Mathematics on a Casio 9860/CG20/CG50

Volume 1: Basics

Chapters 1– 8

Peter McIntyre

School of Science

UNSW Canberra

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Contents

Mathematics on a Casio 9860/CG20/CG50

This, the first of three volumes and a supplement, contains material suitable for students and teachers up to about Year 10. Chapter 2, *Getting Started*, is of course suitable for anyone starting to use a Casio 9860 or equivalent.

Volume 1 Supplement: Activities for Years 9 and 10 contains extra activities for the Volume 1 chapters *Coordinate Geometry* and *Probability and Statistics 1.*

Volume 2 of this book contains topics directly relevant to Calculus and its applications, although *Functions and their Graphs* is of more general relevance and also contains details of how to capture screenshots from your calculator, crop them if desired and insert them into documents. The topics in Volume 2 are: *Functions and their Graphs*; *Graph and Calculus Operations*; *Numerical Integration*; *Taylor Series*; *Differential Equations*; *Population Modelling 2 – Logistic and Epidemic Models*; and *Programming and Program Information.*

Program Information gives a list of all the programs in the book, and full information on copying and using these programs.

Volume 3 of this book contains more advanced topics, relevant to students and teachers of Specialist Mathematics and first-year university Mathematics courses. The topics in Volume 3 are: *Sequences and Series*; *Probability and Statistics 2 – Probability Distributions and Hypothesis Testing*; *Matrices and Vectors*; *Population Modelling 3: Matrix Models*; *Financial Mathematics 2 – TVM Calculations*; and *Complex Numbers.*

Calculator versions

The Casio graphics calculator models CG20 AU and CG50 AU are basically the same as the 9860 used here (except for a higher-resolution colour screen). This is probably true of all Casio graphics calculators one level below the ClassPad. There may be minor differences in how the screen looks and in the menus but they all do the same calculations. There are some extras operations on the CG50 but these are not used here.

Calculations, screenshots and figures were (mostly) done on a Casio fx-9860G AU PLUS. The calculator programs were also written on this calculator, and converted to run on the other calculators. The programs are available at *www.canberramaths.org.au* under *Resources.*

Reference

Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS by Barry Kissane.[1] This book is a real bible on everything a graphics calculator can do and how to do it. A must-have for teachers of Years 10 – 12 using Casio calculators. Still very relevant but sadly now hard to find. Happily, a new electronic version is on the horizon.

Meanwhile, a shorter version (no Finance) is in *Learning Mathematics with Graphics Calculators* by Barry Kissane and Marian Kemp, available at *www.canberramaths.org.au Resources Graphics Calculators* in the *Articles* folder.

[1] *Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS* by Barry Kissane. The Mathematical Association of Western Australia, 2003, ISBN 1 876583 24 X.

1 Graphics Calculators and Mathematics

1.1 Introduction

Mathematics is a visual subject, and graphics calculators can provide the picture in a number of important areas of Mathematics. They are also useful in allowing students to explore mathematics numerically and graphically, and to do realistic mathematical modelling, asking *what if* questions of a model. In fact, any modelling that does not use some sort of technology to do calculations quickly soon becomes very boring. Graphics calculators are portable, powerful and, for what they do, relatively cheap.

1.2 What can graphics calculators do?

- All the features of a scientific calculator plus matrices, statistics, probability and complex numbers.
- Multi-line screen, which displays input and output of calculations simultaneously.
- Recall and editing of previous entries and answers.
- Ability to plot Cartesian, parametric, polar and sequence graphs, and generate tables of function values.
- Graph/Calculus (numerical) operations for finding zeros, maxima and minima, and intersections of functions, derivatives at a specified point and definite integrals.
- Statistical functions for organising, analysing and displaying data; probability distributions.
- Can be linked to other calculators, computers and printers for electronic transfer of programs, data, etc and downloading programs from a computer or the web.
- Programmable, with a large number of programs available for downloading.
- Can be used in conjunction with a calculator-based data logger: this enables easy collection of real data, which can be organised, displayed and analysed on the calculator.
- An emulator for computers is available.

At a more mundane level, graphics calculators are fun. Students pick up the operations very quickly (much faster than teachers), and if you can't get your students to use a graphics calculator, there are heaps of games on the web to tempt them.

Getting started is always the hardest, especially when you have to modify or write new courses, but the experience at UNSW Canberra and most other schools and universities at which graphics calculators have been used for a while, is that graphics calculators should not just be an add-on to a course, but should be integrated fully, including their use in tests and exams. They should enhance student understanding, not replace it.[2] This raises some issues, most of which are resolvable. You might like to read *Graphics calculators in the mathematics curriculum: Integration or differentiation?* by Jen Bradley, Barry Kissane and Marian Kemp about their experiences in WA (available at *canberramaths.org.au* under *Resources*).

[2]This is one reason I much prefer to use the Casio 9860 rather than the ClassPad, which does symbolic manipulation (CAS) and has large numbers of 'black-box' commands in a large network of menus.

1.3 Implications for teaching and learning

- A need to think about classroom dynamics.
- Improved student motivation.
- Enhanced modelling and exploration opportunities.
- The potential for using an 'animated' whiteboard.[3]

The graphics calculator is a tool that can assist teachers, but there is a need to think about its use in the classroom. We need to take care that we don't hand over all of our teaching to the technology. The technology needs to be used to enhance students' understanding, not replace it. It provides a valuable tool for drawing links between various content strands and to complement traditional tools such as pencil and paper.

It is important that teachers remain in control of the learning environment, but the classroom dynamics change — there is more exploration and a problem-solving approach to learning can be encouraged. This may require a change in teaching methodology, the teacher becoming a facilitator of student learning by the use of a wider variety of teaching strategies. The whole process needs to be approached with careful thought, as well as a detemination to persevere if early problems arise.

The use of graphics calculators certainly motivates students. It provides them with different ways (graphical and numerical) of looking at mathematics, is less tedious for a number of necessary tasks once the basics have been learnt, promotes student investigation by allowing them to explore concepts independently and enhances modelling opportunities by doing the basic calculations quickly, perhaps using a program.

1.4 Using graphics calculators in modelling

Modelling with mathematics is problem-solving using practical examples, preferably ones that students have some familiarity with. The importance of problem-solving in Mathematics education was well put by Thelma Perso:[4]

> For too long we have concentrated on teaching students the 'bits' and the 'tools' for applying and solving mathematical problems but have paid little, if any, attention to teaching children how to use them. Someone once said that if we taught English like we teach Mathematics, children would spend all of their time practising spelling, grammar, punctuation and sentence structure without ever doing any creative writing. This is a very powerful analogy: we've spent most of the time teaching children how to add, subtract, multiply, calculate and evaluate but given them little opportunity to use these in a creative way.
>
> Problem-solving, which is the creative 'goal' of mathematics, has too often been used as something 'added on' to the Mathematics lesson;[5] problems are given to

[3] Projection of the screen of a computer using the emulator onto a whiteboard provides an 'animated' whiteboard, on which graphs can be annotated, etc using a whiteboard marker.

[4] *Working Mathematically: What Does It Look Like in the Classroom?* in *Mathematics: Shaping Australia*, Proceedings of the Eighteenth Biennial Conference of The Australian Association of Mathematics Teachers, 2001; available online.

[5] This is reinforced by such problems (often just one) only appearing at the end of the chapter in standard high-school Mathematics texts.

> the academically able students who finish their work early, or they're given to children to do for homework at the end of an exercise. Rarely are they the focus of the Mathematics lesson.

I might add that using the tools to solve problems generally and do modelling in particular is the payback or reward for students, who have spent many years learning all the tools. Too often, the reason given for learning the tools is that you will need them in Mathematics next year, a recursive reason. A better reason is that you will now use them to solve all sorts of interesting, fun and even useful problems that you could not have solved without the tools.

A mathematical model usually requires some input data, a hypothesis to explain the data and some calculations using the model to test the hypothesis or to use the model for prediction. Graphics calculators are valuable in the calculation stage if it requires lengthy or a number of similar calculations; doing these by hand limits what can be done with the model (Item 6 below). However, graphics calculators also have multiple other uses in the modelling process.

1. **As a tool in an investigation:** storing data; as a stopwatch; measuring reaction times; use with a data logger. Requires simple programs but these are readily available and easy to use.
2. **Simulation:** tossing coins, dice, etc (fair or biased); games of chance such as roulette; gambling schemes; random walks; even shuffling cards.
3. **Generating a table or graph of results for interpretation or further analysis.** See, for example, *The Best Shape for a Tin Can*, *Alien Attack*, *Probably Finding* π and *Statistics from Birthdays*; details in Section 3.3.2 of this volume.
4. **Data fitting and interpretation.** Analyse your data with a range of statistical tools, including fitting the data with functions (Chapter 5 here); population models (Chapter 6 here); data from the data logger.
5. **Generating statistics for interpretation or further analysis:** e.g. *Reaction Times and Statistics* (Section 3.3.2 here).
6. **Calculations using a model:** various tools for more complicated calculations. Modelling often requires the changing of parameters to see what happens to the system being modelled. Doing this by hand, even with a scientific calculator, is tedious. Once you have changed a parameter, you want to see its effect, and a graph or table is often the best way to do this; programs simplify this even further, allowing you to concentrate on the interpretation (*Population Modelling 2: Non-Exponential Models* in Volume 2; *Population Modelling 3: Matrix Models* in Volume 3).

1.5 Why not just use computers?

Access: Any tool has to be used frequently to be useful. Graphics calculators can be used by students in all classes and at home, in fact anywhere; they are very portable. Graphics calculators can also be used in tests and exams, so the students see directly the value in using them, and they can then be examined on a much wider range of topics.

User-friendliness: Graphics calculators (some of them anyway) have been designed by teachers for ease of use, and are generally much more user-friendly than computer software. Of course, any device as powerful as a graphics calculator will take a bit of time to master. But not as long as you might think.

Conclusion: Once you start with graphics calculators, you won't want to stop.

2 Getting Started

2.1 Resetting the calculator

The following notes assume that all the default options are set. If the calculator has been used by someone else, it is a good idea to reset the calculator before proceeding.

Turn on your calculator by pressing $\boxed{\text{AC}^{/\text{ON}}}$, in the right-hand column of keys. The MAIN MENU screen should appear. This is where you choose what you want to do. You can always return here by pressing the $\boxed{\text{MENU}}$ key (under $\boxed{\text{F4}}$).

Use the arrow keys to highlight the SYSTEM icon (bottom right). Press $\boxed{\text{EXE}}$. Select $\boxed{\text{F5}}$ (Reset) and press $\boxed{\text{EXE}}$. Press $\boxed{\text{F6}}$ $\boxed{\text{F2}}$ to reset the calculator.

> If the screen is too light or too dark, you should adjust the contrast of the screen. To do this, press $\boxed{\text{F1}}$ (Contrast); press the right-arrow key to make the screen darker, the left-arrow key to make the screen lighter. Press $\boxed{\text{MENU}}$ to return to the MAIN MENU.

If you don't want to reset the calculator, use the SET UP menu to set each of the defaults (see Section 2.14).

2.2 First steps

Notice that many of the keys have yellow words or symbols above them. To access these yellow functions press the yellow $\boxed{\text{SHIFT}}$ key, then press the desired key for your yellow operation. Do not hold the $\boxed{\text{SHIFT}}$ key down; it does not act like a shift key for capital letters.

First we'll do some calculations, so select the RUN-MAT screen with the arrow keys and press $\boxed{\text{EXE}}$, or just press $\boxed{1}$.

Try calculating π^2: press $\boxed{\text{SHIFT}}$, then $\boxed{\pi}$ (on the $\boxed{\text{EXP}}$ key at the bottom), and then $\boxed{x^2}$. Now press $\boxed{\text{EXE}}$.

To access the red letters and characters, first press the red $\boxed{\text{ALPHA}}$ key. The cursor switches to $\boxed{\text{A}}$ (and several options appear at the bottom of the screen). To lock in the letter keys, press $\boxed{\text{SHIFT}}$ $\boxed{\text{ALPHA}}$. Another press of the $\boxed{\text{ALPHA}}$ key returns the cursor to normal.

To clear your screen, press the $\boxed{\text{AC}^{/\text{ON}}}$ key.

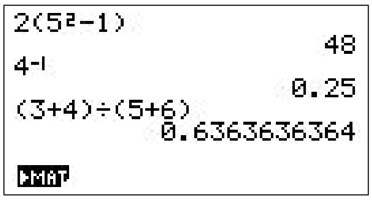
From now on, we will not necessarily mention $\boxed{\text{SHIFT}}$ or $\boxed{\text{ALPHA}}$. We will assume that you know what to do if the character or operation we refer to appears in yellow or red above a key.

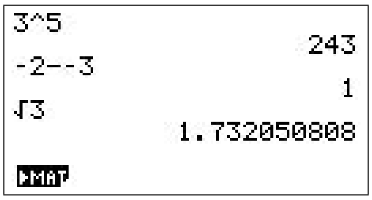
2.3 Syntax

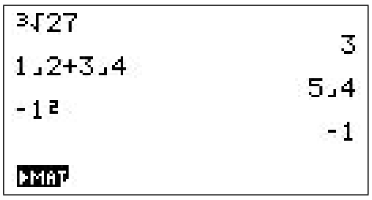
Calculations are performed by constructing an expression in conventional algebraic syntax (including implied multiplication) and then pressing $\boxed{\text{EXE}}$ (which acts as the $\boxed{=}$ key). Brackets are used where necessary.

Some examples of acceptable syntax are given below: each is similar to the way the expressions are conventionally written. Try them on your calculator, observing both the screen display and the final result. It is not necessary to press $\boxed{\text{AC}^{/\text{ON}}}$ (CLEAR) before each calculation. Don't forget the $\boxed{\text{EXE}}$.

$\boxed{2}\ \boxed{(}\ \boxed{5}\ \boxed{x^2}\ \boxed{-}\ \boxed{1}\ \boxed{)}\ \boxed{\text{EXE}}$	for $2(5^2-1)=48$ implied multiplication
$\boxed{4}\ \boxed{x^{-1}}\ \boxed{\text{EXE}}$	for $4^{-1}=0.25$ $\boxed{x^{-1}}$ is $\boxed{\text{SHIFT}}\ \boxed{)}$
$\boxed{(}\ \boxed{3}\ \boxed{+}\ \boxed{4}\ \boxed{)}\ \boxed{\div}\ \boxed{(}\ \boxed{5}\ \boxed{+}\ \boxed{6}\ \boxed{)}\ \boxed{\text{EXE}}$	for $\dfrac{(3+4)}{(5+6)}=0.636\ldots$
$\boxed{\text{F}\leftrightarrow\text{D}}$	for $7 \lrcorner 11 = \dfrac{7}{11}$: fraction $\leftrightarrow$ decimal
$\boxed{3}\ \boxed{\wedge}\ \boxed{5}\ \boxed{\text{EXE}}$	for $3^5=243$
$\boxed{\text{(-)}}\ \boxed{2}\ \boxed{-}\ \boxed{\text{(-)}}\ \boxed{3}\ \boxed{\text{EXE}}$	for $(-2)-(-3)=1$ which $-$? — see below
$\boxed{\sqrt{\ }}\ \boxed{3}\ \boxed{\text{EXE}}$	for $\sqrt{3}=1.732050808$ $\boxed{\sqrt{\ }}$ is $\boxed{\text{SHIFT}}\ \boxed{x^2}$
$\boxed{\sqrt[3]{\ }}\ \boxed{2}\ \boxed{7}\ \boxed{\text{EXE}}$	for $\sqrt[3]{27}=3$ $\boxed{\sqrt[3]{\ }}$ is $\boxed{\text{SHIFT}}\ \boxed{(}$
$\boxed{1}\ \boxed{a^{b/c}}\ \boxed{2}\ \boxed{+}\ \boxed{3}\ \boxed{a^{b/c}}\ \boxed{4}\ \boxed{\text{EXE}}$	for $\dfrac{1}{2}+\dfrac{3}{4}=5 \lrcorner 4=\dfrac{5}{4}$ as a fraction
$\boxed{d/c}$ $(\boxed{\text{SHIFT}}\ \boxed{a^{b/c}})$	to toggle between proper and improper fractions
-1^2 and $(-1)^2$	Which one gives the correct answer?







> The calculator makes a distinction between *negative* and *minus*. When you enter 'negative 2', use the negative key $\boxed{\text{(-)}}$ beside the $\boxed{\text{EXE}}$ key, not the minus key $\boxed{-}$ above the $\boxed{\text{EXE}}$ key.

Note: You can omit final brackets in a calculation.

2.4 Successive commands

You can construct lengthy commands on the screen if you want before pressing $\boxed{\text{EXE}}$, but you can also do chain calculations. The result of the most recent calculation is stored in *Ans*, and can be recalled using $\boxed{\text{SHIFT}}$ $\boxed{\text{(-)}}$.

As an example, try the following key sequences. Watch where *Ans* is automatically recalled.

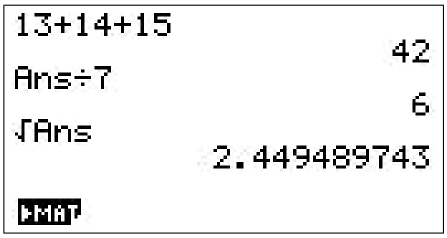
$\boxed{1}$ $\boxed{3}$ $\boxed{+}$ $\boxed{1}$ $\boxed{4}$ $\boxed{+}$ $\boxed{1}$ $\boxed{5}$ $\boxed{\text{EXE}}$

$\boxed{\div}$ $\boxed{7}$ $\boxed{\text{EXE}}$

$\boxed{x^2}$ $\boxed{\text{EXE}}$

$\boxed{x^{-1}}$ $\boxed{\text{EXE}}$

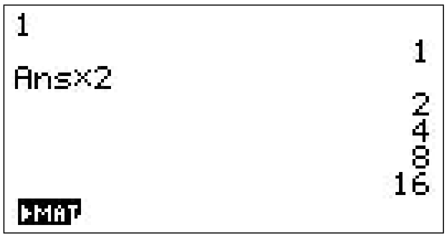
$\boxed{\sqrt{\ }}$ $\boxed{\text{Ans}}$ $\boxed{\text{EXE}}$ key in *Ans* here



What is the affect of the following?

$\boxed{1}$ $\boxed{\text{EXE}}$

$\boxed{\times}$ $\boxed{2}$ $\boxed{\text{EXE}}$ $\boxed{\text{EXE}}$ $\boxed{\text{EXE}}$ ...



If you haven't typed in a new entry, pressing $\boxed{\text{EXE}}$ executes the previous entry.

2.5 Storing and using numbers in variables (memories)

Memories are named alphabetically, as if variables are being given values. To store a specific value in a variable (or memory), first type the value onto your screen, then press $\boxed{\rightarrow}$, type a variable name ($\boxed{\text{ALPHA}}$ followed by a single letter) and press $\boxed{\text{EXE}}$. The value stored in the variable will not change unless you store something else in that variable name.

Let's store 3 in variable/memory P and 4 in Q:

$\boxed{3}$ $\boxed{\rightarrow}$ $\boxed{\text{ALPHA}}$ $\boxed{\text{P}}$ $\boxed{\text{EXE}}$

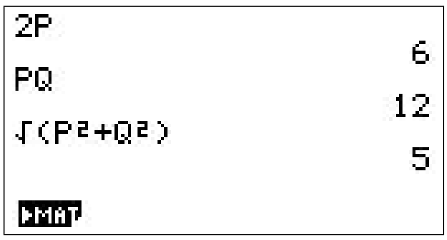
$\boxed{4}$ $\boxed{\rightarrow}$ $\boxed{\text{ALPHA}}$ $\boxed{\text{Q}}$ $\boxed{\text{EXE}}$

Try the following (don't forget the $\boxed{\text{ALPHA}}$).

$\boxed{\text{P}}$ $\boxed{\text{EXE}}$ $\boxed{2}$ $\boxed{\text{P}}$ $\boxed{\text{EXE}}$ $\boxed{\text{P}}$ $\boxed{\text{Q}}$ $\boxed{\text{EXE}}$

$\boxed{3}$ $\boxed{\text{P}}$ $\boxed{x^2}$ $\boxed{\text{Q}}$ $\boxed{\text{EXE}}$

$\boxed{\sqrt{\ }}$ $\boxed{(}$ $\boxed{\text{P}}$ $\boxed{x^2}$ $\boxed{+}$ $\boxed{\text{Q}}$ $\boxed{x^2}$ $\boxed{)}$ $\boxed{\text{EXE}}$

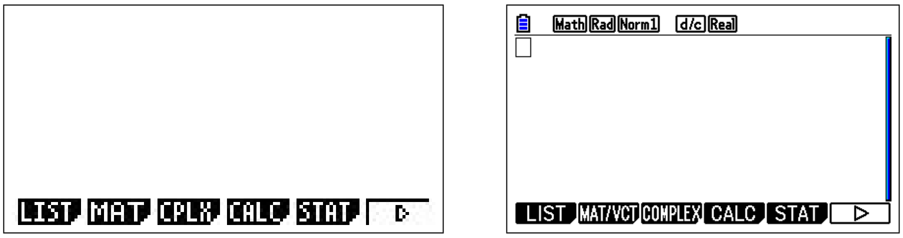


2.6 Recycling expressions

Next, let's try evaluating $\sqrt{P^2+3Q^2}$. Instead of typing the first part of this again, press the left or right arrow. $\sqrt{(P^2+Q^2)}$ should reappear on the screen. Use the arrow keys to change the expression and press $\boxed{\text{EXE}}$. There's no need to move the cursor to the end of the line before pressing $\boxed{\text{EXE}}$.

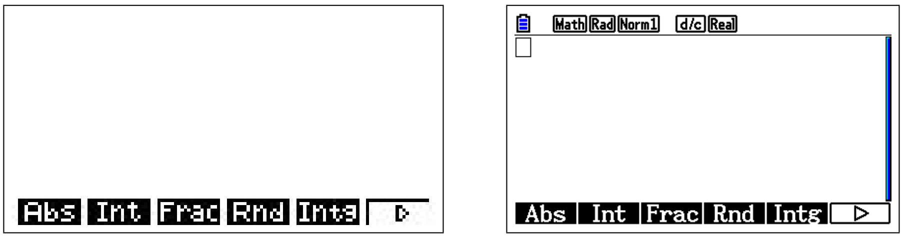
2.7 Using menus

The calculator is mostly menu-driven, which explains why the keyboard is relatively uncluttered. Many of the functions used in maths calculations are accessed by pressing the OPTN key while in the RUN-MAT screen (and most other screens). You then see sub-menus displayed at the bottom of the screen; these you access by pressing the appropriate F key in the top row of keys. F6 is opposite a right arrow, which indicates that there are more sub-menus.



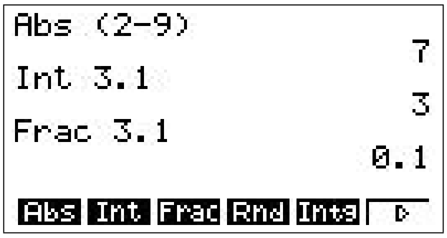
One of the four screens of the 9860 OPTN menu and the corresponding CG50 screen

Press F6 now and select the NUM sub-menu by pressing F4. The commands in the NUM menu are now displayed. We will use some of these in the first few calculations below.



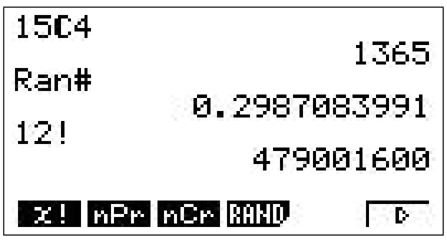
One of the two screens of the NUM menu and the corresponding CG50 NUMERIC screen

Abs (2 − 9) EXE	for $\lvert 2-9\rvert = 7$, the absolute-value function
Int 3 . 1 EXE	for $[3.1] = 3$, the integer part or greatest-integer function
Frac 3 . 1 EXE	for the fraction part of 3.1



Now press EXIT to go back one level of menus. Select the PROB menu for the next few calculations.

1 5 nCr 4 EXE	for $^{15}C_4 = 1365$
RAND Ran# EXE	for a random number on $(0, 1)$
EXIT 1 2 $x!$	for $12! = 479001600$



2.8 Defining functions

The calculator has two display (Input/Output) modes, set in the SET UP menu.

Math tries to display maths as it would appear in proper maths expressions, such as in the lecture notes or textbooks. This looks nice for output, but can complicate input sometimes.

Linear has inputs and outputs on one line. This is easier for inputs, and is used here.

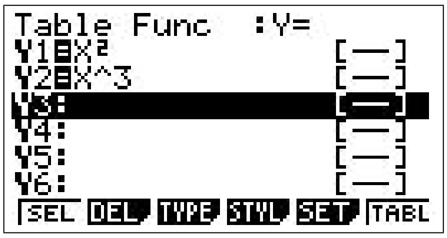
Press MENU to return to the MAIN MENU.

Press 7 or use the arrow keys and EXE to select TABLE. This takes you to the Table Func screen where you enter functions. We could achieve the same thing by selecting GRAPH.[6]

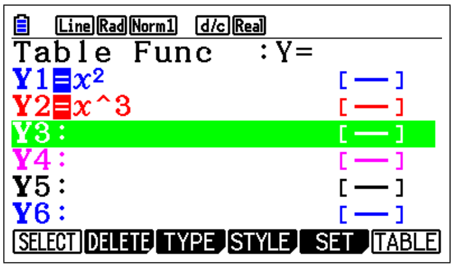
Defining a function is then a simple task. For example, to enter the function $f(x) = x^2$, press X,θ,T x^2. The first key provides whatever independent variable is appropriate (X in this case), and the second squares whatever precedes it. Press EXE to store the function definition.

Note that when you define a function, the equals sign after its name is highlighted. This means the function is 'selected' for tables and for graphing.

Next enter $g(x) = x^3$ into Y2: press X,θ,T $\wedge$ 3 EXE.



9860



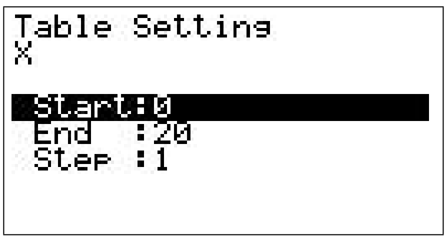
CG50

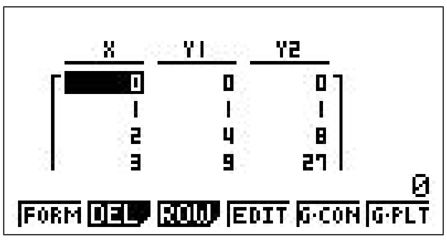
> You can have as many as 20 different functions defined at one time, but you may not want to graph them all at once. Turn off (deselect) any or all of them in the function screen using F1 (SEL(ECT)). Turn them back on the same way. Delete them using F2 (DEL(ETE)).

