000 trees (± a bit; above right). Set $A = 0.2$, $B = 0.5$, $C = 0.3$, $D = 0.2$, $N = 1000$ and graph (below left).





The three populations (table above right) appear to be stabilising at values around 278 baby trees (a_{n+1}), 222 young trees (b_{n+1}) and 111 middle-aged trees (c_{n+1}).

This gives 389 old trees.

15 Programming and Program Information

15.1 Programming

To be brutally honest, if you are starting out to learn programming (coding), I recommend strongly that you don't do it on a Casio calculator. The programming language is clumsy, difficult to use and the outputs not particularly user-friendly. The best device on which to learn programming (including all computers) is a Texas Instruments TI-84, the TI equivalent of the Casio calculators here. The book *Mathematics on a TI-84/CE*,[31] corresponding to this book, has a chapter on programming, and there is plenty more material available.

However, if you already own a Casio calculator and wish to program, it is possible. The many programs accompanying this book attest to this. To assist you, the Articles folder on the website *canberramaths.org.au* under *Resources* contains one of the few helpful articles I have seen, *Basic Programming for the Casio 9850 Series of Calculators* by Marty Schmude. The 9850 was the predecessor of the 9860, and the article is still relevant to all three Casio calculators here.

15.2 Why use programs?

While the first step in learning any new mathematical technique is doing it by hand until you are competent and understand the proceedure, eventually doing all calculations by hand becomes tedious, particularly if the calculation is lengthy or complex (in the general sense). This is a barrier to the next (and more interesting) step, using the technique to explore some mathematical entity or to model some process. This usually entails repeated calculations with different values of the parameters of the problem; with the best will in the world, no-one is going to do a lot of this. Much more interesting is to generate results, numbers or graphs, that one can interpret in the context of the problem.

This is where a program comes in: it does the same steps you would do by hand, but much much faster and probably more accurately. The program can also generate the form of output best suited to the job. You can enter the world of *what if?* and *what does this mean?*. This is the intention of the programs here, not as a black box'but as technology to do the calculations you could do by hand.

PTO

[31] also available at *canberramaths.org.au* under *Resources*

15.3 Copying programs

15.3.1 From the website

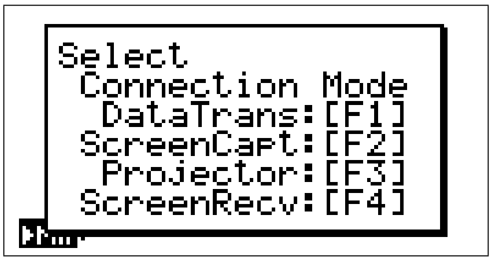
Download the program from the website to your computer desktop or other folder.

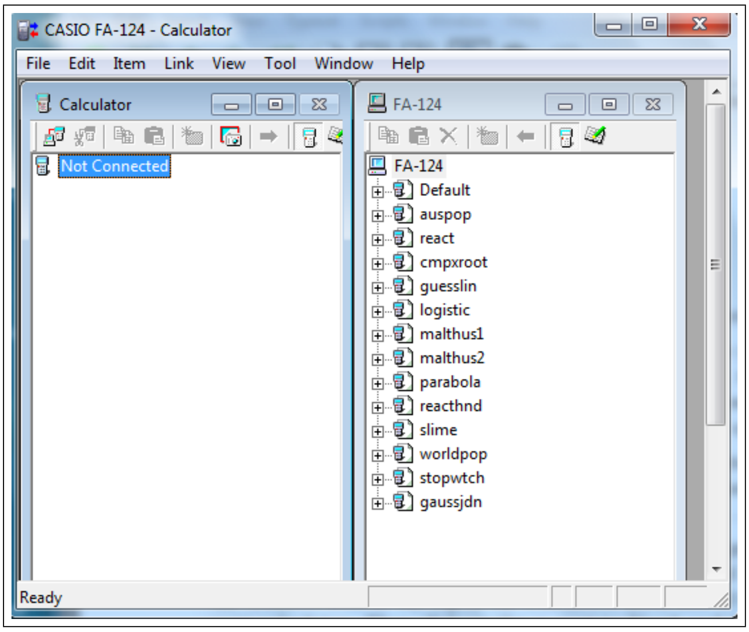
On a 9860

Program names are of the form *name*.g1m or *name*.g2m

You will need *Casio FA-124* (PC only) to transfer the program to your calculator. If you have the original Utilities CD that came with the calculator, use this. Otherwise, *FA-124* is available at *edu.casio.com/forteachers/er/software*. Download the manual there and read the part about installing FA-124 on your particular operating system. Good luck!