2.9 Using a table

Check that you still have both Y1 and Y2 selected. Then press SET (F5) to specify what range of values we want in our table.

Set Start to 0, End to 20, Step to 1, pressing EXE after each. Press EXIT to return to the previous screen, then TABL(E) (F6) to display the table. Scroll with the arrow keys.





[6] Functions for tables are the same as functions for graphs.

Up to three Y columns fit on the screen at once, but if we had more functions selected they would all get tabulated and we could find their values by scrolling to the right.

To change either the function definition or table range, press $\boxed{\text{FORM(ULA)}}$ $\left(\boxed{\text{F1}}\right)$. $\boxed{\text{F5}}$ and $\boxed{\text{F6}}$ allow you to go straight to a graph of the functions, either as a line plot (G(PH)·CON) or a point plot (G(PH)·PLT).

2.10 Graphing functions

Press $\boxed{\text{MENU}}$ and select GRAPH to take you to the Graph Func menu. You should see the functions you defined for the table. Press $\boxed{\text{F6}}$ (DRAW) to display the graph.
If a sensible viewing window was set, you should see both curves.

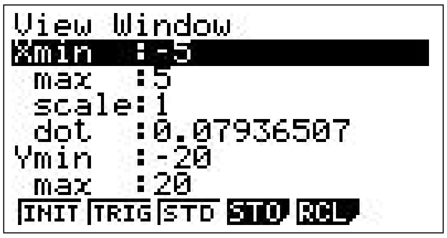
You can change the View Window by pressing $\boxed{\text{F3}}$ (V-Window) if a graph is displayed or $\boxed{\text{SHIFT}}$ $\boxed{\text{F3}}$ if it isn't. Note three preset windows at the bottom. Press each of $\boxed{\text{INIT}}$, $\boxed{\text{TRIG}}$ and $\boxed{\text{STD}}$ to see their effect.

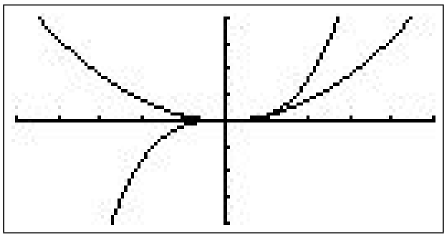
Here we'll do it manually. Make the V-Window go from −5 to 5 in the X direction and from −20 to 20 in the Y direction, pressing $\boxed{\text{EXE}}$ after each new entry (left-hand figure below).

Press $\boxed{\text{EXIT}}$ and $\boxed{\text{DRAW}}$ again to see the same functions graphed in the new window.

You may find that your Y axis now looks a little funny. Go back to the V-Window screen $\left(\boxed{\text{F3}}\right)$. The entries for scale determine where tick marks are placed on the axes. With both set at 1, these marks appear at every integer value in both directions. With the Y range now extending 40 units, that's a lot of tick marks.

Change the Y scale to 5 and graph again (right-hand figure below). Better?



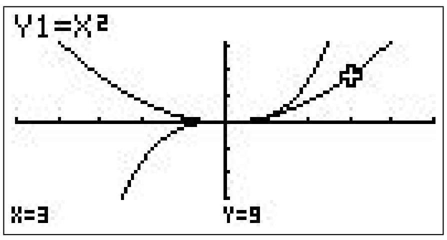


2.11 Tracing and zooming

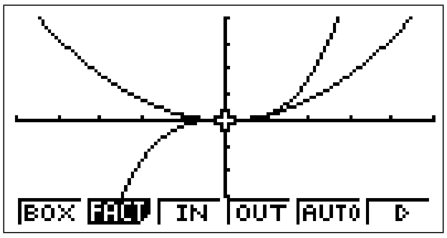
Let's try some other keys. With your graph on the screen, press $\boxed{\text{F1}}$ (Trace). A blinking cursor will appear at the origin. Notice that the coordinates of this point appear at the bottom of the screen (if not, turn on in $\boxed{\text{SET UP}}$; Section 2.14 below).

Use the right arrow to move the cursor along the graph of Y1. Type in an X value followed by $\boxed{\text{EXE}}$ to go to the point on the graph with that X value.

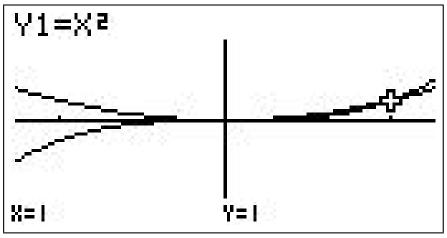
Press the up and down arrows to see what they do.

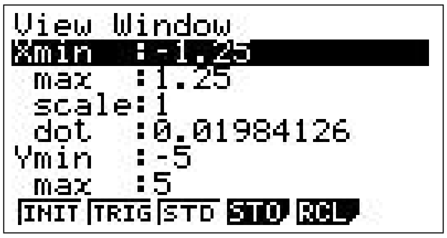


Next press F2 (Zoom) .



Press IN and EXE to zoom in (magnify). The calculator now zooms in on the spot where the cursor was. Look at the new View Window.





Press Trace, move the cursor to a different spot on the graph and press IN again.

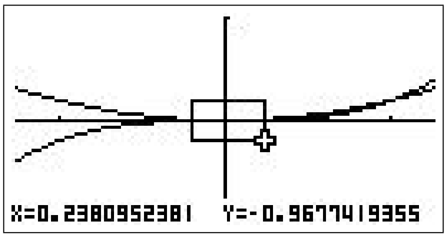
OUT works in the same way, but zooms out.

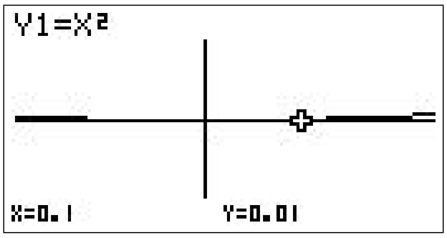
Now try Zoom BOX.

Starting at the origin, move the cursor a bit up, a bit left and press EXE.

Move the cursor down and to the right; the screen displays an outlined box, with the top corner fixed and the cursor at the opposite corner.
Make the box enclose some interesting part of the graph and press EXE again.

The graphs are redrawn with your selected box taking up the whole screen.





There are a number of other options in the Zoom menu.

2.12 Finishing up

If you don't press any keys for five minutes or so, the calculator turns itself off. However, you can also turn it off with (surprise!) SHIFT AC/ON. All settings, function definitions, variable values, etc are remembered when the calculator is off.

2.13 Significant digits and calculations

When we write down a numeric answer to a problem, we have to make some decision as to how many digits we put in the answer. Firstly we have to decide how accurate our answer is — this will depend on the accuracy of the data we use in our calculations and on whether we introduce any further loss of accuracy by our method of calculation. These considerations are discussed below. Secondly, we have to decide what is a *sensible* number of digits to put in our answer. We wouldn't give the distance to a star to a number of digits that takes us down to millimeters, even if we knew it that accurately.

In specifying the accuracy of an answer, we usually give the number of decimal places (DP) or the number of significant digits (SD) that we think are appropriate. We will use significant digits in this course, because it fits in well with scientific notation for specifying numbers, for example 3.56×10^{-5}.

Examples

1. 0.0036 has 2 SD — leading zeros are not significant. We could also write this number as 3.6×10^{-3}, making the number of significant digits clear.

2. 2 may have one SD or may also be an exact number and so implicitly have an infinite number of SD.

3. 2. has 1 SD.

4. 2.00 has 3 SD — trailing zeros after the decimal point are significant.

5. 240 000 000 has 2 SD — trailing zeros before a decimal point are not significant unless specified as being so. Write as 2.4×10^8.

6. Be careful with rounding. If your answer is an approximation, you should specify the number of significant digits which are accurate.

$$
\begin{aligned}
\pi &\approx 3.141592654 \qquad \text{(10 SD)} \\
&\approx 3.14 \qquad \text{(3 SD: rounding down)} \\
&\approx 3.142 \qquad \text{(4 SD: rounding up)} \\
&\approx 3.1416 \qquad \text{(5 SD: rounding down)}
\end{aligned}
$$

There are at least three sources of concern about significance of digits in an answer.

- **The accuracy of available data**

  You should not expect more significant digits in any answer than there are in the least accurate input to the calculation. *Calculation steps never add significant digits, though your calculator will happily add digits!*

  Example: A population of 240 000 000 grows by 2% per year for 3 years. What's the population after 3 years?

  Numerically, the answer is $240\,000\,000 \times (1.02)^3 = 254\,689\,920$. However, the original number had only 2 SD (2 and 4) and possibly 3 — we don't really know about the first 0. The best answer to the question is “about 250 000 000”, or possibly “about 255 000 000” if we thought the first 0 was significant. The answer 254 689 920 is definitely wrong.

- **The finite precision of your calculating device**

  Subtraction of nearly equal numbers can be a real significance killer

  Example: If we subtract two numbers that only differ in the tenth digit, the answer has only one significant digit.

  Adding or subtracting numbers that are very different in magnitude can also lead to inaccuracy

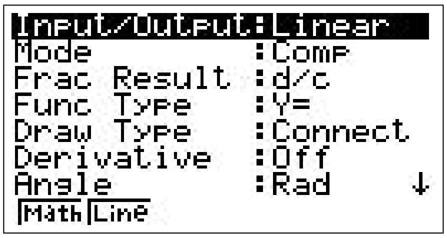
  Exercise: We all know that A + B − A = B.
  Store 10^6 in memory A.[7]
  Store $\sqrt{(2)} \times 10^{-5}$ (correct − sign?) in B.
  Evaluate A + B − A.
  Repeat with A = 10^7, 10^8, 10^9. Explain.

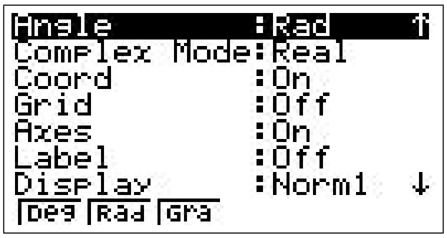
- **Loss of significance due to the way we manipulate numbers**

  Don't discard digits in an intermediate result. The only time you should round off is at the end of a calculation. Preferably use your calculator to do the calculation all in one go — it keeps 15 digits.

2.14 The SET UP menu

Press [MENU]. Select RUN-MAT and press [SET UP] ([SHIFT] [MENU]).





Input/Output
Discussed in Section 2.8.

Mode
Selects the number base you are using in your calculations. **Comp** for general computations is the default.

Frac Result
Selects whether fractions are displayed as proper or improper fractions.

Func Type
Determines the type of function to graph: standard graphs (Y=), polar graphs (r=), parametric graphs, vertical-line graphs (X=C) or inequality graphs (press [F6] to see these).

Draw Type
Sets a connected (line) graph (**Connect**) or a point graph (**Plot**).

[7] [EXP] [6] [→] [ALPHA] [A] [EXE]

Derivative

Determines whether (an approximation to) the derivative will be displayed when using Trace on a graph.

Angle

Set either degrees or radians.

Complex Mode

Determines whether complex results will be displayed and in what form. In *Real* mode, complex calculations will still be carried out if you include an i in the calculation. See the manual for more details.

Coord

Earlier in Trace you used the arrow keys to move along graphs, with the cursor location shown at the bottom of the screen. If you choose **Off** here, the cursor will appear and move, but no X and Y values will appear.

Grid

On tells the calculator to use the tick marks you set on the axes (with scale) to create a grid on the graph screen. Try it now — press MENU 5 and DRAW to display the graph. Return to the SET UP menu when you have finished. **Off** (the default) turns this option off.

Axes

On (the default) tells the calculator to display the X axis when it is between Xmin and Xmax and the Y axis when it is between Ymin and Ymax. **Off** turns off the display of axes.

Label

On turns on the display of axis labels, i.e. the characters x and y. However, the placement of these characters on the screen is not useful, because they can appear in strange locations (try it and see); most of us can figure out which is the x axis and which is the y axis anyway.

Leave it set on the default setting **Off**.

Display

Fix allows you to set the number of decimal places in numerical output; useful in financial calculations,in which the numbers are in dollars and cents.

Sci displays all numbers in scientific notation.

Norm1 displays numbers smaller than 0.01 in scientific notation, *Norm2* numbers smaller than 0.000000001 (10^{-9}). Toggle between the two with Norm.

Eng displays numbers similar to scientific notation but adjusted so that the exponent is always a multiple of 3.

Norm1 preferred unless you like counting zeros.

Simplify

Auto or *Manual*.

Note: The SET UP menu will vary in the different options in the MAIN MENU .

2.15 The graphics screen and accuracy

2.15.1 The graphics screen

Modified from D. Pence, *Calculus Activities for TI Graphic Calculators*, 2nd ed, PWS Publishing, 1994.

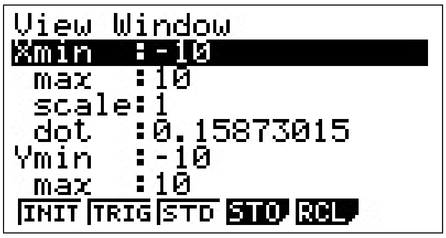
Before graphing a function f on a graphics calculator, you must first specify the range of X values to be considered (i.e. restrict the domain to a finitely bounded interval of numbers) and the range of Y values to be allowed. Setting these ranges defines the scales and locates the coordinate axes on the screen. Generally we will use the estimated range of f as the Y range. However, the true range of f may be an unbounded interval, and so it may not be possible to use it for the Y range.

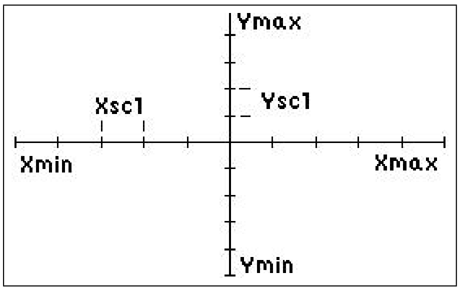
The calculator screen consists of rows and columns of little rectangles, called *pixels*. Compared to a television screen or computer monitor, the pixels on the calculator screen are relatively large. On the 9860, the screen consists of a grid 132 pixels across and 64 pixels high.[8] You can see the individual pixels by turning up the contrast. To do this, press MENU E, then hold down the right-arrow key. Turn the contrast back down again by holding down the left-arrow key until you reach the correct setting for you.

Any point (X, Y) on the screen lies in the range

$$\mathrm{Xmin} \leq \mathrm{X} \leq \mathrm{Xmax}, \qquad \mathrm{Ymin} \leq \mathrm{Y} \leq \mathrm{Ymax},$$

where Xmin, Xmax, Ymin, Ymax are set using V-Window.





The first *column* of pixels on the left has the X coordinate *Xmin*, and the last but one *column* of pixels on the right the X coordinate *Xmax*. Similarly, the *rows* of pixels have equally spaced Y values between *Ymin* and *Ymax*. There is a similar grid of pixels for any computer-drawn coordinate graph, but on a high-resolution monitor or printer the 'dots' or pixels are so small that the eye cannot pick them out.

Each pixel has coordinates (the coordinates of the centre of the pixel) that appear at the bottom of the screen as you move the cursor around with the arrow keys. However, each pixel actually represents a region in the plane (i.e. infinitely many points). If a point to be plotted lies somewhere in the region represented by a pixel, the pixel is turned on (darkened). In Trace, the X value at the bottom of the screen is the X coordinate of the pixel, but the Y value is the value of the function at that X value. This is usually different to the Y coordinate of the pixel, but must lie in the range of Y values covered by that pixel.

The X axis will appear on the screen if $\mathrm{Ymin} \leqslant 0 \leqslant \mathrm{Ymax}$. Scale marks are drawn along the axis at a regular spacing *Xscl*, also set in V-Window. Similarly, the Y axis will appear if

[8] The screen used for graphics is actually 127×63 pixels.

Xmin $\leqslant 0 \leqslant$ Xmax, with scale marks spaced by *Yscl*. No scale marks appear if Xscl/Yscl is set to 0. If Xscl/Yscl is too small, the scale marks will run together and the axis will look as though there is a parallel line running right next to it. This is often confused with the graph of the function you are trying to plot.

There are several commands in the $\boxed{\text{Zoom}}$ menu which provide convenient ways to change the V-Window settings. These commands also contain an implied GRAPH command.

SQR makes the scales on the X and Y axes the same. For equal scales, (Xmax − Xmin) is $127/63 \approx 2.3 \times$(Ymax − Ymin). ZSquare is useful if the shape of the function you are plotting is important, such as a semi-circle.

2.15.2 Function graphers — Getting the picture

Modified from T.P. Dick and C.M. Patton, *Student guide to using technology in calculus*, PWS-Kent, Boston, 1992.

A graph provides a powerful interpretational tool by giving us a visual picture of the input–output pairs a function process produces, but it requires much time and effort to prepare graphs by hand. With the availability of computer and calculator graphics technology, we have a much greater opportunity to exploit graphical representations of functions.

Be forewarned: the graphical evidence provided by a machine can be open to perceptual illusions and therefore to misinterpretations. To make intelligent use of graphical tools, it is important to understand their limitations. In other words, getting the most out of graphics technology requires not only knowing how it can be used, but also how it *can't* be used. Let's look at some of the issues you must be concerned with when using graphing technology.

First of all we need to understand how a graph is produced and displayed by a machine. The screen of a calculator or computer is divided into a rectangular grid of small square picture elements called *pixels*. Each pixel has coordinates corresponding to a single point in the plane, but a pixel does not really represent a point. Rather, a pixel represents a small rectangle containing infinitely many points. The specific point given by the coordinates of the pixel may represent the centre or a corner of the pixel, depending on the particular machine or software. If we want the machine to indicate a certain point in the plane, we have to light up or darken the particular pixel containing that point.

The viewing window

Every graphics package on a computer or calculator has a necessarily limited screen. You might think of this screen as a window from which you can view part of the Cartesian plane. By moving this window around the plane, we can focus our attention on various parts of the graph of a function. This window is also a *lens* through which we can obtain both close-up and distant views of the graph by changing scale. Finding the best window locations and scales are navigational skills for finding our way about a function's graph.

Graphical behaviour can be hidden

- by lying beyond the bounds of the viewing window,
- by scale — zooming in obscures global information about the graph; zooming out obscures local information or detail about the graph,
- by numerical limitations — the choice of which pixels to light up or darken is determined by numerical computations, which in turn are subject to the usual round-off, cancellation, underflow and overflow errors that may occur.

Exercise: Graph the function defined by the formula

$$f(x) = \frac{x^3-1}{x-1},$$

using Y1 and View Window $[-6,5,2]\times[-2,10,2]$. Be careful with brackets.

(a) What is $f(1)$? Check your calculator's reaction to this calculation on the RUN screen (MENU 1) as follows:

First store 1 in memory X: 1 → X EXE .

Then evaluate Y1: VARS F4 F1 1 EXE .

(b) Does this problem show up on your graph? Use Trace to investigate. Explain.

(c) Now change the X View Window to $[-6.3,6.3,2]$ and repeat (b). Why the difference?

(d) Zoom in using Zoom F3 . Move the cursor close to the graph at X = 1 and press EXE .

Do this repeatedly (about 10 times) until an irregularity appears.

Zoom in several more times. Why do you think this might happen? Look at the View Window.

The graph of f has a *hole* at $x=1$, as the function is not defined there. However, a machine plot of this function's graph could have several different appearances near $x=1$:

- the graph may appear to be continuous if $x=1$ falls in between the X coordinates of two adjacent pixels (but not close enough to either to have the function value affected noticeably by numerical imprecision);
- there may be a missing pixel if the X coordinate of a pixel is exactly 1;
- there may be a jagged jump or spike if the X coordinate of a pixel is very close, but not equal, to 1, due to numerical imprecision.

You are more likely to observe the visual effects of numerical imprecision at small scalings. Even a continuous function's machine-plotted graph may break apart under repeated zooms because the function cannot be calculated accurately enough.

Exercise: Graph the function (Radian mode)

$$y = \frac{2\cos(x)-2+x^2}{x^4}$$

using a View Window of:

(a) $[-6.3,6.3,0]\times[-0.1,0.2,0]$;

(b) $[-0.001,0.001,0]\times[-0.1,0.2,0]$.

What do you observe?

This is another example of numerical instability — the calculator cannot calculate $\cos(x)$ accurately enough when its argument is very close to 0.

The dimensions of the viewing window are specified by the View Window parameters *Xmin*, *Xmax*, *Ymin*, *Ymax*. We also know that the 9860 graphing screen is 127 pixels wide by 63 pixels high. (The actual screen is 132×64, but only 95×63 is used for graphing.) From these numbers we can calculate the dimensions of a pixel.

Example: Suppose we specify View Window parameters $[-10, 10]\times[-5, 5]$. Find the dimensions of a pixel.

On the 9860, the coordinates of a pixel correspond to the centre of the pixel. The pixels are (effectively) touching, so that each interval between adjacent pixels is equal to the width of a pixel. Since there are 127 pixels across the width of the graphing screen, there will be 126 pixel-width intervals between the centre of the leftmost pixel representing *Xmin* and the centre of the rightmost pixel (of the 127) representing *Xmax*. Therefore the *width* of a pixel is

$$\Delta\text{X} = \frac{\text{Xmax} - \text{Xmin}}{126} = \frac{10-(-10)}{126} = \frac{20}{126} \approx 0.1587.$$

Similarly, as there are 63 pixels vertically, the *height* of each pixel is

$$\Delta\text{Y} = \frac{\text{Ymax} - \text{Ymin}}{62} = \frac{5-(-5)}{62} = \frac{10}{62} \approx 0.1613.$$

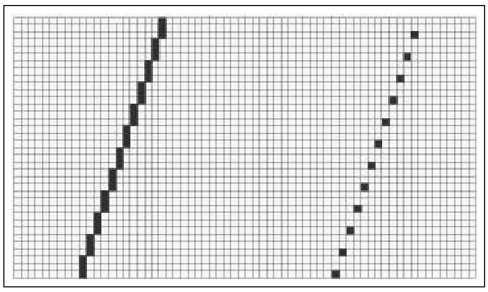
Exercise: If the View Window is $[-6.3, 6.3]\times[-3.1, 3.1]$ (set by INIT(IAL) in V-Window), find the dimensions of each pixel. This is often a useful View Window to use.

When graphing a function, the calculator starts with the first (leftmost) column of pixels. It computes the ordered pair (X, f(X)) using the X coordinate of that column (Xmin) as the value of X, and darkens the pixel in that column whose Y coordinate is closest to f(X), provided f(X) is within the vertical range of the window, i.e. between Ymin and Ymax. This process is then repeated for each column of pixels from left to right.

Lines may be plotted either connected (solid line) or dotted: press SET UP and set *Draw Type* to *Con* (connected) or *Plot* (dotted).

For a dotted line, the calculator will darken at most one pixel in each column (so the function graph on screen will pass the vertical-line test). For a connected line, the function grapher will darken additional pixels to give the visual perception of an unbroken graph. The figure below shows two calculator graphs of the same line; one is plotted in *Connected* mode and the other is plotted in *Dot* mode.

In either case, note that the calculator graph of a function is simply a finite collection of darkened pixels, whereas the true graph generally consists of infinitely many points.



Exercise: Plot Y = 5X, first connected, then dotted (Plot), using a View Window of $[-10, 10, 2]\times[-10, 10, 2]$. The type of graph is set in SET UP.

2.16 Useful websites

charliewatson.com/casio — a variety of useful stuff.

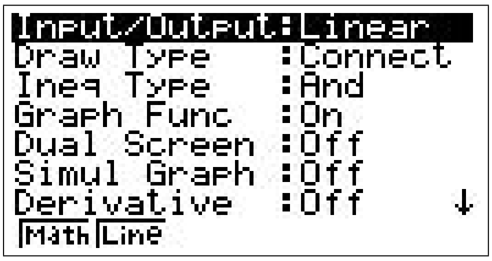
edu.casio.com/en — the Casio education web site.

casioeducation.com.au — the Australian Casio website.

3 Coordinate Geometry

3.1 Setting Up

Press $\boxed{\text{MENU}}$ $\boxed{5}$ for GRAPH mode. Press $\boxed{\text{SET UP}}$ ($\boxed{\text{SHIFT}}$ $\boxed{\text{MENU}}$).[9]



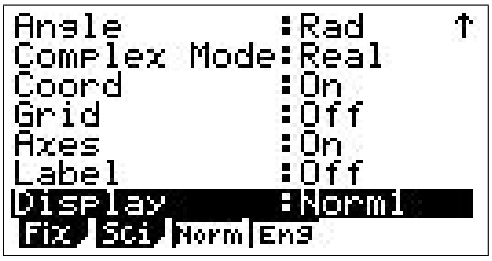
Input/Output: In *Linear* mode, commands are typed on one line, with arguments in brackets. In *Math* mode, the calculator tries to display commands in mathematical notation, with small boxes in the relevant positions for the inputs. *Linear* is used here.

Draw Type: *Connect* means graphs are a continuous line whereas *Plot* plots a set of points (those calculated) which are not connected. *Connect* is better in most cases.

Graph Func: *On* means the equation of the function is displayed *while* its graph is being drawn. Harmless, so leave On.

Simul Graph: *On* means that, if two or more functions are being drawn, they will be drawn simultaneously; *Off* means they are drawn sequentially. Unless you are simulating a race, leave set on *Off*.

Derivative: *On* means that, when a graph is being *traced*, (an approximation to) the value of the derivative at a point will be displayed as well as the function value. Leave *Off* unless required.



Angle: *Rad* (radians) is the appropriate setting for Mathematics.

Coord: *On* means the coordinates of the cursor will be displayed when tracing a graph.

Axes: *On* means Cartesian axes will be drawn on plots where appropriate.

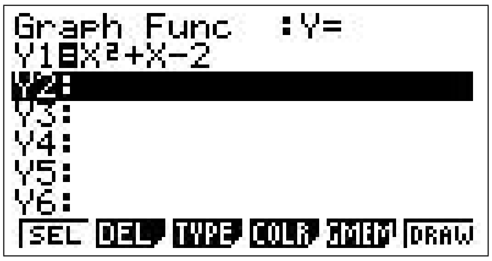
Display: *Fix* allows you to set the number of decimal places in numerical output; useful in financial calculations,in which the numbers are in dollars and cents. *Sci* displays all numbers in scientific notation. *Norm1* displays numbers smaller than 0.01 in scientific notation, *Norm2* numbers smaller than 0.000000001 (10^{-9}). Toggle between the two with $\boxed{\text{Norm}}$. *Eng* displays numbers similar to scientific notation but adjusted so that the exponent is always a multiple of 3. *Norm1* preferred unless you like counting zeros.

[9] The CG50 has several more settings.

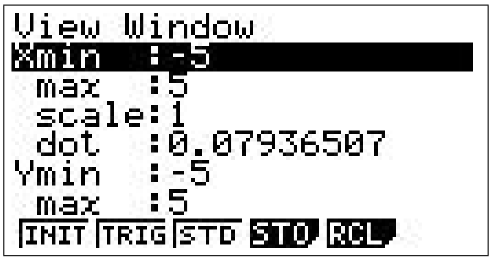
3.2 Basic operations

3.2.1 Graph $f(x) = x^2 + x - 2$ for $-3 < x < 3$

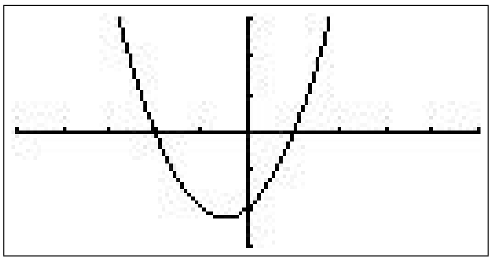
Press $\boxed{\text{MENU}}$ $\boxed{5}$: set $\text{Y1} = \text{X}^2 + \text{X} - 2$.
The key $\boxed{\text{X},\theta,\text{T}}$ gives X.
Note the highlighted = sign — this function will be plotted when you press $\boxed{\text{DRAW}}$.
Use $\boxed{\text{F1}}$ (SEL) to toggle the function off/on.



Press $\boxed{\text{SHIFT}}$ $\boxed{\text{F3}}$ (V-Window): set the View Window as shown in the figure.
Press $\boxed{\text{EXE}}$ after each value, *including the last.*
Note the difference between the $\boxed{-}$ key (*subtract*) and the $\boxed{(-)}$ key (*change sign*).
Xscale and *Yscale* ($= 1$) are the distances between tick marks on the axes (0 gives no tick marks).
Press $\boxed{\text{EXIT}}$ to return to the Graph Func screen.



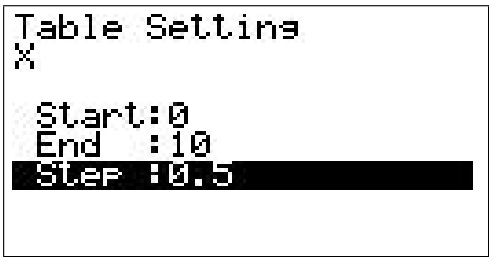
Press $\boxed{\text{F6}}$ (DRAW) to graph the function.



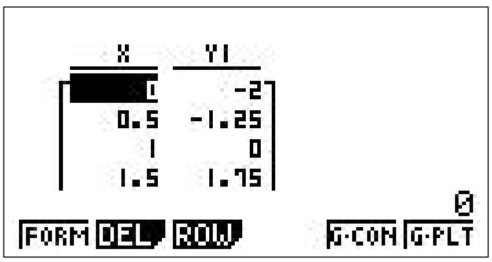
3.2.2 Generating a table of function values

- Press $\boxed{\text{MENU}}$ $\boxed{7}$. Make sure the = signs of the functions you want are highlighted.
- Set the table range using $\boxed{\text{F5}}$ (SET): Start = 0, End = 10, Step = 0.5.

  This generates values automatically, starting at X = 0, incrementing in steps of 0.5, up to X = 10.

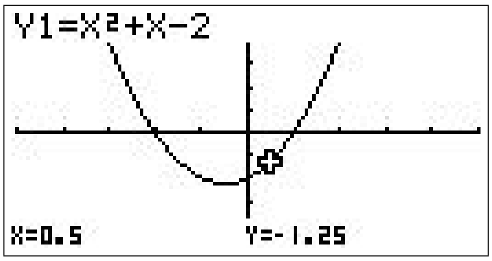


- Press $\boxed{\text{EXIT}}$, then $\boxed{\text{F6}}$ (TABL). The highlighted number is also displayed at the bottom right of the screen. Use the arrows to move around the table.

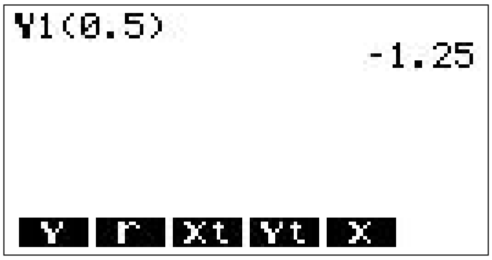


3.2.3 Find (estimate) $f(0.5)$

- **From a table** — see above.
- **On the graph**
  - Press $\boxed{\text{F1}}$ (Trace) and use the left and right arrows to move the cursor along the curve (but note the problem that arises when trying to reach X = 0.5). The up and down arrows move between functions if there is more than one graphed.
  - Type in the X value and press $\boxed{\text{EXE}}$ to move to the desired point on the graph. Note the coordinates at the bottom of the screen.



- **On the RUN screen**
  - Press $\boxed{\text{MENU}}$ $\boxed{\text{1}}$ to go to the RUN screen.
  - The calculator knows f by the name Y1. We need to evaluate Y1(0.5).
    Y is $\boxed{\text{VARS}}$ $\boxed{\text{F4}}$ $\boxed{\text{F1}}$. You can't just type $\boxed{\text{Y}}$. Then put in (0.5) and press $\boxed{\text{EXE}}$ to evaluate Y1.



- *Answer*: $f(0.5) = -1.25$.

PTO

3.2.4 Find the zeros of $f(x) = x^2 + x - 2$

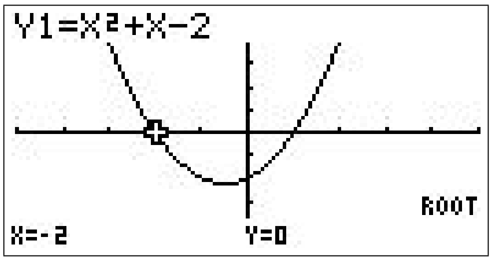
The graphical method in this and all the following operations is usually more meaningful than the corresponding non-graphical numerical operation, which is really just a 'black box'. Numerical operations are in the Appendix.

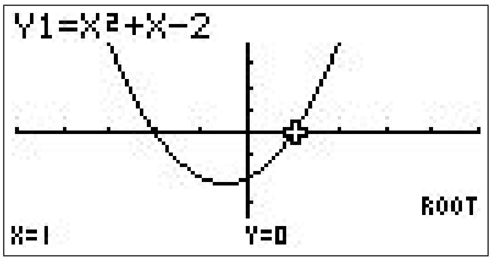
- Graph the function with $\boxed{\text{MENU}}$ $\boxed{5}$ $\boxed{\text{F6}}$.
- For a rough estimate, press $\boxed{\text{F1}}$ (Trace) and move the cursor as close as possible to the zero. Watch the Y coordinate at the bottom of the screen to see when it changes sign.

  Zooming in on the zero — move the cursor to near the zero and press $\boxed{\text{F2}}$ (ZOOM) $\boxed{\text{F3}}$ (IN) — will produce greater accuracy with this method.
- For a more accurate estimate, use $\boxed{\text{F1}}$ (ROOT) in the $\boxed{\text{G-Solv}}$ menu $\left(\boxed{\text{SHIFT}}\ \boxed{\text{F5}}\right)$.

  Press the right arrow to find further zeroes (it works left to right).

  If you have more than one curve plotted, the calculator will ask you to select the one you want with the up/down arrows.



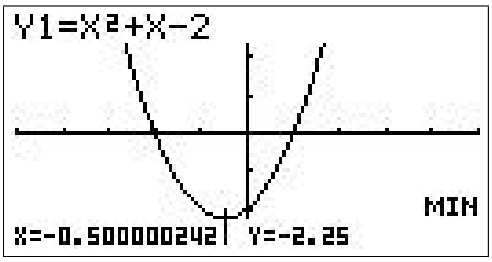


- *Answer*: $f(x) = 0$ when $x = -2$, 1.

3.2.5 Find the minimum of $f(x) = x^2 + x - 2$

- For a rough estimate, press $\boxed{\text{F1}}$ (Trace) and move the cursor as close as possible to the minimum. Watch the Y coordinate at the bottom of the screen.

  Zooming in on the minimum — move the cursor to near the minimum and press $\boxed{\text{F2}}$ (ZOOM) $\boxed{\text{F3}}$ (IN) — will produce greater accuracy with this method.
- For a more accurate estimate, use $\boxed{\text{F3}}$ (MIN) in the $\boxed{\text{G-Solv}}$ menu. MIN works the same way as ROOT above.
- Finding maxima using the MAX command works in exactly the same way.



- *Answer*: the minimum value of $y = -2.25$ occurs at $x = -0.50000$, rounded to 5 decimal places.

3.2.6 Solve $x^2 + x - 2 = \sqrt{x}$

The solution to the equation is the intersection point of the two curves.

- Graph $\mathrm{Y1} = \mathrm{X}^2 + \mathrm{X} - 2$ and $\mathrm{Y2} = \sqrt{\mathrm{X}}$ using a View Window such that the point of intersection of the two curves lies on the screen.

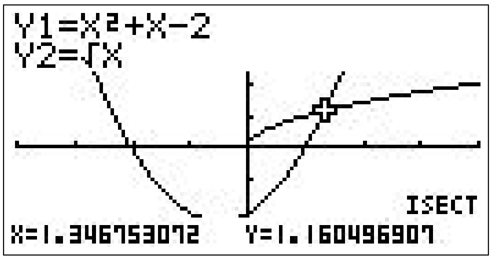
- For a rough estimate of the point of intersection, press $\boxed{\text{F1}}$ (Trace) and move the cursor as close as possible to the intersection. Read the cursor coordinates at the bottom of the screen.

  Zooming in on the intersection — move the cursor to near the intersection and press $\boxed{\text{F2}}$ (ZOOM) $\boxed{\text{F3}}$ (IN) — will produce greater accuracy with this method.

- For a more accurate estimate, use $\boxed{\text{F5}}$ (ISCT) in the $\boxed{\text{G-Solv}}$ menu.

  If there is more than one point of intersection of the two graphs, press the right arrow to find the next one.

  If there are more than two graphs plotted, the calculator will ask you to select which two you want to find the intersection of.



- *Answer*: $x^2 + x - 2 = \sqrt{x}$ when $x \approx 1.347$, $y \approx 1.160$. Answers rounded to 3 decimal places.

PTO

3.2.7 Graphing data points and points joined by lines

There are three possible ways to do this.

Series of points

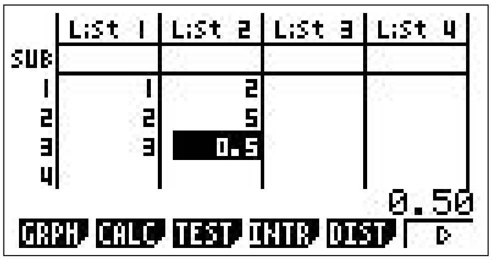
First you need to put the x values of the points in one list, the y values in another. The standard lists on the 9860 are List 1 – List 6, accessed through STAT in the main menu. See Section 8.2 for more details.

Press MENU 2 (STAT).

To clear a list, move the cursor to the list and press F6 F4; respond to the prompt with F1.

Enter the x values into List 1, pressing EXE after each value, *including the last.*

Similarly, put the y values into List 2.

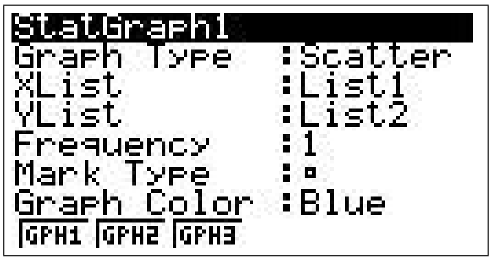


Now you need to tell the calculator what type of plot we want, where the data are and the type of marker for each point.

Press F1 (GRPH). (If you deleted values from a list, you will need to press F6 to see the GRPH menu.) Up to three plots can be displayed.

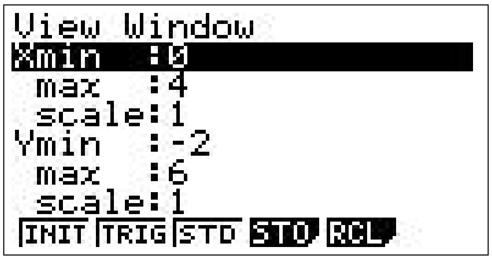
Press F6 to select SET, and select F1 (GPH1) if necessary.

Select the StatGraph1 options as shown in the figure. Press EXIT twice to return to the List Editor.

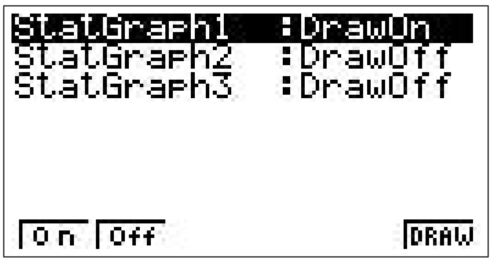


Next the View Window. Press SHIFT F3 (V-Window) and enter the values shown in the figure, pressing EXE after each value, including the last. Press EXIT to return to the List Editor.

Alternatively, set *Stat Wind* in SET UP to *Auto*.

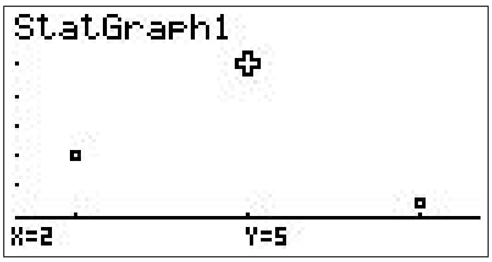


Press F4 (SEL) and turn on *StatGraph1* by pressing F1. Make sure the other two StatGraphs are turned off.



Press F6 (DRAW) to graph the data.

Use Trace and the arrow keys to move around the plot.



Line segment

Use the *F-Line* command: Sketch (SHIFT F4) F6 F2 F2.

From the Run screen: F-Line X1, Y1, X2, Y2.

Example: F-Line 3, 4, 5, 2 draws the line (segment) between (3, 4) and (5, 2).

On a graph (MENU 5)**:** select *F-Line* as above, move the cursor to one end of the line segment you want and press EXE. Move the cursor to the other end of the line and press EXE again.

Point

Use the *Pl–On* command: Sketch (SHIFT F4) F6 F1 F2.

From the home screen: Pl–On (X, Y).

Example: Pl–On (3, 4) plots the point (3, 4).

On a graph: select *Pl–On* as above, move the cursor to the point you want plotted and press EXE. Plot as many points as you like this way.

3.2.8 Shading regions

There are several ways of distinguishing between graphs or solving inequalities using shading under/over graphs. These are selected from the GRAPH menu (MENU 5).

TYPE (F3), then F6**:** shaded above (F1 and F3); shaded below (F2 and F4). Select the type before entering the function.

3.2.9 Appendix: Non-graphical numerical operations

Finding zeros: Use the interactive *Solver*, F3 in the EQUA menu (MENU A). See the manual for details.

Finding minima: from the RUN screen, FMin(Y1, –3, 3): *FMin* is OPTION F4 (CALC) F6 F1. The last two inputs are the bounds.

Y1 can be replaced by a function defined explicitly in terms of X.

Finding maxima using the *FMax* command works in exactly the same way.

Solving equations: Use the interactive *Solver*, F3 in the EQUA menu (MENU A). See the manual for details.

3.3 Activities

3.3.1 Linear Models

Renting a Car

The Rent-a-Wreck Car Rental Company has the cheapest car rentals in town.

You can choose one of two options.

- **Option A** — no flat fee, but a charge of 28c per kilometre.
- **Option B** — a flat rate of $36 per day, plus 18c per kilometre.

(a) We want to hire a car for one day.

Under Option A, how much will it cost in dollars to drive 1 kilometre? 2 kilometres? 10 kilometres? 100 kilometres? x kilometres?

For Option A, what is the equation for the cost in dollars y in terms of the number of kilometres driven x? Check your equation with the numbers you calculated above.

Under Option B, how much will it cost in dollars to drive 1 kilometre? 2 kilometres? 10 kilometres? 100 kilometres? x kilometres?

For Option B, what is the equation for the cost in dollars y in terms of the number of kilometres driven x? Check your equation with the numbers you calculated above.

(b) Graph the equations for the two options, assuming we will drive up to 600 kilometres in a day.

(c) Estimate from the graph how far we have to drive before Option B becomes cheaper.

(d) From the graph, what is the slope of the line for Option A?
Hint: Press $\boxed{\text{Trace}}$ and use the left/right arrows to find two points on the graph; use these work out the slope.

What is the slope of the line for Option B?

What does the slope represent in this problem?

(e) What is the y intercept of the graph of Option A?
Hint: Press $\boxed{\text{G-Solv}}$ $\boxed{\text{F4}}$.

What is the y intercept of the graph of Option B?

What does the y intercept represent in this problem?

(f) Work out *exactly* how far we have to drive before Option B becomes cheaper.

Marketing a Computer Game

You have just written a cool computer game and your company wants to sell it. *How much should it charge?*

If it puts on a high price, the company won't sell as many games, but it will make more money per game sold. If the game is sold for a low price, the company won't make as much money on each game sold, but it will sell more games.

Clearly, the number sold depends on the price. Economists often assume that the number sold and price form a *linear equation.*

After doing some market research, the company thinks that if it sells the game for $160 per copy, it will sell about 800 copies. If the price is dropped to $40 per copy, it should sell about 8000 copies.

(a) Let's use a graph of number sold versus price to help us in our problem. Price will be on the x axis and number sold on the y axis. What are suitable scales for the two axes?

(b) What are the two points that we know on the graph of number sold versus price? Use the Line command (Section 3.2.7) to draw a straight line between these two points. The syntax is (from the home screen) Line (x_1, y_1, x_2, y_2), where (x_1, y_1) and (x_2, y_2) are the two points.

(c) What is the slope of the line? *Hint:* Use two points on the graph to work out the slope.

(d) What is the equation of the line? What extra information about the line do we use here? Graph the line and check that the points you know actually lie on the graph of the line.

(e) Does this graph tell us the answer to the question of what the price should be?

(f) Revenue means total income. It is the product of price and number sold. Write down the equation for revenue as a function of price x. What kind of function is this?

(g) Plot the graph of revenue versus price. Estimate the revenue if the price is $50.

(h) What is the best price to sell the game at? Why is it best? What is the revenue?

(i) What is the revenue if the game is sold at a price of $180? Explain.

Acknowledgement to material from an unknown website.

PTO

Money in the Bank

Brock has $_____ in the bank, has no income, but is spending about $_____ a week on his new girl friend Amber.

His sister Minerva has only $_____ in the bank, but is spending nothing and saving about $_____ a week.

(a) Find the equation of the line that gives how much money each person has in the bank as a function of time in weeks.

(b) What is the slope of each line?

(c) What is the physical interpretation of the slope?

(d) What is the y intercept of each line?

(e) What is the physical interpretation of the y intercept?

(f) When will Minerva and Brock have the same amount of money?

(g) When will Minerva have twice as much money as Brock?

3.3.2 Other Activities

The following Coordinate Geometry activities are collected in the supplementary volume *Supplementary Material: Activities for Years 9 and 10* available at *canberramaths.org.au* under *Resources*. Year levels and subject matter are indicated with each summary. Solutions and teachers' notes are provided with each activity.

Coordinate Geometry Art

A simple picture consisting of straight-line segments is 'coded' using the coordinates of the points of its vertices. These are used 'transmit' the picture to someone else. A graphics calculators is used to 'decode' and check the 'transmitted' picture.
Year 9, Levels 1 and 2; Algebra; Coordinate Geometry.

A Classic Problem — The Hare and Tortoise

The graphs of the distances covered versus time in this classic race are used to answer various questions about the race, such as who won and by how much. A fun exercise in putting questions into maths and solving equations graphically.
Year 10, Level 1; Algebra; Sketching Other Graphs, Simultaneous Equations.

The Best Shape for a Can

Minimising the surface area of a cylinder (can) for a fixed volume. Numerical and graphical techniques, rather than Calculus, are used to find the minimum. Aspects of mathematical modelling are introduced.
Year 10, Level 1; Algebra/Measurement; Sketching Other Graphs/Volume.

Alien Attack

Uses one of Newton's equations of motion to explore properties of quadratic equations both numerically and graphically.
Years 9 and 10, Levels 1 and 2; Algebra; Coordinate Geometry.

Parabolic Aerobics

The first activity investigates the effect of changing the numbers A, B and C on the graphs of the family of parabolas $\mathrm{Y} = \mathrm{A}(\mathrm{X}-\mathrm{B})^2 + \mathrm{C}$. In the second activity, you must guess the numbers A, B and C for the graph of a mystery parabola generated by the PARABOLA program. The calculator checks your answers and keeps score.
Year 9, Levels 1 and 2; Algebra; Coordinate Geometry.

What's My Line

Investigates the connection between a table of values, the line on a number plane and the equation of the line.
Years 9 and 10, Levels 1 and 2; Algebra; Coordinate Geometry — Straight Lines.

Graphing Straight Lines

Uses the equation editor to explore the $y = mx + b$ form of a straight line.
Years 9 and 10, Levels 1 and 2; Algebra; Coordinate Geometry — Straight Lines.

Guess the Line

Guess the equations of straight lines generated by the calculator. The calculator keeps score.
Years 9 and 10, Levels 1 and 2; Algebra; Coordinate Geometry — Straight Lines.

Sketching Quadratics — Intercept Method

Uses $\boxed{\text{Trace}}$ and $\boxed{\text{Zoom}}$ to determine the intercepts and vertex of a parabola. Students should be able to factorise monic quadratics.
Years 9 and 10, Levels 1 and 2; Algebra; Coordinate Geometry.

Starburst

A study of straight lines: slope and intercept.
Year 9, Levels 1 and 2; Algebra; Coordinate Geometry.

Speeding — A Study in Linear Functions

Students apply basic knowledge of linear functions to problems involving speeding tickets.
Year 9, Levels 1 and 2; Algebra; Coordinate Geometry.

Tangrams and Straight Lines

Students make a picture using Tangram pieces, place the picture onto grid paper and use their knowledge of xy coordinates and straight-line equations to redraw the picture on a graphics calculator.
Year 9 , Levels 1–3; Algebra; Coordinate Geometry.

Which Fuel?

Application of linear functions to choosing whether to use petrol or LPG in your car.
Years 9 and 10, Level 1; Algebra; Coordinate Geometry.

Temperature Conversions

Application of linear functions to conversion between degrees Celsius and degrees Fahrenheit.
Year 9, Levels 1 and 2; Algebra; Coordinate Geometry.

3.4 Solutions

Solutions to the three activities in Section 3.3.1.

Renting a Car The Rent-a-Wreck Car Rental Company has the cheapest car rentals in town. You can choose one of two options.

- **Option A** — no flat fee, but a charge of 28c per kilometre.
- **Option B** — a flat rate of \$36 per day, plus 18c per kilometre.

(a) You want to hire a car for one day.

(i) Under Option A, how much will it cost in dollars to drive 1 km? 2 km? 10 km? 100 km? x km?

\$0.28; \$0.56; \$2.80; \$28.00; \$0.28x.

(ii) For Option A, what is the equation for y, the cost in dollars, in terms of x, the number of kilometres driven?
Check your equation with the numbers you calculated in (i).

$y_1 = 0.28x$

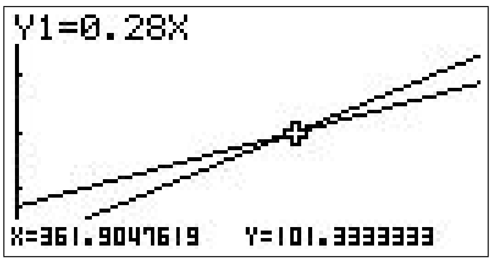
(iii) Under Option B, how much will it cost in dollars to drive 1 km? 2 km? 10 km? 100 km? x km?

\$36.18; \$36.36; \$37.80; \$54.00; 36+0.18x.

(iv) For Option B, what is the equation for y, the cost in dollars in terms of x, the number of kilometres driven?
Check your equation with the numbers you calculated in (iii).

$y_2 = 36 + 0.18x$

(b) Graph the equations for the two options, assuming you will drive up to 600 km in the day.



V-Window $[0, 600, 100] \times [0, 200, 50]$

(c) Estimate <u>from the graph</u> how far you have to drive before Option B becomes cheaper.

Using $\boxed{\text{Trace}}$, you have to drive somewhere between 359 km and 364 km. We could use *ISCT* to find this more accurately but we do it exactly in (f).

(d) From the graph, what is the slope of the line for Option A?

Using $\boxed{\text{Trace}}$, we find points $(100, 28)$ and $(400, 112)$ to be on the graph.

Therefore, slope $= \dfrac{112-28}{400-100} = \dfrac{74}{300} = 0.28$.

Of course, we could have seen this directly from the equation for Option A.

What is the slope of the line for Option B?

Using $\boxed{\text{Trace}}$ again, we find points $(100, 54)$ and $(400, 108)$ to be on the graph.

Therefore, slope $= \dfrac{108-54}{400-100} = \dfrac{54}{300} = 0.18$.

Again, we could have seen this directly from the equation for Option B.

What does the slope represent in this problem?

The slope represents the cost in dollars per kilometre.

(e) What is the y intercept of the graph of Option A?

$y = 0$

What is the y intercept of the graph of Option B?

$y = 36$

What does the y intercept represent in this problem?

The initial cost or flat rate of the rental for one day.

(f) Work out <u>exactly</u> how far you have to drive before Option B becomes cheaper.

We have to find out at what x value $y_1 = y_2$, that is $0.28x = 36 + 0.18x$, with solution $x = 360$.

You have to drive 360 km before Option B becomes cheaper, consistent with our answer to (c).

Marketing a Computer Game

You have just written a fantastic computer game and your company wants to sell it. *How much should it charge?*

If it puts on a high price, the company won't sell as many games, but it will make more money per game sold. If the game is sold for a low price, the company won't make as much money on each game sold, but it will sell more games.

Clearly, the number sold depends on the price. Economists often assume that the number sold and price are related by a *linear equation.*

After doing some market research, the company thinks that if it sells the game for \$160 per copy, it will sell about 800 copies. If the price is dropped to \$40 per copy, it should sell about 8000 copies.

(a) Let's use a graph of number sold versus price to help us in our problem. Price will be on the x axis and number sold on the y axis. What are suitable scales for the two axes?

$0 < x < 160$; $0 < y < 8000$

(b) What is the slope of the line?

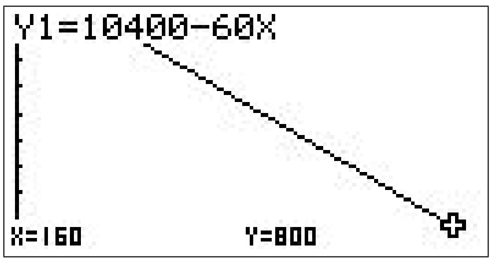
The two points we have are $(160, 800)$ and $(40, 8000)$.