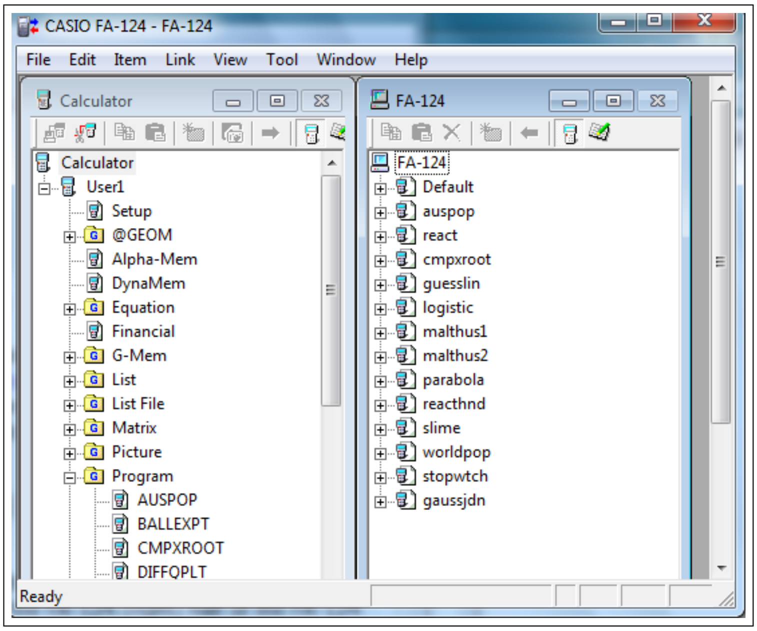
Start *FA-124* and connect your calculator to your computer with the appropriate cord. You will get a message on your calculator asking you what you want to do. Press F1 for data transfer: your screen should now say Receiving. . . [32]





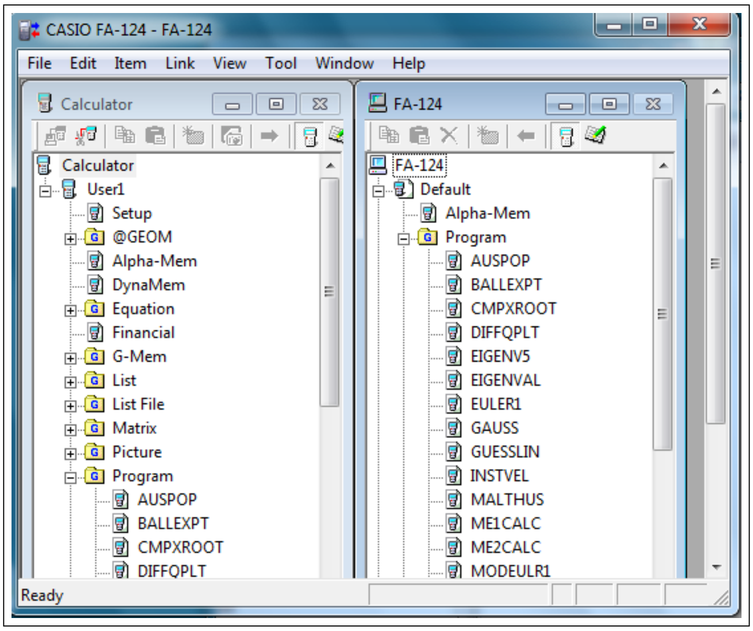
On the menu at the top of the FA-124 screen (above), use *Model Type* in *Tool* to set the calculator type if necessary and *Communications* in *Link* to specify the type of cord, usually USB. Click on the *Connect* (left-hand) icon at the top of the Calculator (left-hand) half of the FA-124 screen to connect your calculator. Then click on the + next to *User1* and the + next to *Program*. That will list all the programs in your calculator (if any) (figure below).

[32] If this does not work, press MENU and select LINK. Make sure the correct cable is selected.



You can read the manual about transferring programs from the computer to the calculator; I did and was none the wiser. Here's what seems to work for me. On the FA-124 (right) half of the FA-124 screen, click on the + next to *Default* and the + next to *Program*. This will list any programs in FA-124 (probably none to start with) (figure below).

Right clicking on a program in this half of the screen gives you the option of deleting it.



Drag the program icon from your desktop (or other folder) onto the FA-124 (right-hand) half of the FA-124 screen; the program name should then appear there. Then drag the program name from the right half to the left half of the screen; the program name should then appear there and be on your calculator.

Dragging a program from left to right stores it on your computer; you can then delete it from your calculator if you wish or drag it on to the desktop and store it in a place of your choosing.

On a CG20/50

Program names are of the form *name*.g3m.

Connect your calculator to your computer with the USB cord. Select USB Flash (F1) on your calculator. You will then get a 'Tips' screen alternating with a 'Caution' screen on your calculator.[33]

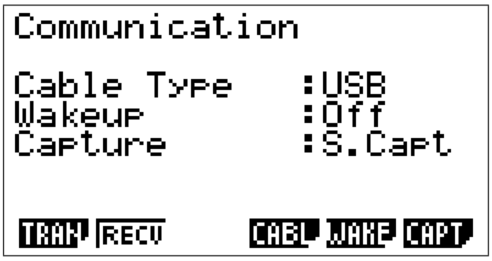
More importantly, a folder should appear on your computer desktop (on a Mac, this is called 'Untitled'); this contains all that is in your calculator memory. Open the folder, click on the arrowhead next to @MainMem, then that next to PROGRAM. Any programs (.g3m) on your calculator will be listed here.

You can copy programs to your desktop by dragging the name onto the desktop; conversely, you copy programs into the calculator by dragging them from the desktop into PROGRAM. If PROGRAM does not exist when you start, dragging a program into @MainMem is supposed to create it.

Remove the directory from your desktop before disconnecting the calculator. The calculator memory will then update.

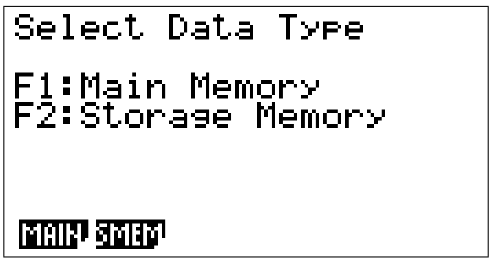
15.3.2 From another calculator

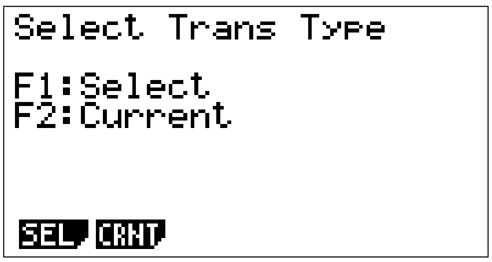
Both calculators need to be in LINK, E in the MAIN MENU. Set the cable type to either USB or 3Pin cable using CABL and Capture to *Memory* using CAPT (the figure below shows *S.Capt* not *Memory* because I am doing a screen capture for this figure). *Wakeup* is *Off* on my calculator.



The receiver presses RECV and waits.

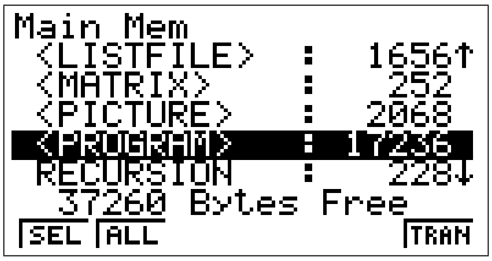
The transmitter presses TRAN, F1 (Main Memory) and F1 again (Select).

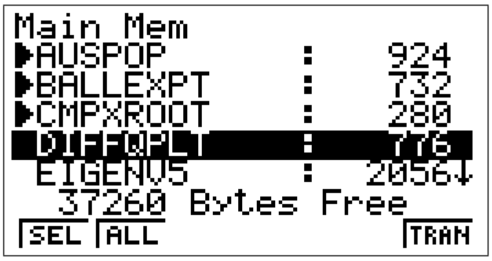




[33] If this does not work, press MENU and select LINK. Make sure the correct cable is selected.

Scroll down to <PROGRAM> and press EXE.





You should then see a list of the programs on your calculator.

Use SEL to select individual programs or ALL to select all of them.

Then press TRAN and F1 to transmit the programs.

15.4 Programs

Note: The programs listed here were all written on a 9860 (.g1m form), then copied directly to a CG50 (.g3m form); the process seems to carry out the necessary modifications of the programs to allow for the higher-resolution screen of the CG50 (and presumably the CG20).

Both forms of the programs are provided in the *Casio Programs* folder.

15.4.1 Activities for Years 9 and 10

Used in activities in the ACT910PROGS folder. Programs are in that folder.

GUESSLIN — guess equation of graphed straight line

Displays a straight line with random slope and Y intercept. You have to guess these values. The calculator keeps score.

Used in *Guess My Line*. Details of the program are given there.

PARABOLA — guess equation of a graphed parabola

Displays a parabola with random coefficients. You have to guess these coefficients. The calculator keeps score. Used in *Parabolic Aerobics*. Details are given there.