The slope of the line is $\dfrac{8000 - 800}{40 - 160} = -60$.

(c) What is the equation of the line? What extra information about the line do we use here other than its slope? Graph the line and check that the points you know actually lie on the graph of the line.

The equation of the line is of the form $y = mx + b$, where m is the slope and b is a constant determined by a point on the line, the extra information. Taking the point $(160, 800)$, we have $800 = -60 \times 160 + b$, so that $b = 800 + 60 \times 160 = 10{,}400$.

The equation of the line is therefore $y = 10{,}400 - 60x$.



View Window $[0, 170, 20] \times [0, 8500, 1000]$

Using $\boxed{\text{Trace}}$ confirms that the two points we know lie on the graph of the line. One of these is shown in the figure.

(d) Does this graph tell us the answer to the question of what the price should be?

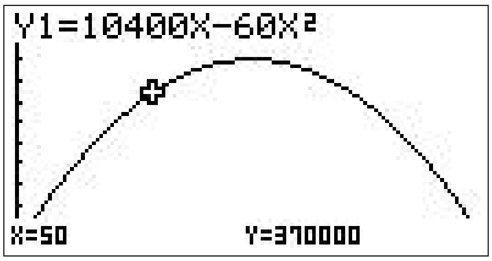
No. There is no clear optimum value.

(e) Revenue means total income. It is the product of price and number sold. Write down the equation for revenue as a function of price x. What kind of function is this?

From our equation above, revenue $R = xy = x(10{,}400 - 60x) = 10{,}400x - 60x^2$.

This is a quadratic function.

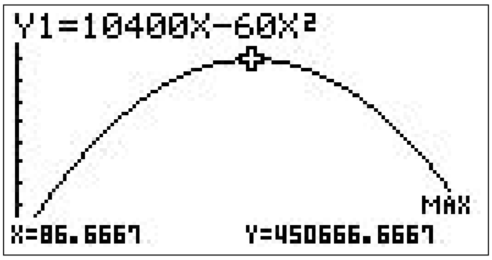
(f) Plot the graph of revenue versus price. Estimate the revenue if the price is \$50.



View Window $[0, 170, 20] \times [0, 550{,}000, 50{,}000]$

If the price is \$50, the revenue is \$370,000 (see the figure).

(g) What is the best price to sell the game at? Why is it best? What is the revenue?



View Window $[0, 170, 20] \times [0, 550{,}000, 50{,}000]$

The best price to sell the game at is the price that maximises revenue; from the graph using MAX in G-Solv, that price is \$86.67 or \$87 to the nearest dollar. The corresponding revenue is \$450,667 to the nearest dollar.

(h) What is the revenue if the game is sold at a price of \$180? Explain.

The revenue predicted by the model if the game is sold at a price of \$180 is $-\$7,000$. Clearly a negative revenue does not make sense; the price of \$180 lies outside the range of the model.

Linear-Model Assignment

The underlined amounts can be changed to whatever you like for individual assignments.

Brock has $\underline{\$100}$ in the bank, has no income, but is spending about $\underline{\$5}$ a week on his new girlfriend Amber.

His sister Minerva has only $\underline{\$10}$ in the bank, but is spending nothing and saving about $\underline{\$3}$ a week.

(a) Find the equation of the line that gives how much money each person has in the bank as a function of time in weeks.

If $y_{\mathrm{B}}(t)$ is the amount of money Brock has in the bank after t weeks and $y_{\mathrm{M}}(t)$ the corresponding amount for Minerva, $y_{\mathrm{B}}=100-5t$ and $y_{\mathrm{M}}=10+3t$.

(b) What is the slope of each line?

The slope of y_{B} is -5, the slope of y_{M} 3.

(c) What is the physical interpretation of the slope?

The slope is the change in the amount of money (\$) in the bank per week.

(d) What is the y intercept of each line?

The y intercept of y_{B} is 100, the y intercept of y_{M} 10.

(e) What is the physical interpretation of the y intercept?

The y intercept is the initial ($t=0$) amount of money in the bank.

(f) When will Minerva and Brock have the same amount of money?

We must solve $y_{\mathrm{B}} = y_{\mathrm{M}}$ for time t

Therefore, $100-5t = 10+3t$.

Therefore, $8t = 90$, so that $t=11.25$ to 2 decimal places.

Minerva and Brock will have the same amount of money between Week 11 and Week 12.

(g) When will Minerva have twice as much money as Brock?

We must solve $y_{\mathrm{M}} = 2y_{\mathrm{B}}$ for time t.

Therefore, $10+3t = 2(100-5t)$.

Therefore, $13t = 190$, so that $t=14.6$ to 1 decimal place.

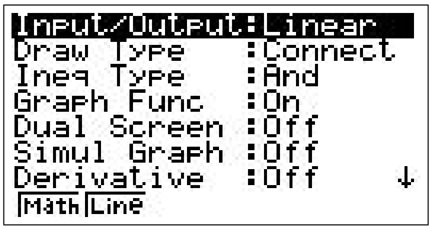
Minerva will have twice as much money as Brock between Week 14 and Week 15.

4 Inequalities and Linear Programming

Numerical, Graphical and Algebraic Approaches

4.1 Setting up

Press $\boxed{\text{MENU}}$ $\boxed{5}$ for GRAPH mode. Press $\boxed{\text{SET UP}}$ ($\boxed{\text{SHIFT}}$ $\boxed{\text{MENU}}$).



Input/Output: In *Linear* mode, commands are typed on one line, with arguments in brackets. In *Math* mode, the calculator tries to display commands in mathematical notation, with small boxes in the relevant positions for the inputs.

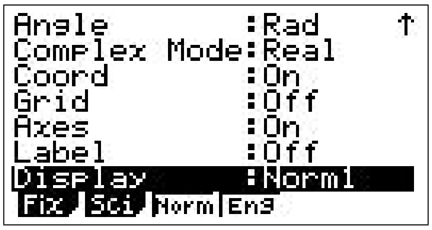
Linear is used here.

Draw Type: *Connect* means graphs are a continuous line whereas Plot provides a set of points (those calculated) which are not connected. *Connect* is better in most cases.

Graph Func: *On* means the equation of the function is displayed *while* its graph is being drawn. Harmless, so leave On.

Simul Graph: *On* means that, if two or more functions are being drawn, they will be drawn simultaneously; *Off* means they are drawn sequentially. Unless you are simulating a race, leave set on *Off*.

Derivative: *On* means that, when a graph is being traced, (an approximation to) the value of the derivative at a point will be displayed as well as the function value. Leave *Off* unless required.



Angle: *Rad* (radians) is the appropriate setting for Mathematics.

Coord: *On* means the coordinates of the cursor wil be displayed when tracing a graph.

Axes: *On* means Cartesion axes will be drawn on plots where appropriate.

Display: *Fix* allows you to set the number of decimal places in numerical output; useful in financial calculations where the numbers are in dollars and cents. *Sci* displays all numbers in scientific notation. *Norm1* displays numbers smaller than 0.01 in scientific notation, *Norm2* numbers smaller than 0.000000001 (10^{-9}). Toggle between the two with $\boxed{\text{Norm}}$. *Eng* displays numbers similar to scientific notation but adjusted so that the exponent is always a multiple of 3.

Norm1 preferred unless you like counting zeros.

4.2 Inequalities

4.2.1 Linear inequalities

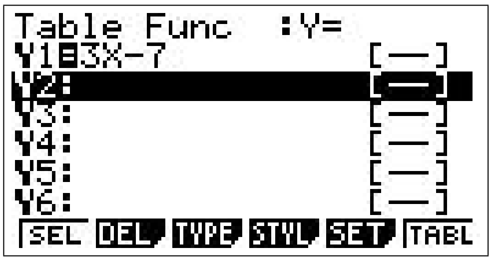
Linear inequalities can be solved numerically, graphically or algebraically. For simple linear inequalities, the algebraic method (manipulating symbols) is probably the quickest but, once the inequalities become more complicated, the graphical method comes into its own. Some may prefer the numerical method but having the big picture of a graph, particularly with the graphics tools in the G-Solv menu, is always useful.

We start with some simple inequalities to demonstrate the three methods before moving on to more complicated examples.

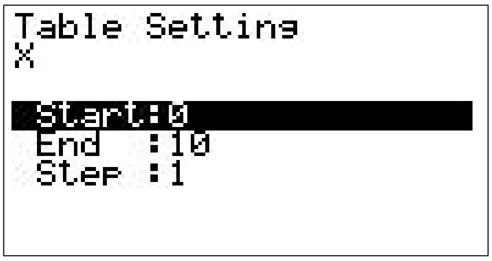
Example 1: Find all values of x for which $3x-7 \leqslant 2$.

Numerical method

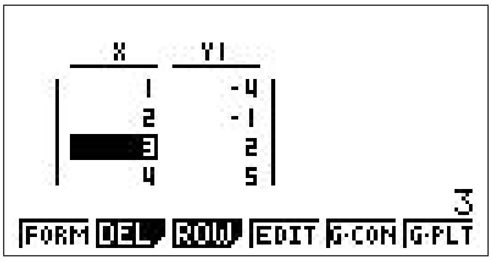
- Press MENU 7 (TABLE) and enter the left-hand side of the inequality into Y1.



- Press SET (F5) and set the values as shown. The table will start at $x=0$ and increment in steps of 1.
- Press EXIT, then TABL (F6).



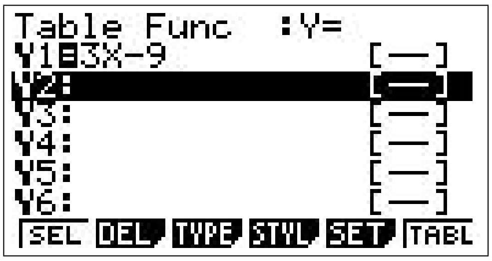
- Scroll in the X column if necessary to find where $\mathrm{Y1} \leqslant 2$.
- You should find that $\mathrm{Y1} \leqslant 2$ when $x \leqslant 3$.
- Therefore the solution is $x \leqslant 3$.



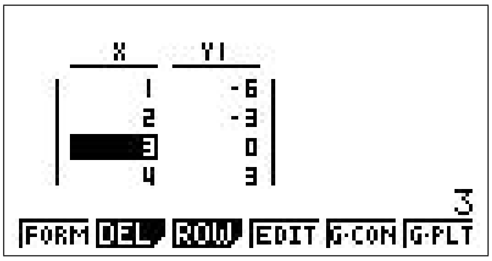
PTO

Sometimes it is easier to see when a function is greater than or less than 0, rather than comparing values of the two functions.

- Rewrite the inequality as $3x-9 \leqslant 0$.
- Set Y1 = 3X−9



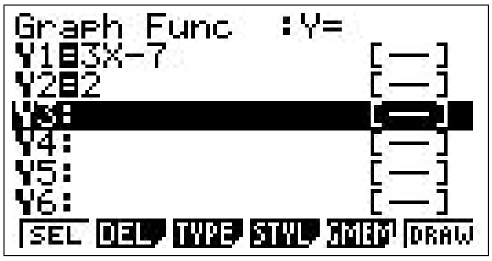
- Press TABL to see where Y1⩽0. Verify the solution for the inequality that you found above.

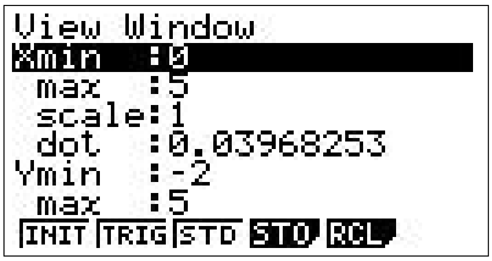


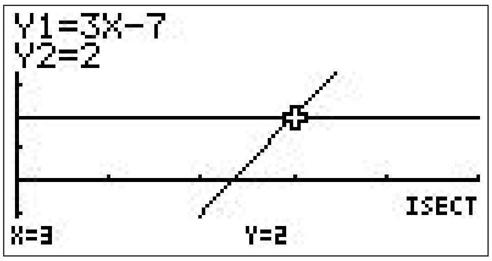
Graphical method

- Set Y1 = 3X−7 again using MENU 5 (GRAPH). Set Y2 = 2, an appropriate V-Window, press EXIT, then DRAW.
- We must find the region (in terms of x) for which points on Y1 have smaller y values than the corresponding points on Y2.

  Find the point of intersection of the two lines, the point at which the y values are equal, by pressing F5 (G-Solv) F5 (ISCT) EXE.
- Observe that for all values of x less than or equal to 3, the left-hand side of the inequality (Y1) is less or equal to than the right-hand side (Y2).
- Therefore, the solution is $x \leqslant 3$.







PTO

Algebraic method

Here we manipulate symbols in the same way as when we want to isolate a variable in an equation. However, keep in mind that if you divide or multiply an inequality by a negative number the inequality sign must be reversed. For this reason, you should never multiply or divide by a variable when solving inequalities.

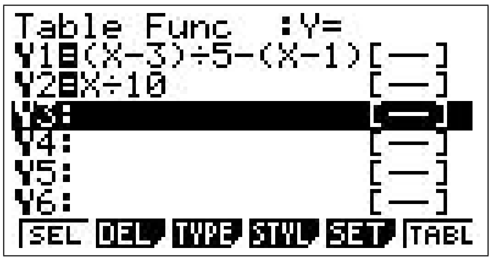
$$3x - 7 \leqslant 2$$

- Add 7 to both sides: $3x \leqslant 9$.
- Divide by 3 (positive): $x \leqslant 3$.
- Therefore the solution is $x \leqslant 3$.

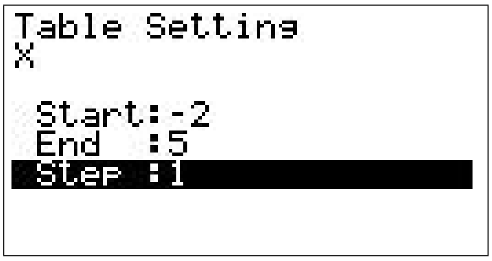
Example 2: Find all values of x for which $\dfrac{x-3}{5} - \dfrac{x-1}{2} \leqslant \dfrac{x}{10}$.

Numerical method

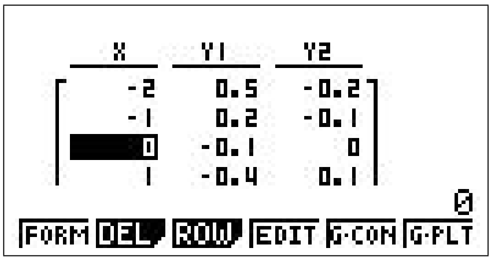
- Press MENU 7 and enter the left-hand side of the inequality into Y1 and the right-hand side into Y2.



- Press SET and set the values as shown. The table will start at $x = -2$ and increment in steps of 1.



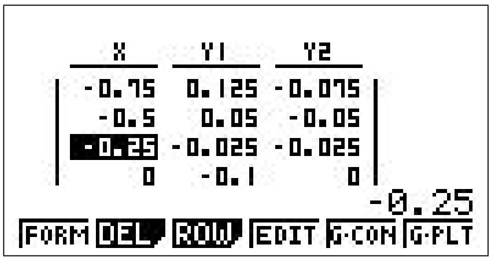
- Press EXIT, then TABL.
- Here scroll down in the X column to find when Y1 is less than or equal to Y2.



PTO

- When you have narrowed down the point at which the two are equal to between two integers (here between -1 and 0), press $\boxed{\text{EXIT}}$, then $\boxed{\text{SET}}$ again. Set *Start* to the smaller of the two integers (-1) and *Step* to 0.25. Press $\boxed{\text{EXIT}}$, then $\boxed{\text{TABL}}$ again.

  You should find that Y1$\leqslant$Y2 when $x \geqslant -0.25$.



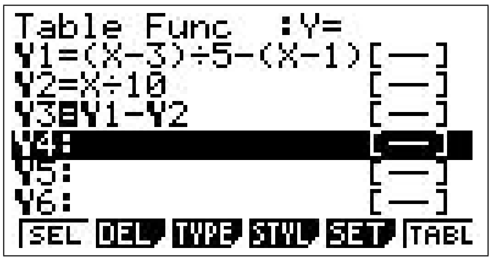
- Therefore the solution is $x \geqslant -0.25$.

Again, it is sometimes easier to see when a function is greater than or less than 0, rather than comparing values of the two functions.

- Rewrite the inequality as

$$\frac{x-3}{5} - \frac{x-1}{2} - \frac{x}{10} \leqslant 0,$$

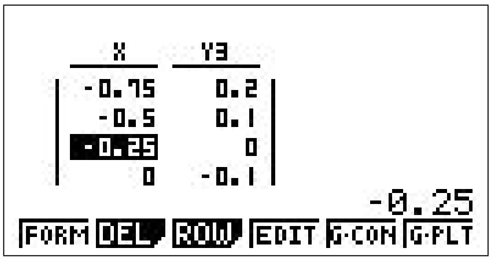
  or, in calculator terms Y1 $-$ Y2 $\leqslant 0$.



- Set Y3 = Y1 $-$ Y2.

  Y is $\boxed{\text{VARS}}$ $\boxed{\text{GRPH}}$ $\boxed{1}$.

  Turn off Y1 and Y2 by moving the cursor over the equation and pressing $\boxed{\text{SEL}}$.

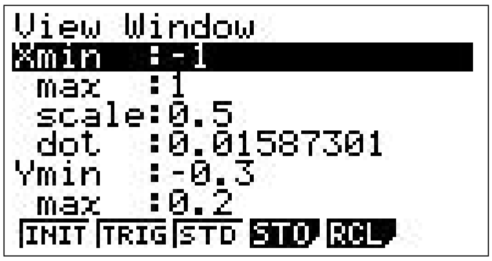


- Press $\boxed{\text{TABL}}$ to see the values of Y3. Verify the solution for the inequality that you found above.

PTO

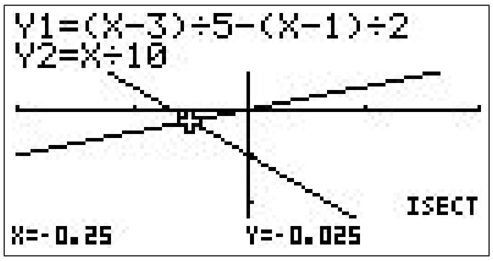
Graphical method

- The two sides of the inequality are already entered into Y1 and Y2. Go to $\boxed{\text{MENU}}$ $\boxed{5}$, turn these two functions back on, turn off or delete Y3, set an appropriate $\boxed{\text{V-Window}}$, and press $\boxed{\text{EXIT}}$, then $\boxed{\text{DRAW}}$.



- We must find the region (in terms of x) for which points on Y1 have smaller y values than the corresponding points on Y2.

  Find the point of intersection of the two lines, the point at which the y values are equal, by pressing $\boxed{\text{G-Solv}}$ (F5) $\boxed{\text{ISCT}}$ (F5 again).



- Observe that for all values of x greater than -0.25, the left-hand side of the inequality (Y1) is less than the right-hand side (Y2).
- Therefore the solution is $x \geqslant -0.25$.

Algebraic method

$$\frac{x-3}{5} - \frac{x-1}{2} \leqslant \frac{x}{10}$$

- Multiply both sides by 10 to clear the fractions: $\dfrac{10(x-3)}{5} - \dfrac{10(x-1)}{2} \leqslant \dfrac{10x}{10}$.
- Cancel out factors: $2(x-3)-5(x-1) \leqslant x$.
- Expand brackets and simplify: $-3x-1 \leqslant x$.
- Add $3x$ to both sides: $-1 \leqslant 4x$.
- Divide by 4: $-\frac{1}{4} \leqslant x$ or $x \geqslant -\frac{1}{4}$.

Therefore the solution is $x \geqslant -0.25$.

4.2.2 Absolute-value inequalities

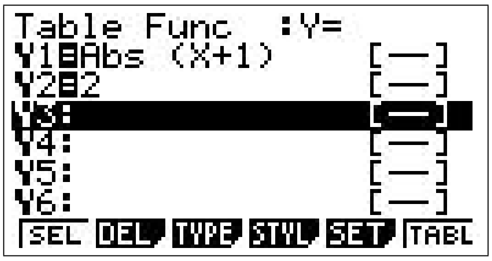
The same three methods can also be used with absolute-value inequalities. Recall that the absolute value of x or mod x, written $|x|$, is **defined** as

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0. \end{cases}$$

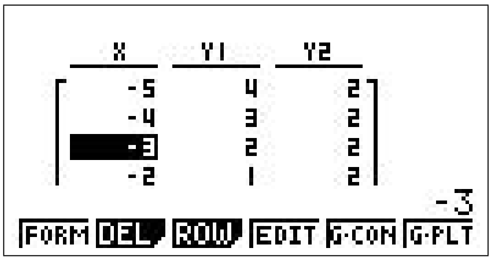
Example 3: Solve the inequality $|x+1| < 2$.

Numerical method

- Press $\boxed{\text{MENU}}$ $\boxed{7}$: set Y1 = Abs (X+1) and Y2 = 2.
  Abs is $\boxed{\text{OPTN}}$ $\boxed{\text{NUM}}$ $\boxed{\text{F1}}$.
- Press $\boxed{\text{SET}}$: set Start = −5, End = 5 and Step = 1.

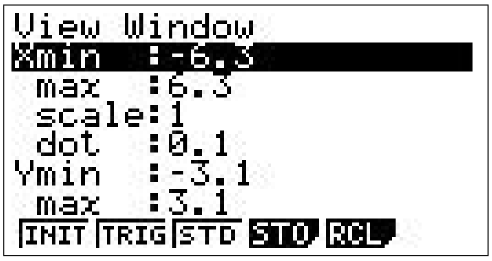


- Press $\boxed{\text{EXIT}}$, then $\boxed{\text{TABL}}$.
- Scroll down in the X column to find when Y1 < Y2: $-3 < x < 1$.
- Alternatively, set Y1 = Abs (X+1)−2 and use $\boxed{\text{TABL}}$ to find where Y1 < 0.



Graphical method $|x+1| < 2$

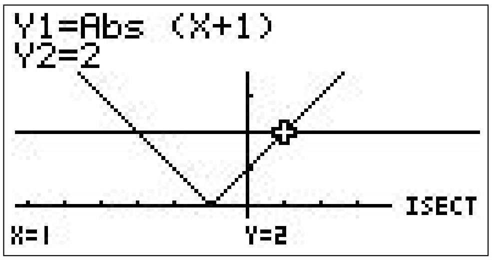
- Press $\boxed{\text{MENU}}$ $\boxed{5}$. If necessary, put the functions in Y1 and Y2 as described above in *Numerical method*.
- Press $\boxed{\text{V-Window}}$ $\boxed{\text{F1}}$ (INIT) to set the window, then $\boxed{\text{EXIT}}$ and $\boxed{\text{DRAW}}$ to graph the functions. Then press $\boxed{\text{F1}}$ (Trace) and move the cursor along the graph. Note the values of the cursor co-ordinates — a result of the INIT option.



PTO

- Find the two intersection points using *ISCT*, as described on page 41. Press the right arrow to find the second intersection point.

  Unfortunately, the functions obscure the intersections: in V-Window, add 2 to Ymin and Ymax to get a better view (as in the figure).



- From the points of intersection we see that, for all points to the right of -3 and to the left of 1, the graph of $y=|x+2|$ has $y<2$.

  Thus, the solution is $-3<x<1$.

Algebraic method $|x+1|<2$

From the definition of $|x|$, the inequality $|x-b|<a$ means that[10]

$$x-b<a \quad \textbf{and} \quad -(x-b)<a.$$

Here, we have $a=2$ and $b=-1$, so that $|x+1|<2$ means that

$$x+1<2 \quad \textbf{and} \quad -(x+1)<2.$$

Multiplying the second inequality by -1 and changing the inequality direction gives

$$x+1<2 \quad \textbf{and} \quad x+1>-2,$$

so that $x<1$ **and** $x>-3$. These can be combined to give $-3<x<1$.

The graphical and numerical methods can be used to solve any inequality, whereas the algebraic method soon becomes difficult once the functions become more complicated.

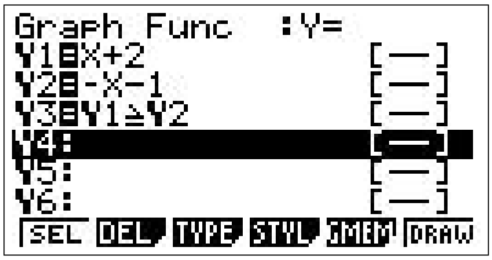
[10]Alternatively, $-a<x-b<a$.

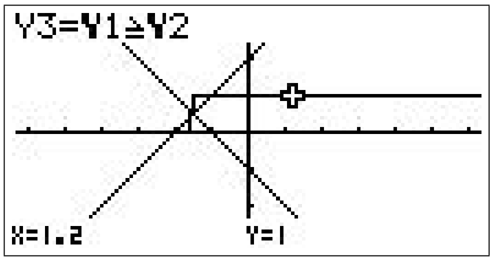
4.2.3 Using logic functions to solve inequalities

An interesting way to solve inequalities graphically is to use logic functions or relational operators.

Example 4: Solve the inequality $x+2 \geqslant -x-1$.

- Set Y1 = X+2 and Y2 = −X−1. Watch − signs here.
- Press V-Window F1 (INIT) and graph the the lines. Adjust the V-Window until the intersection point of the lines lies somewhere near the middle of the screen.
- Set Y3 = Y1 ≥ Y2. The ≥ symbol is in the CATALOG.
- Press DRAW and a new line will now appear on the graph.
- Press Trace and the down arrow until you select the new line. Move the cursor along it and you will see that the line has y values of 1 or 0. If $y = 0$, the corresponding value of x does not satisfy the inequality; if $y = 1$, it does.
- We can see that for all values of x greater than −1.5, the y value is 1, so the solution must be $x \geqslant -1.5$.





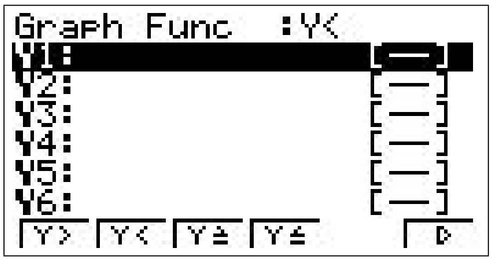
A conclusion from the use of the three methods is that the algebraic method is quickest for simple linear inequalities, the graphical method for more-complicated inequalities. As in most mathematics, a diagram (graph) is always useful.

4.2.4 Graphing regions

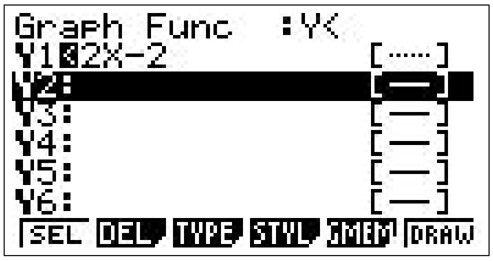
The line $ax + by + c = 0$ divides the number plane into two regions (known as half-planes). All the points that satisfy the inequality $ax+by+c<0$ lie on one side of the line, while all the points that satisfy the opposite inequality $ax+by+c>0$ lie on the other side. When given such an inequality to solve, the answer can be shown by shading the appropriate region on a graph.

The inequality has to be rewritten with y on one side, so it can be entered into the calculator.

Example 5: Graph the region satisfying the inequality $2 < 2x-y$ for $-10<x<10$.

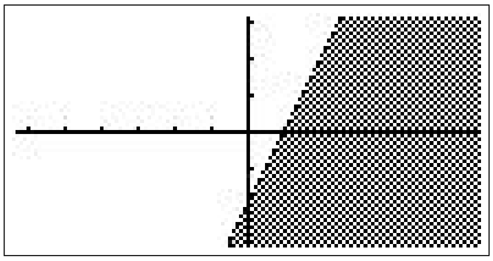


- Rewrite the inequality as $y < 2x-2$.
- Move the cursor on to Y1 and press $\boxed{\text{TYPE}}$. Press $\boxed{\text{F6}}$ then $\boxed{\text{F2}}$ (Y<: shading below the line). Then type in 2X−2 and press $\boxed{\text{EXE}}$.



- Press $\boxed{\text{V-Window}}$ $\boxed{\text{F1}}$ to set a viewing window, $\boxed{\text{EXIT}}$ and $\boxed{\text{DRAW}}$ to graph the function.

  In general, you will have to experiment with the $\boxed{\text{V-Window}}$ parameters to find the best settings for any given inequality.



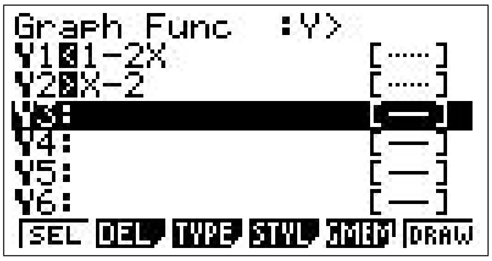
Pick a point in the shaded region, say $(4, 0)$, to verify that y is less than $2x-2$ there, but not at points in the unshaded region. The boundary line here, $y=2x-2$, is not in the region because of the $<$ sign.

4.2.5 Compound inequalities

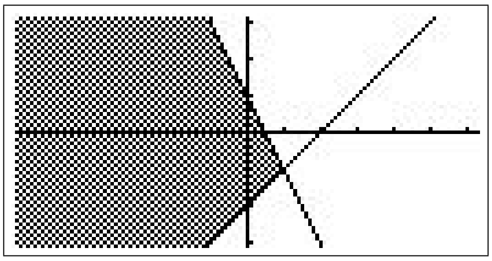
While a single inequality divides the number plane into two regions when graphed, two inequalities graphed simultaneously divide the number plane into three or more regions. The graphics calculator can be used to shade the regions and thus see if there is a region which satisfies both inequalities.

Example 6: Find the solution region to the inequalities $2x+y<1$ **and** $y>x-2$.

- First rearrange $2x+y<1$ as $y<1-2x$.
- Select shading *below* for Y1 (Y<: a lesser-than inequality), as in the previous example, then enter 1−2X and press $\boxed{\text{EXE}}$. Similarly, select shading *above* for Y2 (Y>: a greater-than inequality), then enter X−2 and press $\boxed{\text{EXE}}$.



- Press $\boxed{\text{V-Window}}$ $\boxed{\text{INIT}}$ and graph the functions.
- The shaded area, in which both inequalities are satisfied, is the solution region: any values/points (x,y) lying in that region satisfy both inequalities, as you can verify by substituting coordinates of points in that region into the inequality.



Note: Unfortunately, because of the way the 9860 plots the inequalities here (only shading the region satifying both inequalities), it is not possible to have the region satifying both inequalities the only region unshaded.

4.2.6 Exercises

Answers are provided here, full solutions in Section 4.5.1.

In 1–4, solve the following inequalities using any or all of the three approaches.

1. $2+5x \geqslant 1$. $(x \geqslant -0.2)$

2. $5+4x > 2x+1$. $(x > -2)$

3. $|2-5x| \leqslant 3$. $(-0.2 \leqslant x \leqslant 1)$

4. $|2x-3| \leqslant |2-x|$. $(1 \leqslant x \leqslant \frac{5}{3})$

5. Graph the region satisfying the inequality $1 < x+2y$ for $-10 < x < 10$.

6. Graph the region satisfying the inequalities $x-3y<2$ **and** $y>2x-1$.

4.3 Linear Programming

4.3.1 Method

Inequalities are particularly useful with problems of allocation, in which there is a region of possible answers, and you must select the most effective or profitable one. In its simplest form, this is called linear programming.

Graphics are essential here, both for understanding the process and doing the problems. The graphs can be done by hand, and should be once or twice, but this becomes rather boring. Graphics calculators allow one to concentrate on the process. The method is illustrated by the following example.

Example 7: Two foods provide the main source of carbohydrate and protein for a diet. Food A costs \$4 per kilogram and Food B costs \$5 per kilogram. The problem is to determine the cheapest mix of the two foods that provides enough protein and carbohydrate, given the following information.

- Each kilogram of Food A contains 10 units of protein and 30 units of carbohydrate.
- Each kilogram of Food B contains 20 units of protein and 20 units of carbohydrate.
- Daily requirements: protein — at least 30 units; carbohydrate — at least 60 units.

Let the amount of Food A you eat be x kg and the amount of Food B be y kg. The constraints can then be written

$10x+20y \geqslant 30$ amount of protein

$30x+20y \geqslant 60$ amount of carbohydrate

$x,\ y \geqslant 0$ physical constraints (you can't have negative amounts of food).

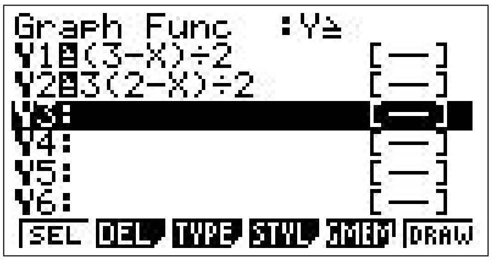
Arrange these so that y is the subject.

$$y \geqslant (30-10x)/20 = (3-x)/2$$

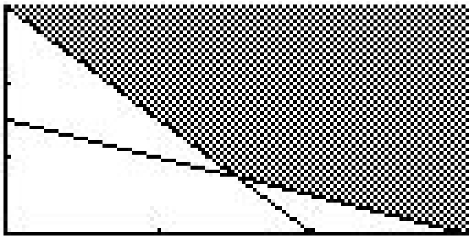
$$y \geqslant (60-30x)/20 = 3(2-x)/2$$

Enter the right-hand sides of these inequalities into Y1 and Y2 respectively.

Set an appropriate $\boxed{\text{V-Window}}$ (note the third constraint) and graph the functions.



The two curves intersect at the point $(1.5, 0.75)$ (G-Solv ISCT).



V-Window $[0, 3, 1] \times [0, 3, 1]$

Any point in the shaded area provides enough protein and carbohydrate. In order to determine which of the possible solutions (which point in the shaded area) is best, we need to create a cost line for our model.

Food A costs \$4 per kilogram and Food B \$5 per kilogram. Therefore, the total cost C (in dollars) = 4 × amount of Food A + 5 × amount of Food B, i.e.

$$C = 4x + 5y.$$

Rewriting this expression to make y the subject, we obtain the equation of the cost line

$$y = \frac{C}{5} - \frac{4}{5}x.$$

Each value of C gives a cost line — all points lying on a given cost line will give the same total cost. Moreover, all the cost lines are parallel: they all have a slope of $-4/5$. We want to find the cost line with the smallest C that still intersects the shaded area.

The lines on the graph, starting at the top right, have $C=20$, $C=15$, $C=12$ and $C=9.75$; it is clear that the further the lines are to the lower left, the smaller C and the lower the total cost.

The cost line that passes through the point of intersection of the two lines (Y1 and Y2)[a] defining the shaded region will therefore give the smallest C; this point is a vertex of the shaded quadrilateral.

V-Window $[0, 3, 1] \times [0, 3, 1]$

[a]Points on these lines satisfy the $\geqslant$ inequalities.

With the intersection point $(1.5, 0.75)$ (this point satisfies the $\geqslant$ inequalities),

$$C = 4 \times 1.5 + 5 \times 0.75 = 9.75.$$

Therefore, the cheapest way of getting your dietary requirements is to eat 1.5 kg of Food A and 0.75 kg of Food B per day, at a total cost of \$9.75.

The theory of linear programming says that the solution lies at a vertex of the region created by the inequalities.

4.3.2 Activities

Answers are provided here, full solutions in Section 4.5.2.

The fruit basket

The produce manager of a grocery is making up fruit baskets to sell as gifts. They are to sell for no more than \$5, and contain only apples and oranges. She wants to get 24c per orange, 12c per apple and 68c for the basket. No more than 26 pieces of fruit will fit in the basket. Suppose she uses x oranges and y apples.

(a) Show that we are looking for those x and y values that satisfy the inequalities

$$\begin{aligned} x &\geq 0 & \qquad x+y &\leq 26 \\ y &\geq 0 & \qquad 2x+y &\leq 36 \end{aligned}$$

(b) When the equality signs are used we have the equations for four lines. Draw a diagram showing those four lines and then find an area of the xy plane in which useful x and y values must be found. Shetch that area. *Remember to explain what you are doing.*

(c) Which of these (x, y) values could the manager use?

$$(5,10) \quad (10,5) \quad (5,25) \quad (20,5) \quad (-2,10)$$

(d) If she makes a profit of 3 cents on every orange sold and 2 cents on every apple sold, what is the equation for the total profit P? Draw in the $P = 30$ and $P = 45$ lines on your diagram. Can you see how to get the maximum profit? *Explain your reasoning.*

Answer: Maximum profit 62 cents; 10 oranges and 16 apples.

Market garden

The area available for crops on a farm is $1000\,\text{m}^2$. Two crops grow well in the area and are being considered for planting. Each crop takes a different amount of time per m^2 to prepare the soil and plant. The total cost of this soil preparation and planting should not exceed \$2,500.

- Beans (per m^2): cost \$3; profit \$2.
- Spinach (per m^2): cost \$2; profit \$1.50.

What areas of each should be planted to maximise profit?

Answer: $500\,\text{m}^2$ of each crop; profit \$1750.

Manufacturing #1

A factory manufactures doodads and whirligigs. It costs \$2 and takes 3 person hours to produce a doodad. It costs \$4 and takes 2 person hours to produce a whirligig. The factory has \$220 and 150 person hours a day to produce these products. If each doodad sells for \$6 and each whirligig sells for \$7, then how many of each product should be manufactured each day in order to maximise profit?

Answer: 20 doodads and 45 whirligigs; profit \$215.

Manufacturing #2

A manufacturer has 750 m of cotton fabric and 1000 m of polyester fabric. Production of a sweatshirt requires 1 m of the cotton and 2 m of the polyester, while production of a shirt requires 1.5 m of the cotton and 1 m of the polyester. The sale prices of a sweatshirt and a shirt are \$30 and \$24, respectively. How many of each type of shirt should be produced to maximise the return on sales (assuming all are sold)?

Answer: 375 sweatshirts and 250 shirts; return \$17,250.

Brainbuilding

A student is looking for supplement protein bars to help his brain work faster, and there are two available products: protein bar A and protein bar B.

Each protein bar A contains 15 g of protein and 30 g of carbohydrates and has total of 200 calories. Each protein bar B contains 30 g of protein and 20 g of carbohydrates and has total of 240 calories.

According to his nutritional plan, this student needs at least $20,000$ calories from these supplements over the month, which must comprise of at least $1,800$ g of protein and at least $2,200$ g of carbohydrates.

If each protein bar A costs \$3 and each protein bar B costs \$4, what is the least possible amount of money he can spend to meet all his one-month requirements?

Answer: 70 of protein bar A and 25 of protein bar B; total cost \$310.

Feeding the cows

A farmer feeds his cows a feed mix to supplement their foraging. The farmer uses two types of feed for the mix. Corn feed contains 100 g of protein per kg and 750 g starch per kg. Wheat feed contains 150 g protein per kg and 700 g starch per kg. Each cow should be fed at most 7 kg of feed per day. The farmer would like each cow to receive at least 650 g of protein and 4000 g of starch per day. If corn feed costs \$0.40/kg and wheat costs \$0.45/kg, what is the optimal feed mix that minimises cost? Round your answers to the nearest gram.

Answer: 3.412 kg of corn feed and 2.059 kg of wheat feed per cow; total cost \$2.29.

PTO

Pizza profits

Modified from *Profiteering from Pizza* in *The Atomic Project*, V. Geiger, J. McKinlay and G. O'Brien (eds), AAMT, 1997.

A company produces two frozen pizzas, the Gluttono and the Carnivore. The three main ingredients used in both pizzas are cheese, tomato and meat. The quantity of these ingredients required for each pizza and the weekly supply of these are listed in the table below.

Ingredient	**Grams required per Gluttono**	**Grams required per Carnivore**	**Weekly supply (kg)**
cheese	70	80	96
tomato	60	40	70
meat	10	60	48

The company makes a profit of \$4 on each Gluttono and \$5.50 on each Carnivore.

(a) How many of each type of pizza should the company produce each week to maximise the profit?

Answer: 564 Gluttonos and 705 Carnivores.

(b) What quantity of ingredients, if any, are left unused when the maximum profit is generated?

Extensions

(c) If the amount of cheese available increases to 110 kg per week, what is the effect on production, given the company still strives to maximise its profit?

(d) What is the effect of a 10% decrease in the profit made on a Gluttono pizza?

(e) What change in profit on the Gluttono pizza would alter the optimal solution?

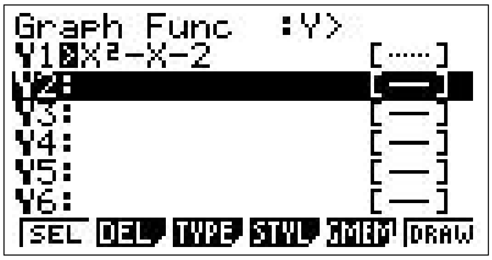
4.4 Quadratic inequalities

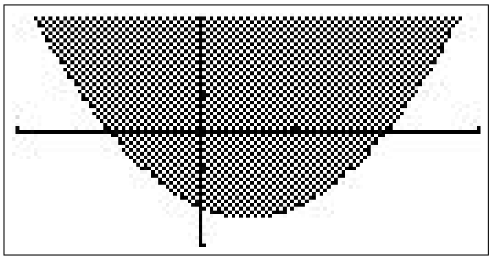
4.4.1 Methods

Example 8: Sketch the region in the xy plane for which $y > x^2 - x - 2$.

Graphical approach

- Select shading *above* for Y1 (Y>: a greater-than inequality), as in Example 5 on page 45, then enter $X^2 - X - 2$ and press EXE.
- Set an appropriate V-Window. You may need to experiment.
- Press EXIT then DRAW to show the region, the points in which satisfy the inequality.
- Pressing Trace puts the cursor on the curve dividing the two regions. The curve $y = x^2 - x - 2$ does not lie in the region specified by the inequality because of the $>$ sign.





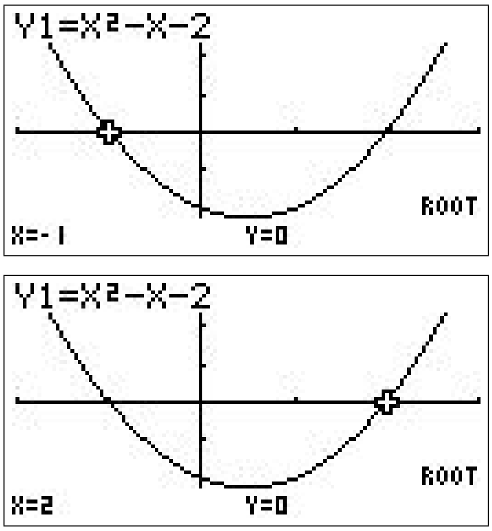
V-Window $[-2, 3, 1] \times [-3, 3, 1]$

Example 9: Find all values of x for which $x^2 - x - 2 < 0$.

Graphical approach

- Enter the left-hand side of the inequality into Y1.
- Set an appropriate V-Window.
- Graph the curve.
- Use ROOT in the G-Solv menu to find the (two) points where the curve crosses the x axis. Press the right arrow to find the second zero.

  $x^2 - x - 2 < 0$ for $-1 < x < 2$.



V-Window $[-2, 3, 1] \times [-3, 3, 1]$

PTO

Algebraic approach

The inequality can only be satisfied if the quadratic has two real roots.[11] This is the case if $b^2-4ac > 0$ in the quadratic formula; here, $b^2-4ac = 8 > 0$.

To solve the inequality, we need to factorise the quadratic: $x^2 - x - 2 = (x+1)\,(x-2)$.

$$\text{Then, } (x+1)\,(x-2) < 0 \quad \Longrightarrow \quad \begin{array}{c} x+1 < 0 \ \textbf{ and } \ x-2 > 0 \qquad (1) \\ \textbf{or} \\ x+1 > 0 \ \textbf{ and } \ x-2 < 0. \qquad (2) \end{array}$$

Equation (1) gives $x<-1$ **and** $x>2$, which has no solution.

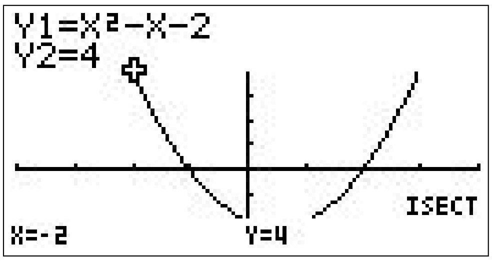
Equation (2) gives $x>-1$ **and** $x<2$, that is $-1<x<2$, which is the algebraic solution.

Example 10: Find all values of x for which $x^2-x-2 \leqslant 4$.

Graphical approach

- Enter the left-hand side of the inequality into Y1, the right-hand side into Y2.
- Set an appropriate $\boxed{\text{V-Window}}$.
- Graph the curves.
- Use $\boxed{\text{ISCT}}$ in the $\boxed{\text{G-Solv}}$ menu to find the (two) points where the curve crosses $y = 4$. Press the right arrow to find the second intersection point.

  $x^2 - x - 2 \leqslant 4$ for $-2 \leqslant x \leqslant 3$.



V-Window $[-3, 4, 1]\times[-3, 5, 1]$

Algebraic approach

Rewrite the inequality with 0 on the right-hand side: $x^2 - x - 6 \leqslant 0$.

This inequality can only be satisfied if the quadratic has two real roots or one real root. This is the case if $b^2-4ac \geqslant 0$ in the quadratic formula; here, $b^2-4ac = 25 > 0$.

To solve the inequality, factorise the quadratic: $x^2 - x - 6 = (x+2)\,(x-3)$.

$$\text{Then, } (x+2)\,(x-3) \leqslant 0 \quad \Longrightarrow \quad \begin{array}{c} x+2 \leqslant 0 \ \textbf{ and } \ x-3 \geqslant 0 \qquad (1) \\ \textbf{or} \\ x+2 \geqslant 0 \ \textbf{ and } \ x-3 \leqslant 0. \qquad (2) \end{array}$$

Equation (1) gives $x\leqslant-2$ **and** $x\geqslant 3$, which has no solution.

Equation (2) gives $x\geqslant-2$ **and** $x\leqslant 3$, that is $-2\leqslant x\leqslant 3$, which is the algebraic solution.

[11] If the inequality were $\leqslant$, it would be satisfied if there were only one real root: $b^2 - 4ac = 0$.

4.4.2 Exercises

Answers are provided here, full solutions in Section 4.5.3.

Solve the following inequalities using any or all of the three approaches.

1. $x^2 - 2x - 3 \leqslant 0.$ $(-1 \leqslant x \leqslant 3)$

2. $x^2 - 2x - 3 \leqslant 5.$ $(-2 \leqslant x \leqslant 4)$

3. $20x^4 - 4x^3 - 143x^2 + 46x + 165 < 0.$ $(-2.5 < x < 1 \text{ or } 1.5 < x < 2.2)$

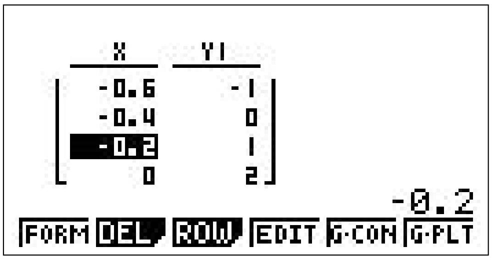
4. $|x^2 - 2x - 3| \leqslant 2.$ $(-1.449 \leqslant x \leqslant -0.4142 \text{ or } 2.414 \leqslant x \leqslant 3.449)$

4.5 Solutions

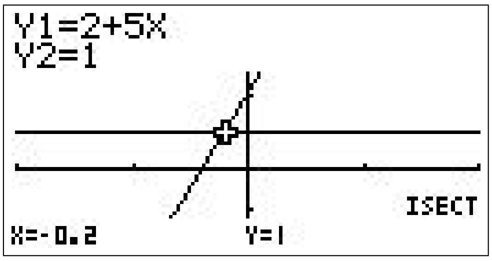
4.5.1 Exercises (Section 4.2.6)

1. Solve the inequality $2+5x \geqslant 1$ using the three approaches.

Numerically: setting Y1 = 2+5X, Start = −1, End = 1 and Step = 0.2, we obtain the table shown in the figure. Scrolling down, we see that $2+5x \geqslant 1$ if $x \geqslant -0.2$.



Graphically: setting Y1 = 2+5X and Y2 = 1, we obtain the graph in the figure. Using *ISCT*, we see that $2+5x \geqslant 1$ if $x \geqslant -0.2$.

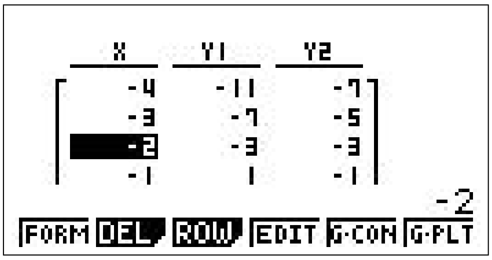


V-Window $[-2, 2, 1] \times [-2, 4, 1]$

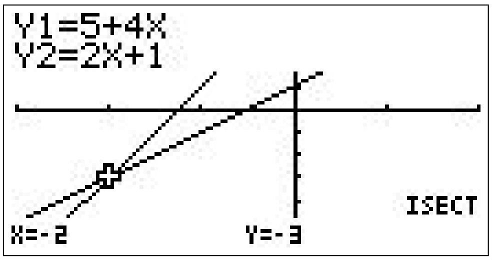
Algebraically: subtracting 2 from both sides of the inequality and dividing by 5, we have $x \geqslant -0.2$.

2. Solve the inequality $5+4x > 2x+1$ using the three approaches.

Numerically: with Y1 = 5 + 4X and Y2 = 2X+1, Start = −4, End = 0 and Step = 1, we obtain the table shown in the figure. From this, we see that $5+4x > 2x+1$ (Y1>Y2) if $x > -2$.



Graphically: we obtain the graph shown in the figure. Finding the intersection point of the two lines, we see that $2+5x > 2x+1$ (Y1>Y2) if $x > -2$.

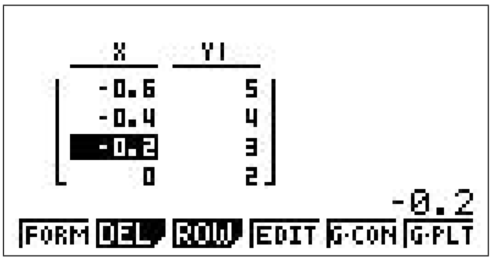


V-Window $[-3, 2, 1] \times [-6, 4, 1]$

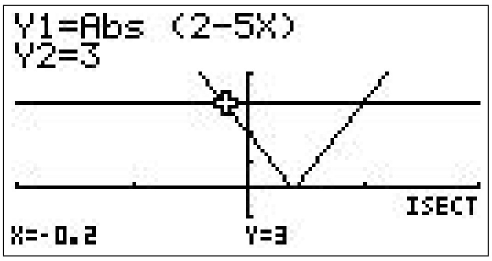
Algebraically: subtracting $2x$ and 5 from both sides of the inequality, we have $2x > -4$, so $x > -2$.

3. Solve the inequality $|2-5x| \leqslant 3$ using the three approaches.

Numerically: setting Y1 = Abs(2−5X), Start = −1, End = 2 and Step = 0.2, we obtain the table shown in the figure. From this, $|2-5x| \leqslant 3$ if $-0.2 \leqslant x \leqslant 1$.



Graphically: setting Y2 = 3, we obtain the graph shown in the figure. Finding the intersection points of the two graphs, we see that $|2-5x| \leqslant 3$ if
$-0.2 \leqslant x \leqslant 1$.



V-Window $[-2, 2, 1] \times [-2, 6, 1]$

Algebraically

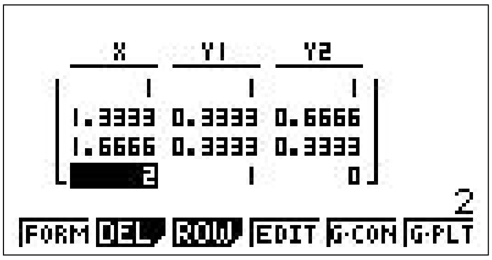
The inequality $|2-5x| \leqslant 3$ means $2-5x \leqslant 3$ **and** $-(2-5x) \leqslant 3$.

Multiply the second inequality by -1 and change the inequality direction to give $2-5x \leqslant 3$ **and** $2-5x \geqslant -3$, so that $-1 \leqslant 5x$ **and** $5 \geqslant 5x$.

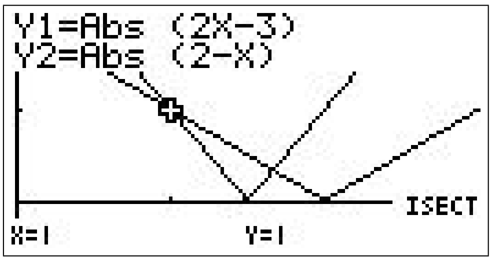
Dividing both by 5 and combining gives $-0.2 \leqslant x \leqslant 1$.