REACT — reaction times

The program measures reaction times in different scenarios. Used in *Reaction Times and Statistics*. Details of the program are given there.

REACTHND — reaction times

The program measures reaction times using left and right hands.

Used in *Reaction Times and Statistics*. Details of the program are given there.

SLIME — simulates a projectile shot straight up

Used in *Alien Attack*. Details of the program are given there.

15.4.2 Differential equations

Used in *Differential Equations* in this volume. Details are given there. Programs are in the DIFFEQNS folder.

SLPFIELD

Plots slope fields.

EULER1

Uses Euler's Method to plot approximate solutions to a DE; also gives final values.

MODEULR1/ME1CALC

Uses the modified Euler's Method to plot approximate solutions to a DE; also gives final values. ME1CALC just calculates the final values (no plots).

MODEULR2/ME2CALC

Uses the modified Euler's Method to plot approximate solutions to two coupled first-order DEs or a second-order DE; also gives final values. ME2CALC just calculates the final values (no plots).

15.4.3 Numerical integration

Use in *Numerical Integration* in this volume. Details are given there.

The programs calculate approximate values for $\int_{\mathrm{A}}^{\mathrm{B}} f(\mathrm{X})\,d\mathrm{X}$.

NINTGRPH

Approximates the integral using the *Left-Endpoint Rule*, the *Right-Endpoint Rule*, the *Trapezoidal Rule*, the *Midpoint Rule* and *Simpson's Rule* with **N sub-divisions** (N+1 if N is odd, for Simpson's Rule), and draws the corresponding approximations to the function on each subinterval.

NUMINT

Approximates the integral using the *Left* (**L**) and *Right* (**R**) *Endpoint Rules*, the *Trapezoidal Rule* (**T**) and the *Midpoint Rule* (**M**), all with **N sub-divisions**, and *Simpson's Rule* (**S**) with **2N sub-divisions** (to ensure an even number of sub-intervals).

GLQUAD

Gauss-Legendre quadrature (integration) of orders 9 and 11.

15.4.4 Population modelling

Programs are in the POPN folder.

The programs AUSPOP, MALTHUS and WORLDPOP are meant to be used by teachers as a class demonstration or to show teachers the sorts of activities students could do with the data and the implications thereof.

The other programs are tools for use in particular population models

AUSPOP — Australian population 1906 – 1996

Used in *Population Modelling 1* in Volume 1. Details are given there.

Fits exponential, logistic and Australian Bureau of Statistics (ABS) curves to the population data and allows the models to be used for extrapolation into the future.

LOGISTIC — populations of bacteria or kangaroos

Used in *Population Modelling 2* in this volume. Details are given there.

Sets up the graphics for a population of bacteria obeying the discrete logistic equation or for a population of kangaroos obeying the discrete logistic equation with hunting.

MALTHUS — models the US population 1790 – 1990

Used in *Population Modelling 1* in Volume 1. Details are given there.

Fits an exponential curve to the population for 1790 – 1830, the years for which Malthus had data, and exponential and logistic curves to the population for 1790 – 1990. It also allows the models to be used for extrapolation into the future.

POP — matrix muliplication

Used in *Population Modelling 3* in Volume 3. Details are given there.

WORLDPOP — world population 1940 – 2000

Used in *Population Modelling 1* in Volume 1. Details are given there.

Fits exponential, logistic and US Bureau of Statistics (USBS) curves to the population data and allows the models to be used for extrapolation into the future.

15.4.5 Other programs

CMPXROOT — roots of complex numbers

Used in *Complex Numbers* in Volume 3. Details are given there.

Displays the rectangular values (X, Y) and plots the Nth roots of the complex number $\mathrm{A} + i\mathrm{B}$.

DIFFQPLT — local linearity

Used in *Graph and Calculus Operations* in this volume. Details are given there.

EIGENV5 — eigenvalues and eigenvectors of a matrix

Used in *Matrix and Vector Operations* in Volume 3. Details are given there.

GAUSS — Gauss and Gauss-Jordan reduction

Used in *Matrix and Vector Operations* in Volume 3. Details are given there.

PERIODIC — draws periodic functions

A periodic function is one whose graph over some interval, the period, is repeated over all other such intervals. Used in *Functions and their Graphs* in this volume. Details are given there.

VECTOR — vector operations

Calculates scalar multiple, dot product, norm and projection for vectors of arbitrary length; cross product and triple scalar product for three-dimensional vectors. Used in *Matrix and Vector Operations* in Volume 3. Details are given there.