4. Solve the inequality $|2x-3| \leqslant |2-x|$ using the three approaches.

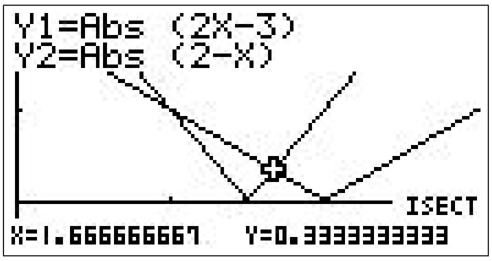
Numerically: with Y1 = Abs(2X−3), Y2 = Abs(2−X), Start = −1, End = 2 and Step = 1/3, we obtain the table shown in the figure.
$|2x-3| \leqslant |2-x|$ (Y1 ⩽ Y2) if $1 \leqslant x \leqslant \frac{5}{3}$.



Graphically: finding the intersection points of the two graphs, we see that $|2x-3| \leqslant |2-x|$ (Y1 ⩽ Y2) if $1 \leqslant x \leqslant \frac{5}{3}$.



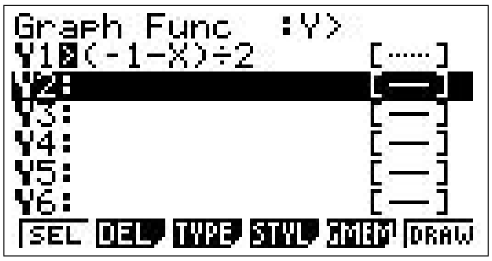
Algebraically: this gets messy. Better to use either of the other two methods.

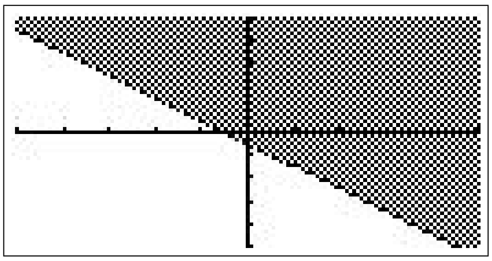


V-Window $[0, 3, 1] \times [-0.5, 2, 1]$

5. Graph the region satisfying the inequality $1 < x+2y$ for $-10 < x < 10$.

Rewrite as $y > \dfrac{-1-x}{2}$. Set Y1 > (−1−X)/2 ($\boxed{\text{TYPE}}$).





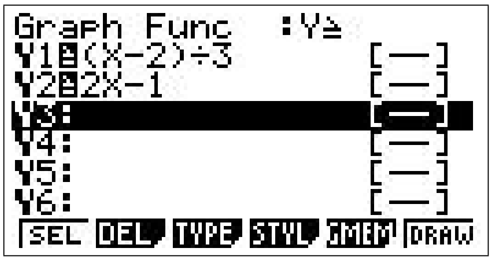
V-Window $[-10, 10, 2] \times [-5, 5, 1]$

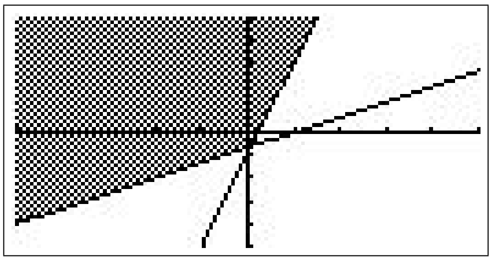
The point $(-3, 0)$ in the unshaded region does not satisfy the inequality, showing that the shaded region does. Points on the boundary line $y = (-1-x)/2$ do not satisfy the inequality because of the $<$ sign.

6. Graph the region satisfying the inequalities $x - 3y \leqslant 2$ **and** $y \geqslant 2x - 1$.

Rewrite the first inequality as $y \geqslant \dfrac{x-2}{3}$.

Set Y1 ⩾ (X−2)/3, shade above, and Y2 ⩾ 2X−1, shade above ($\boxed{\text{TYPE}}$).





V-Window $[-10, 10, 2] \times [-5, 5, 1]$

The shaded region is the solution region. Check with a point in the region such as $(-2, 0)$.

Points on the boundary lines do satisfy the inequalities because of the $\leqslant$ and $\geqslant$ signs.

4.5.2 Activities (Section 4.3.2)

The fruit basket

The produce manager of a grocery is making up fruit baskets to sell as gifts. They are to sell for no more than \$5, and contain only apples and oranges. She wants to get 24c per orange, 12c per apple and 68c for the basket. No more than 26 pieces of fruit will fit in the basket. Suppose she uses x oranges and y apples.

(a) Show that we are looking for those x and y values that satisfy the inequalities

$$\begin{aligned} x &\geq 0 & \qquad x+y &\leq 26 \\ y &\geq 0 & \qquad 2x+y &\leq 36 \end{aligned}$$

The numbers of oranges and apples are zero or positive, so that

$$x \geq 0 \qquad y \geq 0.$$

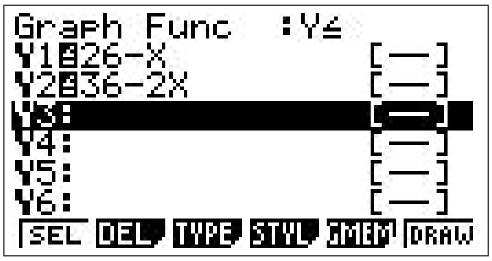
There are to be no more than 26 pieces of fruit, so that $x+y \leq 26$.

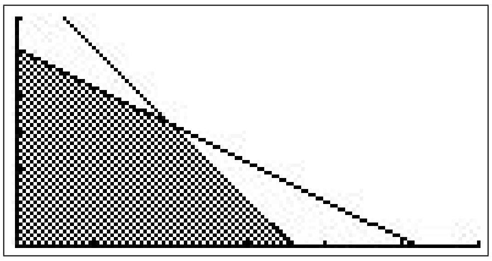
The total cost of fruit plus basket is to be no more than \$5, so that

$$24x + 12y + 68 \leq 500$$

(b) When the equality signs are used we have the equations for four lines. Draw a diagram showing those four lines and then find an area of the xy plane in which useful x and y values must be found. Shetch that area.

$x = 0$ and $y = 0$ give the axes. Rewrite the other two inequalities as $y \leqslant 26 - x$ and $y \leqslant 36 - 2x$ for graphing; set shaded below to give a shaded solution region. The shaded area bounded by the axes and the two lines contains the (x, y) values that satisfy all the inequalities. The two lines intersect at $(10, 16)$ (G-Solv ISCT).





V-Window $[0, 30, 5] \times [0, 30, 5]$

(c) Which of these (x, y) values could the manager use?

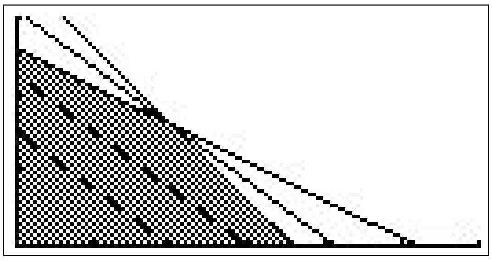
$$(5, 10) \quad (10, 5) \quad (5, 25) \quad (20, 5) \quad (-2, 10)$$

$(5, 10)$ and $(10, 5)$ are in the shaded area and so satisfy all the inequalities; these (x, y) values could be used. The other (x, y) values are in the unshaded area and could not be used.

(d) If she makes a profit of 3 cents on every orange sold and 2 cents on every apple sold, what is the equation for the total profit P? Draw in the $P=30$ and $P=45$ lines on your diagram. Can you see how to get the maximum profit?

The profit is given by $P=3x+2y$, which we rewrite for graphing as $y=\frac{1}{2}(P-3x)$.

The lines $P=30$, $P=45$ and $P=62$ are shown in the figure below, starting bottom left.



V-Window $[0,30,5]\times[0,30,5]$

All profit lines will be parallel to those. The largest value of P is for the profit line farthest to the right and still intersecting the shaded region, i.e. the line through the corner point (vertex) of the region $(x,y) = (10,16)$, giving $P = 62$ from the profit equation.

The maximum profit is therefore 62 cents, obtained with 10 oranges and 16 apples in a basket.

Market garden

The area available for crops on a farm is $1000\,\text{m}^2$. Two crops grow well in the area and are being considered for planting. Each crop takes a different amount of time per m^2 to prepare the soil and plant. The total cost of this soil preparation and planting should not exceed \$2,500.

- Beans (per m^2): cost \$3; profit \$2.
- Spinach (per m^2): cost \$2; profit \$1.50.

What areas of each should be planted to maximise profit?

Let x (m^2) be the area of beans planted and y (m^2) the area of spinach. Then,

$x + y \leqslant 1000$ total area,

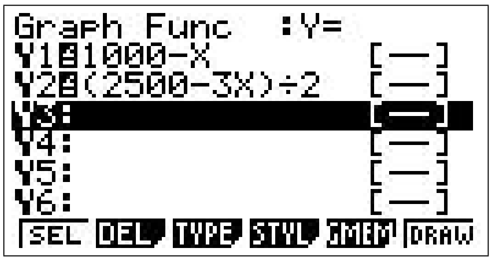
$3x + 2y \leqslant 2500$ total cost (\$),

$x,\ y \geqslant 0$ areas can't be negative;

profit P(\$) is given by $P = 2x + 1.5y$.

PTO

Rewrite the first two inequalities as $y \leqslant 1000-x$ and $y \leqslant \frac{1}{2}(2500-3x)$; set shaded below to give a shaded solution region. Graphing the four inequalities gives the figure below. The two curves intersect at the point $(500, 500)$ $\big($ $\boxed{\text{G-Solv}}$ $\boxed{\text{ISCT}}$ $\big)$.



V-Window $[0, 1000, 500] \times [0, 1300, 500]$

Rewriting the profit equation gives $y = \frac{2}{3}(P-2x)$. Profit curves with $P = 500$, $P = 1000$ and $P = 1750$ are shown in the figure below, starting bottom left. The profit curve with the greatest profit is the one that passes through the intersection point $(500, 500)$ (a vertex of the shaded quadrilateral), giving $P = 1750$.

V-Window $[0, 1000, 500] \times [0, 1300, 500]$

Therefore, $500\,\text{m}^2$ of each crop should be planted, giving a profit of \$1750.

Manufacturing #1

A factory manufactures doodads and whirligigs. It costs \$2 and takes 3 person hours to produce a doodad. It costs \$4 and takes 2 person hours to produce a whirligig. The factory has \$220 and 150 person hours a day to produce these products. If each doodad sells for \$6 and each whirligig sells for \$7, then how many of each product should be manufactured each day in order to maximise profit?

Let x be the number of doodads produced and y the number of whirligigs. Then,

$2x + 4y \leqslant 220$ total cost,

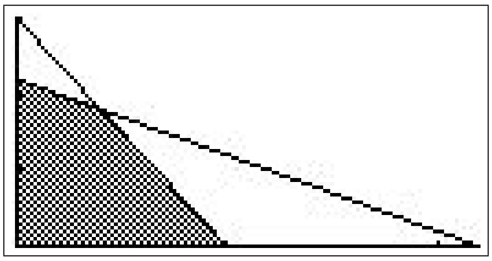
$3x + 2y \leqslant 150$ total hours,

$x,\, y \geqslant 0$ numbers produced can't be negative;

profit $P(\$)$ is given by $P = 4x + 3y$.

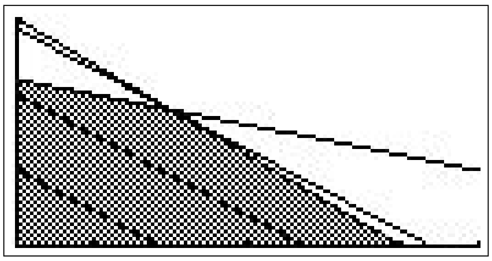
Rewrite the first two inequalities as $y \leqslant 55 - \dfrac{x}{2}$ and $y \leqslant 75 - \dfrac{3x}{2}$, and set shaded below to give a shaded solution region.

Graphing the two inequalities gives the figure below. The two curves intersect at the point $(20, 45)$ ($\boxed{\text{G-Solv}}$ $\boxed{\text{ISCT}}$).



V-Window $[0, 110, 50] \times [0, 75, 25]$

Rewriting the profit equation gives $y = \frac{1}{3}(P - 4x)$. Several profit curves are shown in the figure below: $P = 75$, $P = 150$ and $P = 215$, starting bottom left. The profit curve with the greatest profit is the one that passes through the intersection point $(20, 45)$ (a vertex of the shaded quadrilateral), giving $P = 215$.



V-Window $[0, 60, 10] \times [0, 75, 25]$

Therefore, 20 doodads and 45 whirligigs should be produced each day, giving a profit of \$215.

Manufacturing #2

A manufacturer has 750 m of cotton fabric and 1000 m of polyester fabric. Production of a sweatshirt requires 1 m of the cotton and 2 m of the polyester, while production of a shirt requires 1.5 m of the cotton and 1 m of the polyester. The sale prices of a sweatshirt and a shirt are \$30 and \$24, respectively. How many of each type of shirt should be produced to maximise the return on sales (assuming all are sold)?

Let x be the number of sweatshirts produced and y the number of shirts. Then,

$x + 1.5y \leqslant 750$ amount of cotton used,

$2x + y \leqslant 1000$ amount of polyester used,

$x, y \geqslant 0$ numbers produced can't be negative;

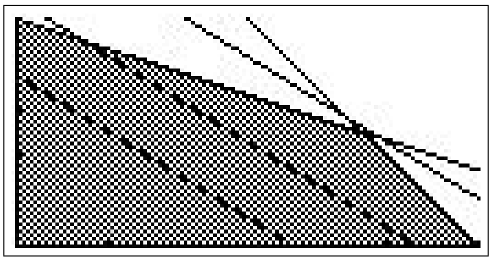
the return on sales R(\$) is $R = 30x + 24y$.

Rewrite the first two inequalities as $y \leqslant \frac{2}{3}(750 - x)$ and $y \leqslant 1000 - 2x$, and set shaded below to give a shaded solution region.

Graphing the two inequalities gives the figure below. The two curves intersect at the point $(375, 250)$ ($\boxed{\text{G-Solv}}$ $\boxed{\text{ISCT}}$).

V-Window $[0, 500, 100] \times [0, 500, 100]$

Rewriting the return equation gives $y = \frac{1}{24}(R - 30x)$. Several return curves are shown in the figure below, starting at the bottom left: $R = 9000$; $R = 13,000$; and $R = 17,250$. The curve with the greatest return is the one that passes through the intersection point $(375, 250)$ (a vertex of the shaded quadrilateral), giving $R = 17,250$.



V-Window $[0, 500, 100] \times [0, 500, 100]$

Therefore, 375 sweatshirts and 250 shirts should be produced, giving a total return of \$17,250.

Brainbuilding

A student is looking for supplement protein bars to help his brain work faster, and there are two available products: protein bar A and protein bar B.

Each protein bar A contains 15 g of protein and 30 g of carbohydrates and has a total of 200 calories. Each protein bar B contains 30 g of protein and 20 g of carbohydrates and has a total of 240 calories.

According to his nutritional plan, this student needs at least 20,000 calories from these supplements over the month, which must comprise of at least 1800 g of protein and at least 2200 g of carbohydrates.

If each protein bar A costs \$3 and each protein bar B costs \$4, what is the least possible amount of money he can spend to meet all his one-month requirements?

PTO

Assume he buys x of protein bar A and y of protein bar B in a month. Then,

$200x + 240y \geqslant 20,000$ total calories,

$15x + 30y \geqslant 1800$ total protein (g),

$30x + 20y \geqslant 2200$ total carbohydrate (g),

$x,\ y \geqslant 0$ quantities must be non-zero;

the total cost $C(\$)$ is given by $C = 3x + 4y$ ($).

Rewrite the first three inequalities as $y_1 \geqslant \frac{1}{240}(20,000-200x)$, $y_2 \geqslant \frac{1}{30}(1800-15x)$ and $y_3 \geqslant \frac{1}{20}(2200-30x)$. Set shaded above to give a shaded solution region.

Graphing these gives the figure below. Two intersection points of the three curves are vertices of the shaded region; one of these must give the solution. y_1 and y_2 intersect at the point $(70, 25)$, y_1 and y_3 at the point $(40, 50)$ $\left(\boxed{\text{G-Solv}}\ \boxed{\text{ISCT}}\right)$.

V-Window $[0, 100, 50]\times[0, 100, 50]$

At this stage, knowing that one of these points must provide the solution, we could simply substitute both points into the cost equation to see which gives the lower cost.

The point $(40, 50)$ gives $C=3\times 40+4\times 50=320$.

The point $(70, 25)$ gives $C=3\times 70+4\times 25=310$, which is therefore the solution.

However, plotting a few cost curves shows us why this is the solution.

Rewriting the cost equation gives $y=\frac{1}{4}(C-3x)$. Several cost curves are shown in the figure below: $C = 500$; $C = 400$; and $C = 310$, starting top right. The curve that passes through the intersection point $(70, 25)$ has $C = 310$, that through the intersection point $(40, 50)$ has $C=320$.

V-Window $[0, 100, 50]\times[0, 100, 50]$

Therefore, he should buy, monthly, 70 of protein bar A and 25 of protein bar B, at a total cost of $310.

Feeding the cows

A farmer feeds his cows a feed mix to supplement their foraging. The farmer uses two types of feed for the mix. Corn feed contains 100 g of protein per kg and 750 g starch per kg. Wheat feed contains 150 g protein per kg and 700 g starch per kg. Each cow should be fed at most 7 kg of feed per day. The farmer would like each cow to receive at least 650 g of protein and 4000 g of starch per day. If corn feed costs \$0.40/kg and wheat costs \$0.45/kg, then what is the optimal feed mix that minimises cost? Round your answers to the nearest gram.

Assume he feeds each cow x kg of corn feed and y kg of wheat feed each day. Then, noting the different inequalities,

$x + y \leqslant 7$ total amount fed per cow (kg),

$100x + 150y \geqslant 650$ total protein (g),

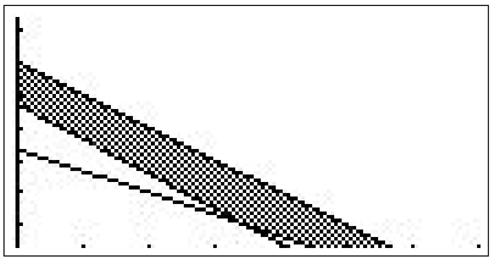
$750x + 700y \geqslant 4000$ total starch (g),

$x,\, y \geqslant 0$ quantities must be non-zero;

the total cost $C(\$)$ is given by $C = 0.4x + 0.45y$.

Rewrite the first 3 inequalities as $y \leqslant 7 - x$, $y \geqslant \frac{1}{150}(650 - 100x)$ and $y \geqslant \frac{1}{700}(4000 - 750x)$.

On graphing the inequalities, shaded appropriately to give a shaded solution region, we obtain the figure here.
The bottom two curves intersect at the point $(3.412, 2.059)$ (G-Solv ISCT) which, according to the theory, must be the solution because it is the only vertex (intersection) of the shaded region.



V-Window $[0, 7, 1] \times [0, 7, 1]$

Checking graphically, rewriting the cost equation gives $y = \frac{1}{0.45}(C - 0.4x)$. Several cost curves (heavy dotted lines) are shown in the figure below, starting at the top right: $C = 4$; $C = 3$; and $C = 2.29$. Clearly, the curve with the least cost is the one that passes through the intersection point of the bottom two curves $(3.412, 2.059)$ (a vertex of the shaded quadrilateral), giving $C = 2.29$.

V-Window $[0, 7, 1] \times [0, 7, 1]$

Therefore, the farmer should feed each cow 3.412 kg of corn feed and 2.059 kg of wheat feed each day (rounded to the nearest gram), at a total cost of \$2.29 per cow.

Pizza profits

A company produces two frozen pizzas, the Gluttono and the Carnivore. The three main ingredients used in both pizzas are cheese, tomato and meat. The quantity of these ingredients required for each pizza and the weekly supply of these are listed in the table below.

Ingredient	Grams required per Gluttono	Grams required per Carnivore	Weekly supply (kg)
cheese	70	80	96
tomato	60	40	70
meat	10	60	48

The company makes a profit of \$4 on each Gluttono and \$5.50 on each Carnivore.

(a) How many of each type of pizza should the company produce each week to maximise the profit?

Let x and y be the numbers of Gluttono and Carnivore pizzas produced per week. Then,

$70x + 80y \leqslant 96,000$ quantity of cheese (g),

$60x + 40y \geqslant 70,000$ quantity of tomato (g),

$10x + 60y \geqslant 48,000$ quantity of meat (g),

$x,\ y \geqslant 0$ quantities must be non-zero;

the total profit P(\$) is given by $P = 4x + 5.5y$.

Rewrite the inequalities as $y_1 \leqslant \frac{1}{80}(96000-70x)$, $y_2 \leqslant \frac{1}{40}(70000-60x)$ and $y_3 \leqslant \frac{1}{60}(48000-10x)$.

On graphing the inequalities, shaded appropriately to give a shaded solution region, we obtain the figure here.
y_1 and y_3 intersect at the point $(564.7, 705.9)$; y_1 and y_2 at the point $(880, 430)$ (G-Solv ISCT).

V-Window $[0, 1200, 200] \times [0, 1200, 200]$

Rewriting the profit equation gives $y = \frac{1}{5.5}(P-4x)$. Several cost curves are shown in the figure below, starting at the bottom left: $P = 4000$; $P = 5000$; and $P = 6000$.

Clearly, the curve with the greatest profit is the one that passes through the intersection point of y_1 and y_3, $(564.7, 705.9)$ (a vertex of the shaded quadrilateral), giving $P = 6141$.

V-Window $[0, 1200, 200] \times [0, 1200, 200]$

Therefore, the maximum profit of \$6141 is generated if 564.7 Gluttonos and 705.9 Carnivores are produced each week.

These numbers should be rounded down to 564 and 705 to be appropriate to the problem and still fit the constraints. The maximum profit is then \$6133.50.

(b) What quantity of ingredients, if any, are left unused when the maximum profit is generated?

Substituting $x = 564$ and $y = 705$ back into the original constraint equations gives the amount of cheese used as 95.88 kg, so that only 120 g of cheese is unused at the end of the week. Similarly, 8 kg of tomatoes and 60 g of meat are unused.

This seems cutting it a bit fine for cheese and meat, so maybe some adjustment is needed.

Extensions

(c) If the amount of cheese available increases to 110 kg per week, what is the effect on production, given the company still strives to maximise its profit?

Cheese no longer comes into consideration. Only one intersection point (Y_2 and Y_3) at $(712.5, 681.25)$, giving 712 Gluttonos and 681 Carnivores and a profit of \$6593.50.

(d) What is the effect of a 10% decrease in the profit made on a Gluttono pizza?

Same optimum point but profit now \$5907.90.

(e) What change in profit on the Gluttono pizza would alter the optimal solution?

For the other intersection point (Y_1 and Y_2) to be the optimal solution, we need the profit-line slope $-P_G/P_C$ steeper than that of Y_1, which is $-7/8$. Therefore, with $P_C = 5.5$, need $P_G > 4.8125$.

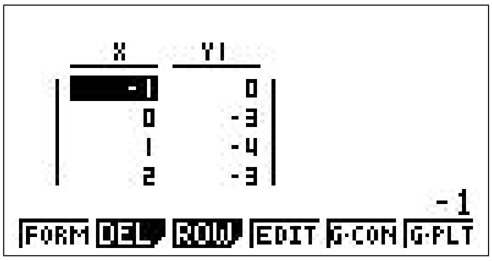
The profit on the Gluttono pizza would have to be greater than \$4.82 to alter the optimal solution.

4.5.3 Quadratic inequalities (Section 4.4.2)

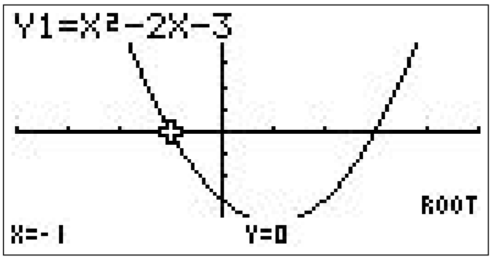
Solve the following inequalities using any or all of the three approaches.

1. $x^2 - 2x - 3 \leqslant 0$.

Numerically: setting Y1 = X^2 − 2X − 3, Start = −3, End = 4 and Step = 1, we obtain the table shown in the figure. From this, $x^2 - 2x - 3 \leqslant 0$ if $-1 \leqslant x \leqslant 3$.



Graphically: we obtain the graph in the figure. Finding the zeros ($\boxed{\text{G-Solv}}$), we see that $x^2 - 2x - 3 \leqslant 0$ if $-1 \leqslant x \leqslant 3$.



V-Window $[-4, 5, 1] \times [-5, 5, 1]$

Algebraically

To solve the inequality, we then need to factorise the quadratic:

$x^2 - 2x - 3 = (x + 1)(x - 3)$.

Then, $(x+1)(x-3) \leqslant 0 \quad \Longrightarrow \quad x + 1 \leqslant 0$ **and** $x - 3 \geqslant 0$ (1)

or

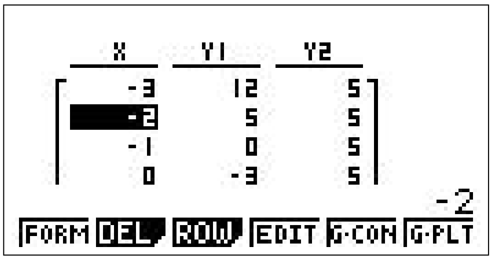
$x + 1 \geqslant 0$ **and** $x - 3 \leqslant 0$. (2)

Equation (1) gives $x \leqslant -1$ **and** $x \geqslant 3$, which has no solution.

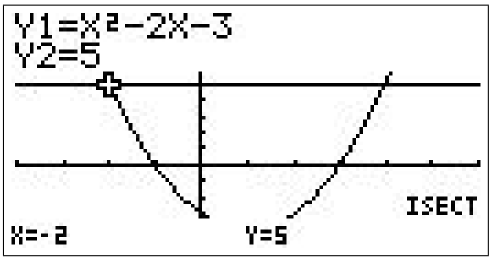
Equation (2) gives $x \geqslant -1$ **and** $x \leqslant 3$, that is $-1 \leqslant x \leqslant 3$, which is the algebraic solution.

2. $x^2 - 2x - 3 \leqslant 5$.

Numerically: setting Y1 = X^2−2X−3, Y2 = 5 and Step = 1, we obtain the table shown in the figure. From this, $x^2 - 2x - 3 \leqslant 5$ if $-2 \leqslant x \leqslant 4$.



Graphically: setting Y1 = X^2−2X−3 and Y2 = 5, we obtain the graph shown in the figure. Finding the intersections ($\boxed{\text{G-Solv}}$), we see that $x^2 - 2x - 3 \leqslant 5$ if $-2 \leqslant x \leqslant 4$.



V-Window $[-4, 6, 1] \times [-5, 7, 1]$

Algebraically

First, rewrite the inequality as $x^2 - 2x - 8 \leqslant 0$.

To solve the inequality, we then need to factorise the quadratic:

$x^2 - 2x - 8 = (x+2)(x-4)$.

$$\text{Then, } (x+2)(x-4) \leqslant 0 \quad \Longrightarrow \quad \begin{array}{cr} x+2 \leqslant 0 \ \textbf{ and } \ x-4 \geqslant 0 & (1) \\ \textbf{or} & \\ x+2 \geqslant 0 \ \textbf{ and } \ x-4 \leqslant 0. & (2) \end{array}$$

Equation (1) gives $x \leqslant -2$ **and** $x \geqslant 4$, which has no solution.

Equation (2) gives $x \geqslant -2$ **and** $x \leqslant 4$, that is $-2 \leqslant x \leqslant 4$, which is the algebraic solution.

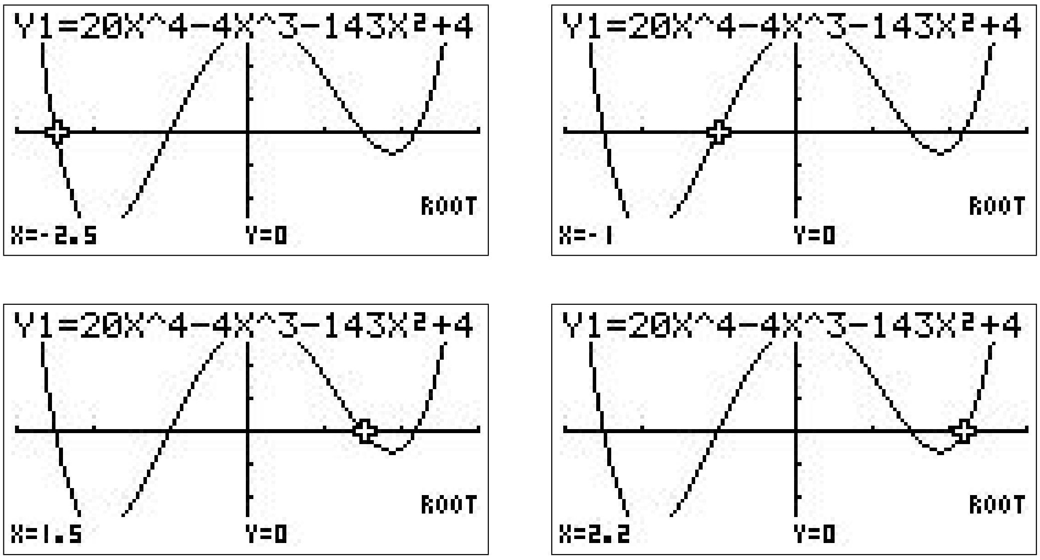
3. $20x^4 - 4x^3 - 143x^2 + 46x + 165 < 0$.

The only sensible way to do this is graphically.

Set Y1 = 20X^4−4X^3−143X^2+46X+165 and graph with an appropriate $\boxed{\text{V-Window}}$.

As this is a fourth-degree polynomial, we expect up to 4 zeros.

Using $\boxed{\text{G-Solv}}$ $\boxed{\text{ROOT}}$:

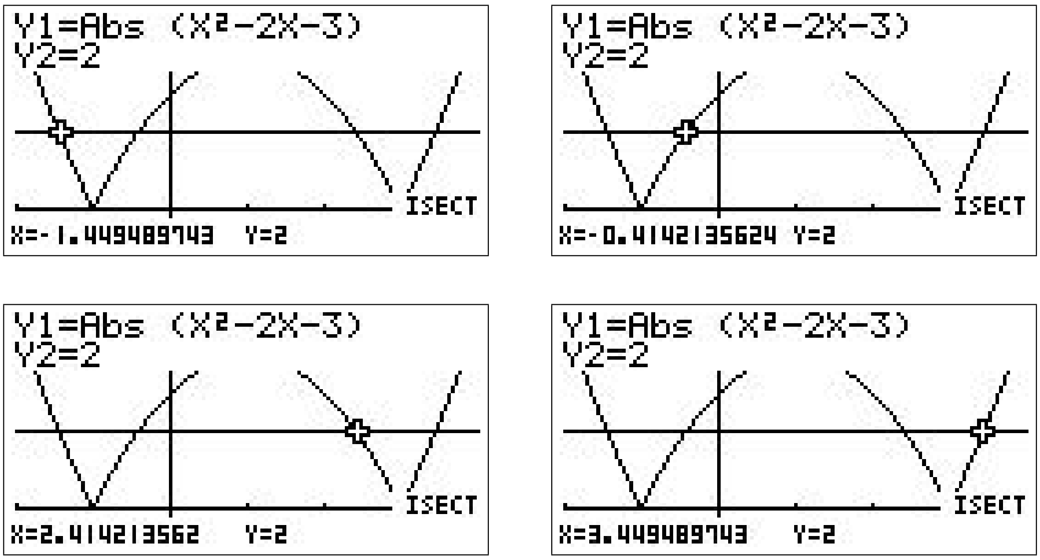


V-Window $[-3, 3, 1] \times [-170, 170, 50]$

From the graphs, $20x^4 - 4x^3 - 143x^2 + 46x + 165 < 0$ if $-2.5 < x < 1$ or $1.5 < x < 2.2$.

4. $|x^2 - 2x - 3| \leqslant 2$.

Again, the only sensible way to do this is graphically. Set Y1 = Abs(X^2−2X−3), Y2 = 2 and graph with an appropriate $\boxed{\text{V-Window}}$. Using $\boxed{\text{G-Solv}}$ $\boxed{\text{ROOT}}$:



V-Window $[-2, 4, 1] \times [-1, 5, 1]$

Therefore, $|x^2 - 2x - 3| \leqslant 2$ if $-1.449 \leqslant x \leqslant -0.4142$ or $2.414 \leqslant x \leqslant 3.449$, approximately.

5 Fitting Curves to Data

5.1 Introduction

Much scientific and other research involves data. Fitting a function to the data is a way of summarising the data; if the fit is good, the fitted curve can be used instead of the data in further calculations, especially useful if Calculus is involved. Sometimes the function chosen is guided by the theory involved, for example motion under gravity. At other times, the choice is empirical: the function which gives the best fit is used.

The material here is especially relevant to the second case. The functions available on the 9860 are given here, together with an example of each in fitting a given dataset.

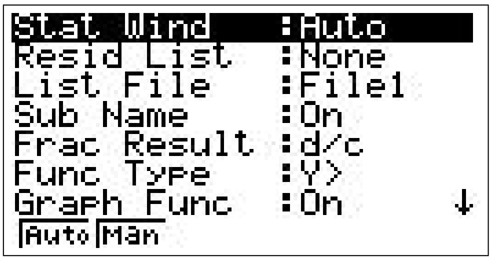
5.2 Setting up

Launch the STAT application [MENU] [2]. This takes you to the List Editor.

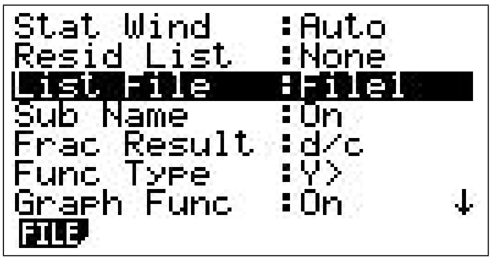
Enter the [SET UP] menu ([SHIFT] [MENU]).

Most of the following options will already be set.

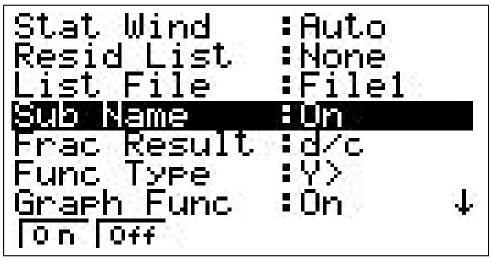
When *Stat Wind* is set to *Auto*, the calculator will set axis scales on a plot that are appropriate for the data. You can always change these later. Set on *Auto*.



With *List File* set to *File1*, the List Editor displays the 26 lists (scroll across) that are collectively called File1. Choose FILE to change the set of lists displayed.



Sub Name ON lets you give lists a text name. Choose ON.

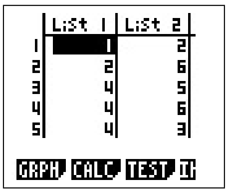


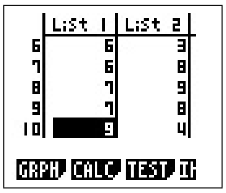
The defaults for the other options in SET UP are usually the ones you want.

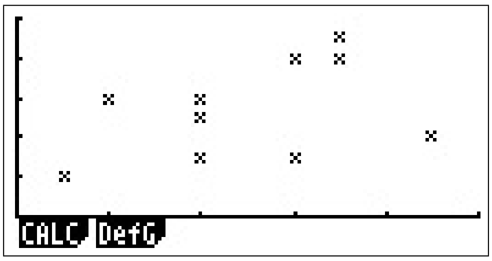
Press [EXIT] to return to the List Editor.

5.3 Operations on the calculator

The data shown below left and centre, and graphed (see Section 3.2.7) below right are used for all the fits here. List 1 contains the X values, List 2 the Y values.

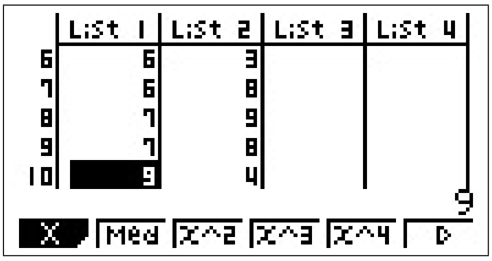


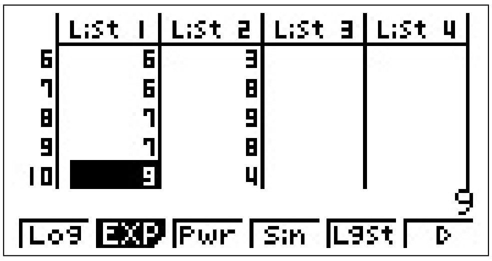




data V-Window $[0,10,2]\times[0,10,2]$

The curve-fitting commands are in the CALC REG menu (screens below: F6 has been pressed after the first screen to show the remaining commands).





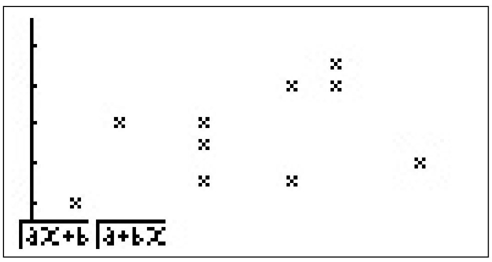
General procedure: Graph the data as above. Then press CALC (F1) and choose which regression to use. Pressing the appropriate key will give the result. Pressing F6 (DRAW) then plots the regression line/curve over the data points. DefG takes you to the function screen to look at the equation of the regression line/curve.

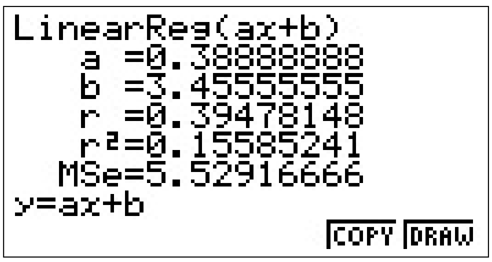
5.3.1 Linear functions

Linear regression, the fitting of straight lines to data, is the workhorse of simple data fitting. Provided the regression coefficient r is not exactly zero, all sorts of claims are made that the data are linear. The other methods presented here allow you to go beyond linear regression easily. If some other type of function gives a much better fit, you can start thinking whether there might be a good reason for this.

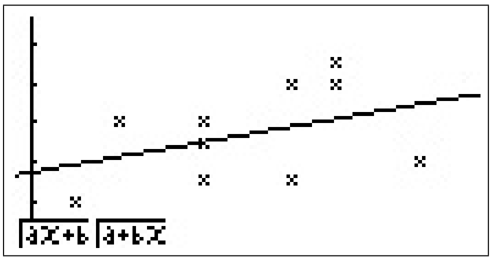
Graph the data as above. Then press CALC (F1) and X (F1) to show the two linear regressions (below left).

Pressing $aX{+}b$ fits the model equation $y = ax + b$ to the data using a least-squares fit. It displays values for a (slope) and b (y-intercept); it also displays values for r, r^2 and mean standard error (MSe) (below right).





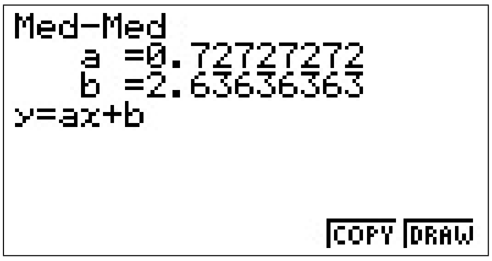
Pressing $\boxed{\text{F6}}$ (DRAW) then plots the straight line over the data points (below).

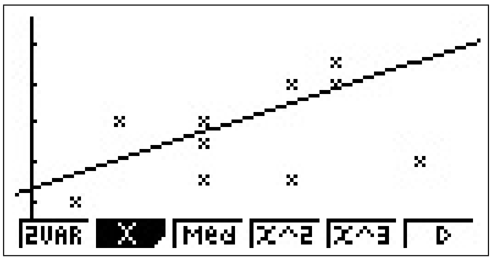


Pressing $\boxed{a{+}bX}$ similarly fits the model equation $y{=}a{+}bx$ to the data using a least-squares fit. The output is the same as for $ax{+}b$) above but with a and b interchanged.

The least-squares regression line above is relatively easy to calculate and has a sound theoretical foundation to justify its use, but outliers can have a large effect on the line. The median-median regression line is a more-resistant alternative. To obtain the line, the data points are divided into three equal groups by size of x.[12] For the low and high groups, the medians of the x and y values are obtained separately, giving one summary point for each group. The median-median regression line is the line joining these two points.

$\boxed{\text{Med}}$ (median-median) fits the model equation $y = ax + b$ to the data using the median-median regression technique and displays values for a (slope) and b (y-intercept).

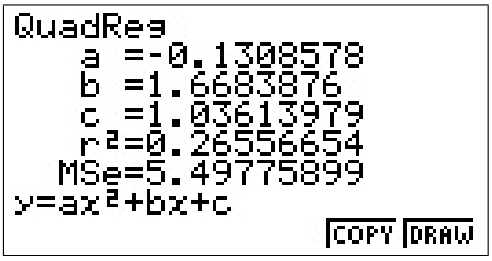


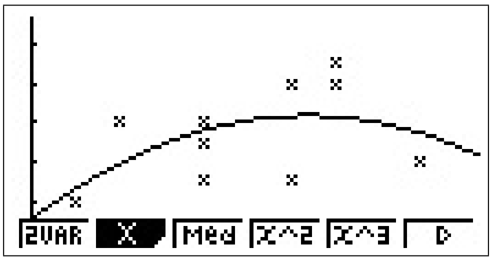


5.3.2 Quadratic functions

Quadratic functions occur in many areas involving data, with perhaps the best-known being the position of a body moving vertically under gravity. As in this case, theory often directs you to using a quadratic fit. Any set of data with a single (local) maximum or minimum is a candidate for a quaratic fit.

$\boxed{x\wedge 2}$ (quadratic regression) fits the second-degree polynomial $y = ax^2 + bx + c$ to the data. It displays values for a, b and c, r^2 and MSe. For three data points, the equation is a polynomial fit; for four or more, it is a polynomial regression. At least three data points are required.





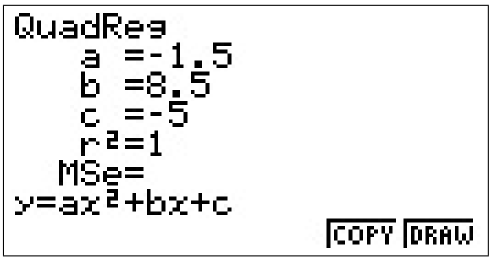
[12]If there is one extra point, the middle group is made one larger; if there are two extra points, the low and high groups are each made one larger.

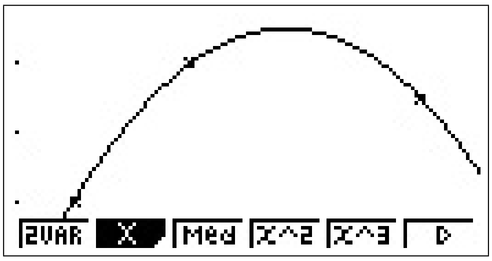
Finding maxima or minima of data

A quick way of finding (an approximation to) the maximum (minimum) of a set of data is to fit a quadratic function to the top (bottom) three points (and store the quadratic in Y1 if you want to calculate the y value of the maximum/minimum). The maximum (minimum) of the data is then given approximately by the maximum (minimum) of the quadratic: $x_m = -b/(2a)$ and $y_m = \mathrm{Y1}(x_m)$.

This works best if the highest (lowest) point is the middle point.

Example: Consider the first three points in the data used so far. A (perfect) quadratic fit to these points gives the result below left. Immediately, $x_m = -8.5/(2\times -1.5) = 2.83$ and $y_m = \mathrm{Y1}(x_m) = 7.04$. Graphing the fitted parabola shows this maximum point (below right).

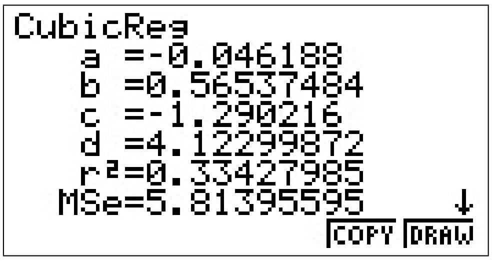


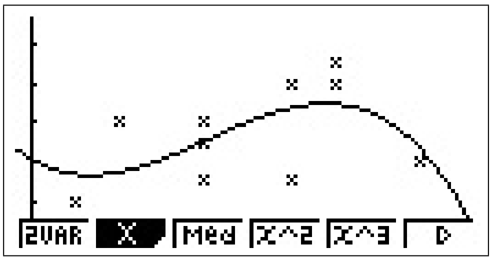


5.3.3 Cubic functions

Less common than quadratics, there are nevertheless theories that predict cubic behaviour. Any set of data with a (local) minimum and (local) maximum is a candidate for a cubic fit.

$\boxed{x\wedge 3}$ (cubic regression) fits the third-degree polynomial $y = ax^3 + bx^2 + cx + d$ to the data. It displays values for a, b, c and d, and r^2 and MSe. For four points, the equation is a polynomial fit; for five or more, it is a polynomial regression. At least four points are required.

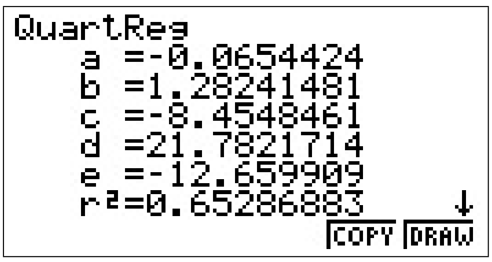


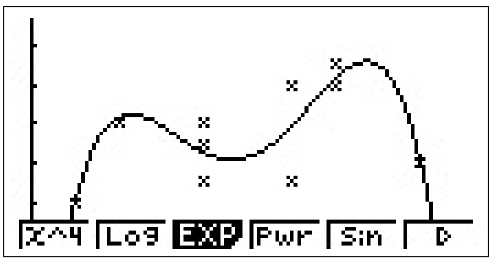


5.3.4 Quartic functions

Quartic functions, in general, are characterised by either one (local) minimum and two (local) maxima (negative a; as in the figure below) or vice versa (positive a).

$\boxed{x\wedge 4}$ (quartic regression) fits the fourth-degree polynomial $y = ax^4 + bx^3 + cx^2 + dx + e$ to the data. It displays values for a, b, c, d and e, and r^2 and MSe. For five points, the equation is a polynomial fit; for six or more, it is a polynomial regression. At least five points are required.

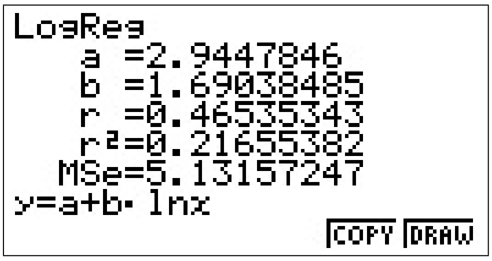


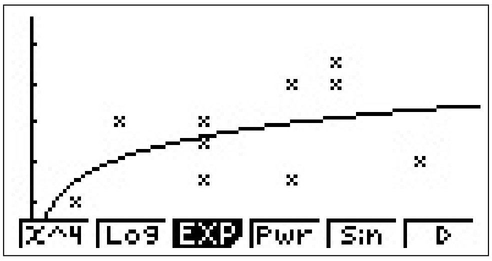


5.3.5 Logarithmic functions

Logarithmic functions $y = a + b\ln(x)$ with $b < 0$ appear to approach some finite value asymptotically but actually increase without bound; those with $b > 0$ tend asymptotically but slowly to zero.

$\boxed{\text{Log}}$ (logarithmic regression) fits the equation $y = a + b\ln(x)$ to the data using a least-squares fit. It displays values for a and b, and r, r^2 and MSE. The fit is done by turning the logarithmic function into a linear function by defining a new independent variable $X = \ln(x)$.

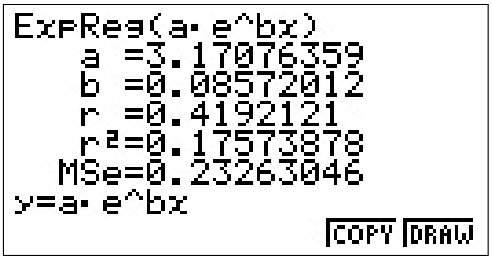


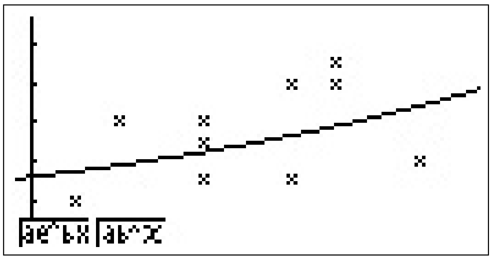


5.3.6 Exponential functions

Exponential functions are common in mathematical modelling and associated data; the most common form is *the* exponential e^{ax}. With $a < 0$, it models decay (for example, radioactive decay or declining populations), absorption (for example, of light) and slowing down (for example, air resistance and friction). Positive a is less common, as this represents exponential growth, which is unsustainable over any distance or time.

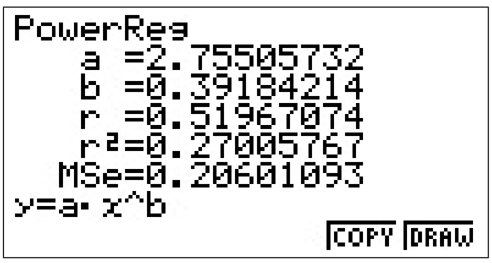
$\boxed{\text{EXP}}$ (exponential regression) fits the model equation $y = ae^{bx}$ (figure below) or $y = ab^x$ to the data using a least-squares fit. It displays values for a and b, and r, r^2 and MSe. The fit is done by taking natural logs of both sides of the equation, then turning this into a linear function by defining a new dependent variable $Y = \ln(y)$.

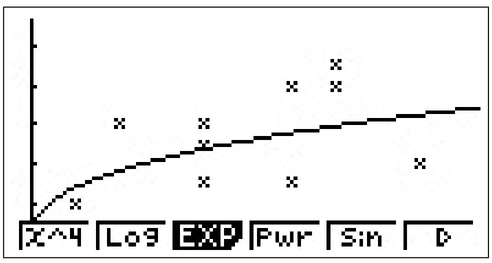




5.3.7 Power functions

$\boxed{\text{Pwr}}$ (power regression) fits the model equation $y = ax^b$ to the data using a least-squares fit with transformed values $\ln(x)$ and $\ln(y)$. It displays values for a and b, and r, r^2 and MSE. The fit is done by taking natural logs of both sides of the equation, then turning this into a linear function by defining new variables $Y = \ln(y)$ and $X = \ln(x)$. Log-log plots are often used to display data with a large range of values, particularly in Biology. A log-log plot turns a power function into a straight line of slope b, the transformation here.

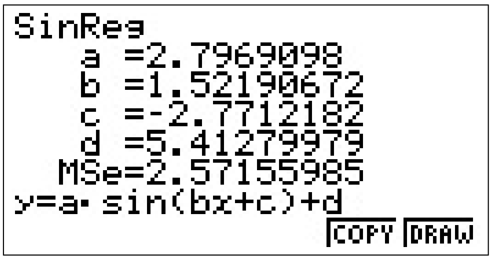


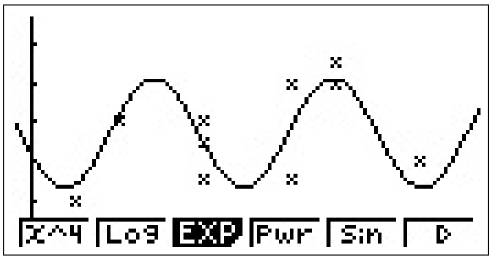


5.3.8 Sine functions

Sine (and cosine) functions, along with exponentials, are the most common functions in modelling; they model oscillations. Together, sines and exponentials model damped oscillations, such as the motion of a pendulum, and form the basis for all models of oscillating and vibrating systems.

$\boxed{\text{Sin}}$ (sinusoidal regression) fits the model equation $y = a\sin(bx+c) + d$ to the data using an iterative least-squares fit. It displays values for a, b, c and d, and MSE. At least four data points are required. At least two data points per cycle are required in order to avoid aliased frequency estimates.





6 Population Modelling 1: Exponential Growth

6.1 Introduction to population modelling

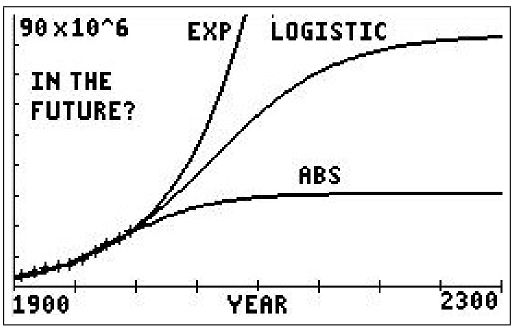
When mathematicians talk about playing with a model, chances are they don't mean a model plane or boat. They are probably talking about a mathematical model — a set of equations that describe in mathematics how a particular system works. There are mathematical models for many things, such as the planets revolving about the sun, heating iron ore in a blast furnace, pollution in a lake, how prices vary on the stock exchange, the spread of diseases and how populations (people, animals, bacteria, viruses, etc) change with time.

Population modelling started a long time ago, and one of the earliest modellers was Fibonacci (1170–1250). In his book *Liber abaci*, he modelled a rabbit population, starting with one pair of baby rabbits. If each adult pair of rabbits produces only one pair of baby rabbits each month, and if baby rabbits take one month to become adults, the numbers of pairs of rabbits in successive months are given by the famous Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, and so on. The next number is found by adding the previous two numbers. Fibonacci numbers are also found elsewhere in Nature. If you look at a pine cone, you will find the 'petals' spiral in two directions. The number of petals it takes to get once around is almost always a Fibonacci number. The same thing occurs in pineapples, sunflowers and many other flowers.

Much later, Thomas Malthus (1766–1834) in England startled the world by predicting that food would run out sometime in the future, because of the rapid increase in the human population. Based on the data he had at the time, Malthus predicted that the world population would increase exponentially, doubling every 40 years, thereby increasing at a faster and faster rate (40 years is the current doubling time of the world's population). If you start with the number 1 and keep doubling it, you will see an example of exponential growth.

The models of Fibonacci, Malthus and some other scientists all predict that the population will grow faster and faster. This is an alarming prospect, but does not seem to happen in experiments performed when there are limited resources, such as food and space to live in. Experiments with small animals and fungi in the laboratory, and with larger animals in fenced areas in the field show that as the resources start to run out, the reproduction rate reduces and the rate of growth slows down. The Belgian scientist Pierre Verhulst (1804–1849) while at the Belgian military school, the Ecole Royale Militaire, developed a model, called the logistic model, which took into account these observations. He introduced the idea of a 'carrying capacity' or maximum sustainable population that the environment will support.

We can illustrate the Malthus exponential model and the Verhulst logistic model by looking at the population of Australia since 1900. The small dark crosses on the graph below show the Australian Bureau of Statistics figures for the number of people in Australia (in millions) up until 1996. If we model these data with an exponential curve, we get the top curve in the figure. The middle curve is the logistic model. Both these curves fit the population numbers up to the present time well, but predict quite different future populations.



According to the exponential (Malthus) model, the population will continue to grow at a faster and faster rate, with a predicted population of about 109 million people in the year 2100, and about 587 million people in 2200. The logistic (Verhulst) model predicts that the population will keep on growing, but at a slower and slower rate; the predicted population in 2100 is about 57 million people, and the population would level off eventually at about 83 million people.

The Australian Bureau of Statistics (ABS) uses a mathematical model to predict the population of Australia well into the future to assist in planning for the number of people who will be living here. The predictions of their model are shown as the bottom curve in the figure. It has the shape of a logistic curve, but levels out much faster than the middle curve, predicting a population in 2100 of about 30 million people, and a maximum population of about 31 million.

Prediction is one powerful aspect of a mathematical model. By putting in the numbers we know, such as for the Australian population, we can predict what a population will be in the future, according to our model. Of course, the accuracy of our predictions depends on how good our model is, that is how well it describes the phenomena that affect population growth.

Another important use of a population model is to predict what will happen to the population if something changes, for example if the birth rate drops, if the number of immigrants is decreased, or if, say in a war, many people die. Predicting changes in a population is particularly relevant to populations of animals, insects and plants which have become serious pests after being brought into Australia from overseas. These include rabbits (see the picture below), foxes, mice, cane toads and European carp among the animals, and prickly pear, Paterson's curse, salvinia, mimosa and scotch thistle, to name but a few of the plants. The populations of some of these have reached very high levels at times, causing serious problems for farmers and the environment.

How do we control such pests? Often there are a number of possible ways, but which one is best? Population models can be modified to include the effect of the release of a predator, the spread of a disease in the pest population, the effect of poisoning or some other control measure. It is then possible to use the mathematical models to predict what would happen to the population if the different control strategies were tried. The models can also be used to find the best way of carrying out a particular control measure. Sometimes the modelling is done together with small-scale experiments, but often only the model can be used because the experiments are too risky or too expensive.

Rabbits drinking at a waterhole before the introduction of the myxomatosis virus.

In using a population model, we put the starting conditions and parameters (number of animals, how quickly they breed, etc) into our equations and predict the population at some later time. What if we change the starting conditions only slightly? We will end up with nearly the same final answer, right? Not necessarily. In some models, for example a variation on the Verhulst logistic model, with particular parameters, we find that the population does not change steadily towards some ultimate population, as we saw in modelling the Australian population, but changes rapidly and unpredictably with time. We say the model exhibits chaos: it loses its ability to predict, because a small change in the starting conditions produces a large change in how the population varies with time.

6.2 Population problems

Mathematically, the problems here are about *iteration* and about *exponential processes.*

Iteration is the process of carrying out the same operation over and over again. Let's take a simple example, that of multiplying by 2. Start with the number 1. Multiply it by 2 to give 2. Multiply the answer 2 by 2 again to give 4. Multiply 4 by 2 to give 8, and so on.

To do the calculation here, try this: press $\boxed{1}$ $\boxed{\times}$ $\boxed{2}$ $\boxed{\text{EXE}}$, then just press $\boxed{\text{EXE}}$ to multiply by 2 each time. You'll have to keep count of how many times you have multiplied by 2.

Iterating by multiplying by a constant (2 here) is an example of an *exponential process.* You may have heard the term exponential growth, which many people interpret to mean 'grow quickly'. But exponential growth has a precise mathematical meaning, and some interesting properties which we shall explore shortly.

Exponential iteration models a number of processes such as radioactive decay, population growth and absorption of light. If the constant we multiply by is larger than 1, we get exponential growth; if it is less than 1 (but greater than 0), we get exponential decay.

The use of scientific notation makes writing down our calculations much easier. For example, if we start with 5 and multiply it by 2 three times, we get $5\times2\times2\times2$, written as $5\times2^3=40$. 2^3 means three 2s multiplied together. If we multiply 5 by 2 ten times, we have $5\times2^{10}=5120$. 2^{10} means ten 2s multiplied together. The exponentiation key is $\boxed{\wedge}$ so, to calculate 2^3, we would press $\boxed{2}$ $\boxed{\wedge}$ $\boxed{3}$ $\boxed{\text{EXE}}$. The $\boxed{\text{EXP}}$ key gives 10^ so, to enter 2×10^6, press $\boxed{2}$ $\boxed{\text{EXP}}$ $\boxed{6}$.

6.2.1 Exponential iteration

Write down the results of the first 10 iterations of multiplying by 2, starting with 1.

6.2.2 Lots and lots of bacteria

Bacteria multiply (increase in number) by dividing — into two. One type of bacterium, *Streptococcus exponentiae*, divides every minute. If we start with 1 bacterium, it divides into 2 bacteria after 1 minute. Each of these 2 bacteria divides after 1 more minute, and so on. *The number of bacteria grows exponentially.*

Make up a table with time in the first column and the number of bacteria in the second column. *How many bacteria are there after 10 minutes? after 20 minutes? after 1 hour? after n minutes? Why isn't the Earth covered metres deep in these bacteria?*

6.2.3 Malthus and exponential growth

Thomas Robert Malthus (1766–1834) made some worrying predictions for the world population, and his name is often associated with the idea of exponentially growing populations. Look up Malthus to find out the details of his ideas. Why was he worried about the world's population?

Malthus looked at the United States population to try to verify his ideas. He concluded the growth was exponential. *From the numbers in the table below, can you tell if he was correct for the years until he died?*[13] *What about the population growth after about 1860?*

Year	**Population** (millions)	**Year**	**Population** (millions)	**Year**	**Population** (millions)
1790	3.90	1860	31.4	1930	123
1800	5.30	1870	38.6	1940	132
1810	7.20	1880	50.2	1950	151
1820	9.60	1890	62.9	1960	179
1830	12.9	1900	76.0	1970	203
1840	17.1	1910	92.0	1980	227
1850	23.2	1920	106	1990	249

Why might the populations not continue to increase exponentially?

Note: The graphics-calculator programs in Section 6.5 make teachers' lives easier for this problem, and for modelling the Australian and world populations.

If students are to do the data plotting and curve fitting, they should do it manually[14] rather than using a program. The data (contained in the programs) could be transferred from the teacher's calculator.

6.2.4 Cane toads

The Hawaiian cane toad *Bufo marinus* was introduced into Australia to control sugar-cane beetles. From the original 101 toads released in north Queensland in June 1935, the population grew rapidly and spread across the countryside. The table below shows the total land area of Australia colonised by cane toads for the years 1939 to 1974.

Year	**Area** (1000 km^2)	**Year**	**Area** (1000 km^2)
1939	33.8	1959	202
1944	55.8	1964	257
1949	73.6	1969	301
1954	138	1974	584

Is exponential growth a good model here? You can get a rough idea by the process we used for the Malthus data — finding ratios of successive values — but a plot of the data together with an exponential fit (graphics calculator required) will provide a better answer. *What is the exponential equation of best fit?*

Given that the area of Queensland is 1728 thousand km^2 and the area of Australia is 7619 thousand km^2 when, according to the exponential model, will (did) the cane toads colonise all of Queensland? all of Australia?

The cane growers were warned by Walter Froggart, President of the NSW Naturalist Society, that the introduction of cane toads was not a good idea and that the toads would eat the native ground fauna. He was immediately denounced as an ignorant meddlesome crank. He was also dead right.

[13] *Hint*: If the growth is exponential, each population should be a constant multiple of the previous value. Try a multiplier of 1.35, meaning the population increased by 35% every 10 years. The numbers you obtain only need to be close to the actual numbers, not exactly the same.

[14] see Chapter 5 for how to do this

6.3 Other exponential problems

6.3.1 Piles of paper

A ream of paper (500 sheets) is about 50 mm thick, so that one sheet is about 0.1 mm thick. Take a sheet of paper, cut it in half and put the two halves one on top of the other. Cut this pile of 2 pieces in half and make a pile of 4 pieces. Keep cutting the pile in half and stacking the pieces up.

Now suppose you could make 42 cuts altogether (you'd need big scissors!). *How high would your final pile be?* Try making up a table like the one below to keep track of your pile. *Where would your pile reach to?* Kilometres might be a good unit to use eventually. *Write down a formula that tells you the height after n cuts. What units will you use?* Be careful!

Cut number	**Height of pile in sheets**	**in mm**
1	2	0.2
2	4	0.4
3	8	0.8
4	16	1.6

This is an example of where maths lets you find an answer to something you can't actually do in real life.

6.3.2 Shoeing a horse

A rich man sends his horse to the blacksmith to have 4 new horseshoes put on. Each shoe needs 5 nails. The blacksmith offers to charge either \$100 per nail (they're gold!), or 1c for the first nail, 2c for the second nail, 4c for the third nail, and so on, the cost doubling each nail. *Which offer should the rich man take?*

Think first which offer *you* would take. Then do some calculations. Don't forget to add up the total cost at the end. A calculator might be useful. *Did you pick the better offer? Was there much difference?*

Perhaps you might like to write down a function for the cost of the second offer after n nails and graph it. Write down a new version of the problem if each shoe needed 6 nails. What could the first offer be in this case?

6.3.3 Interest rates

Once you have some money in the bank, you start to think about interest, and you might want to answer a question like the one below to work out how much money you will have some time in the future.

> *If the annual interest rate on a bank account is 12% compounded monthly and you deposit \$10, how much money will you have after 1 year? after 5 years? after 10 years?*

What does this mean? In simpler terms, it means that every month the bank will pay you an amount of interest equal to 1% (an annual interest rate of 12% means a monthly interest rate of $12\%/12 = 1\%$ of the amount you have in the account at the end of the month. So, after the first month, the bank will pay into your account 1% of \$10 or $0.01 \times \$10 = \$0.10$ = 10c in interest, and you will then have $\$10 + \$0.10 = 1.01 \times \$10 = \$10.10$ in your account.

After the second month, the interest will be 1% of \$10.10 or $0.01 \times \$10.10 = \$0.101$ = 10.1c, and you will then have $\$10.10 + \$0.101 = 1.01 \times \$10.10 = 1.01 \times (1.01 \times \$10) = \$10.201$ in your account. Although the bank won't pay you the 0.1c, they leave it in for future calculations.

Can you see a pattern? At the start of each month, the new amount in your account will be the amount you had last month times 1.01.

Now can you answer the question above? Can you write down a formula using exponential notation for the amount in your account after n months? How long before you have \$15? \$30?

6.4 Solutions

Exponential Iteration

$1 \times 2 = 2 = 2^1$
$1 \times 2 \times 2 = 4 = 2^2$
$1 \times 2 \times 2 \times 2 = 8 = 2^3$
$1 \times 2 \times 2 \times 2 \times 2 = 16 = 2^4$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 = 32 = 2^5$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64 = 2^6$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128 = 2^7$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256 = 2^8$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 512 = 2^9$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 1024 = 2^{10}$

Lots and lots of bacteria

You should end up with a table containing the numbers above and lots more. From your table, you can read off the answers to the questions. If you are using a calculator, it will probably switch to scientific notation when the numbers become large enough; powers of 10 are just like powers of 2, but much easier to write down. 10^3 is a 1 followed by three zeros, 10^{10} a 1 followed by 10 zeros, and so on.

Time	**Number of Bacteria**
after 1 minute	2
after 10 minutes	1024 $(=2^{10})$
after 20 minutes	$1,048,576$ $(=2^{20})$
after 1 hour	about 1.15×10^{18} $(=2^{60})$

The number of bacteria after n minutes is 2^n.

The Earth isn't covered with these bacteria because in reality the growth is not exponential but has a limit: environmental conditions ultimately limit growth and scientists continue to discover new antibiotics.

Malthus and exponential growth

Thomas Malthus was worried about the world's population because he believed that population growth increased exponentially such as in the examples above, but that the food supply would only grow linearly (as a straight line — much more slowly). In other words, left unchecked, the human population would eventually exceed its food and other resources, leading to overcrowding, poverty, malnutrition, disease, crime and war. You can find out more about Malthus at *www.igc.org/desip/malthus*.

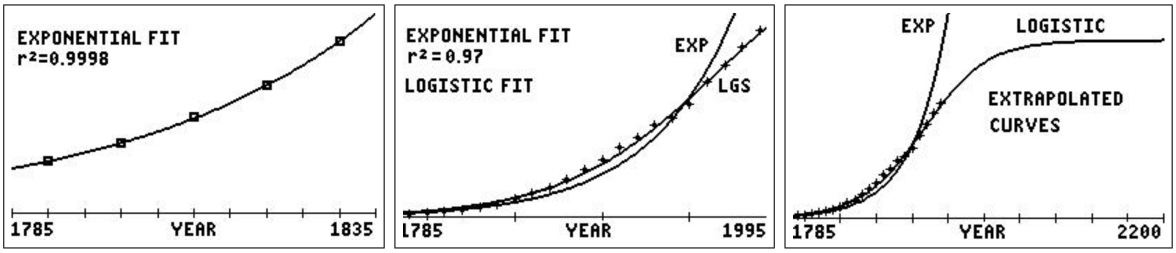
Starting at 1790, multiplying each number in the table by 1.35 gives a number close to the next number, up until 1860: the US population increased by about 35% every ten years from 1790 to 1860. The ten-year growth rate then decreased to values in the range 20–30%, and finally down to around $9\frac{1}{2}\%$ in the decade to 1990. A marked decline in growth occurred between 1910 and 1950, during the two world wars and the Great Depression. The growth rate picked up a bit after the wars (the baby boom: 1950–60), then slowly declined again.

PTO

The left-hand figure below shows the exponential fit to the US population for the years 1790–1830. The fit is excellent (coefficient of determination r^2 very close to 1).

The middle figure shows exponential and logistic fits to the full data set 1785–1990. The exponential fit ($r^2=0.97$) is not quite as good now and doesn't follow the trend in later years. The logistic fit is much better.

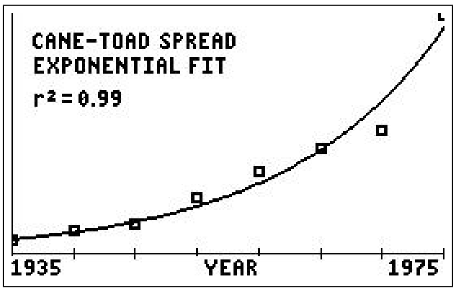
The right-hand figure shows an extrapolation of the two curves to the year 2200. The exponential model predicts a US population in 2200 of more than 29 billion, the logistic model about 386 million, eventually stabilising at about 388 million.



Cane toads

The ratio of successive terms jumps about a bit, between 1.17 and 1.94, with a mean of 1.34. The population is therefore increasing roughly exponentially.

The exponential fit to the data looks reasonable (the value from 1969 is a bit low: a drought period from 1964 to 1969?). The curve of best fit is $y = 36.9 \times 1.08^t$ or, using *the* exponential function, $y=36.9e^{0.0774t}$, where t is years since 1939.



Use a graphics calculator to find when $y=1728$ and $y=7619$ on the curve of best fit.[15] According to the exponential model, Queensland was overrun by cane toads between 1988 and 1989, Australia between 2007 and 2008. Clearly, and fortunately, there are some factors that restrict the spread of the cane toad. *Find out more?*

PTO

[15]Graph $y=1728$ and $y=7619$, and use *ISCT* to find in what year these lines intersect the curve of best fit.

Piles of paper

1 sheet of paper = 0.1 mm thick.

After 1 cut, 2 (2^1) sheets of paper = 0.2 mm thick.
After 2 cuts, 4 (2^2) sheets of paper = 0.4 mm thick.
After 3 cuts, 8 (2^3) sheets of paper = 0.8 mm thick.

$\vdots$

After 42 cuts, $4,398,046,511,104$ (2^{42}) sheets of paper $\approx 439,804,651,110$ mm thick, $439,805$ km — a little more than the distance from the Earth to the Moon.

Shoeing a horse

A horse needs 4 new horseshoes with 5 nails in each.

Nail	Cost of nail
1st	1c
2nd	2c
3rd	4c
4th	8c
5th	16c
6th	32c
7th	64c
8th	$1.28
9th	$2.56
10th	$5.12
$\vdots$	$\vdots$
19th	$2,621.44
20th	$5,242.88
Total	$10,485.75

If the man pays $100 per nail, it will cost him $2,000 to shoe the horse.

If he pays by the nail as in the table above (the total cost is the sum of all the costs in the second column), it will cost him $10,485.75, more than 5 times the flat rate. The choice is now obvious, and demonstrates clearly how rapidly exponential functions can increase.

Interest rates

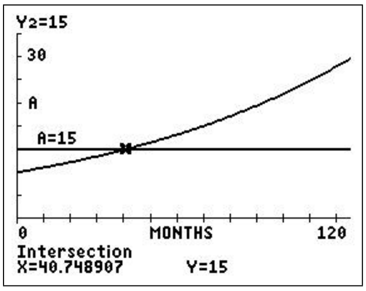
Deposit	Annual Interest Rate	Time Period	Value
$10.00	12%	after 1 year	$11.27 (=$\$10.00 \times 1.01^{12}$)
	compounded monthly	after 5 years	$18.17 (=$\$10.00 \times 1.01^{60}$)
		after 10 years	$33.00 (=$\$10.00 \times 1.01^{120}$)

A formula for the amount A in a bank account, given compound interest, is therefore

$$A(n) = D\left(1+\frac{i}{100}\right)^n,$$

where D is the initial deposit, $i\%$ the interest rate per month and n the number of months the money has been left in the bank.

It takes 41 months or 3 years 5 months before you have $15, and 111 months or 9 years 3 months before you have $30. Find these values by trial and error, by using the table feature on the graphics calculator or by graphing the function A with the number of months n as the independent variable, and finding where A equals 15 (figure below).



Amount in bank $A(n)$ versus month n,
showing when the amount reaches \$15.

6.5 Population-modelling programs

These programs are available at *canberramaths.org.au* under *Resources.*

AUSPOP: Australian population 1906–1996

This program plots the population of Australia, fits and plots an exponential function and a logistic function, and plots a logistic curve from the Australian Bureau of Statistics (ABS). It also shows extrapolation of the curves to predict the Australian population in the future.

Use: Run the program and press EXE. The screen tells you which keys can be used at any given time. Press MENU 2 at the end to see the data and model values.

MALTHUS: US population 1790–1990

This program plots the data for 1790–1990 from the table of US population on page 79 and fits an exponential function and a logistic function. It also shows extrapolation of the curves to predict the US population in the future.

Use: Run the program and press EXE. The screen tells you which keys can be used at any given time. Press MENU 2 at the end to see the data and model values.

WORLDPOP: World population 1940–2000

This program (not used in the problems here) plots the world population, fits and plots an exponential function and a logistic function, and plots a logistic curve from the US Bureau of Statistics (USBS). It also shows extrapolation of the curves to predict the world population in the future.

Use: Run the program and press EXE. The screen tells you which keys can be used at any given time. Press MENU 2 at the end to see the data and model values.

After you have finished running any of these programs, clear Lists 1–5 and and functions Y1–Y3.

7 Financial Mathematics 1: Compound Interest

7.1 Introduction

As with all mathematics, some simple interest calculations should be done by hand first, so that students understand what they are are calculating. Section 7.2, *Compound Interest: The Basics* (suitable for Year 9/10), leads on from hand calculations to interest calculations using the calculator. However, hand calculations very quickly become tedious once regular payments are involved; Section 7.3,, *Using Sequences* includes these in a logical development of the methods of Section 7.2.

To be able to do some useful financial modelling, such as comparing loans, etc, the TVM (Time Value of Money) Solver is essential; this is the subject of *Financial Mathematics 2*) (Chapter 4 in Volume 3).

7.2 Compound interest: The basics

Based on an article for students in Year 9 by Steve Arnold.

The 9860/CG20/CG50 is a very powerful tool, more like a palm-top computer than a calculator. However, unlike Maths teachers, computers and calculators are very unforgiving. If you don't give them exactly the right information, you will probably get the wrong answer. So be careful!

The process — what you do — is crucial. Compare your results with the person beside you and ask the teacher if neither of you is sure. Solutions to the questions here are in Section 7.4.

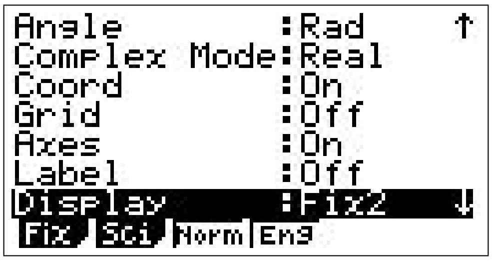
Set the calculator to display numbers rounded to two decimal places, as we are working in dollars and cents.

From the RUN-MAT screen, press SET UP.

Scroll down to *Display*.

Press Fix 2 EXE.

Press EXIT to return to the RUN screen.

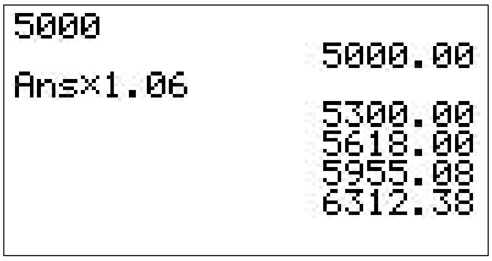


Question 1 *Solutions on page 92*

(a) If you invest $5000 at an annual rate of 6% compounded annually, how much money will you have after 5 years? after 10 years?

Method A: Repeated multiplication

Type	See	Result
5000 EXE	5000	5000.00
×1.06 EXE	Ans ×1.06	5300.00
EXE		5618.00
EXE		5955.08
⋮		⋮



Don't forget to count the EXEs: one EXE = 1 year.

(b) What calculation does the calculator perform each time you press EXE (except for the first time)?

(c) Write out the calculation steps as the calculator does them to find the amount of money after 5 years. Turn this into a formula involving a number raised to power 5 and hence do the calculation on the calculator the normal way to check your answer.

Method B: Using a formula

The compound-interest formula is

$$A = P\left(1 + \frac{R}{100}\right)^N,$$

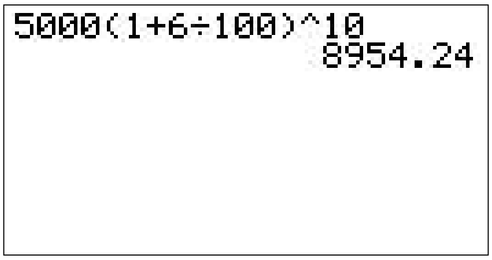
where A is the amount of money after N years, P is the principal or starting amount and R is the annual interest rate expressed as a percentage.

For our question, $P = 5000$, $R = 6$ and $N = 10$.

On the RUN screen, type the formula as

5000(1+6÷100) ^ 10

and press EXE.



Question 2: How long does it take to double your money?

Clearly we could use Method A to answer this question by continuing to press EXE until the result reaches 10,000. What about Method B? You have to find the smallest (integer) value of N that gives a value of A greater than 10,000.

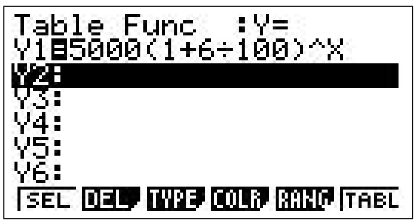
(a) Press the left arrow. This recalls the previous command. Change the value of N to one which you think will give an answer greater than 10,000 and press EXE to re-calculate the formula.

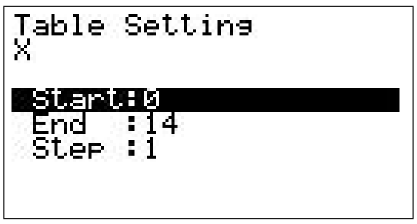
Make up a table of the values of N you tried and the amount of money you found with each N. Identify which N answers the question and show that it does.

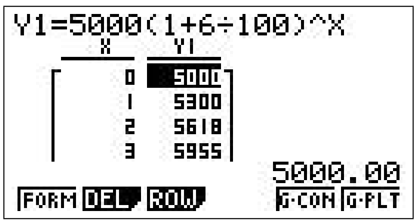
Method C: Using a table

Press MENU 7 and set Y1 = 5000(1+6÷100) ^ X. Y1 represents A, the amount of money, and X (press X,θ,T) represents N, the time in years.

Now press TABL (F6).



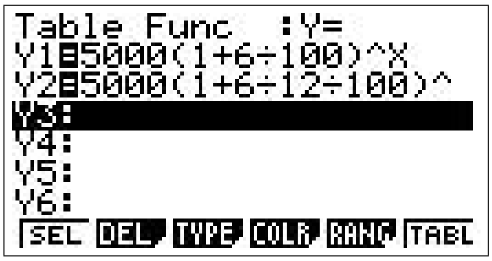




If your table does not start at X = 0 and go up in steps of 1, press EXIT SET (F5) to go to the Table Setting screen or table 'window'. With the cursor and EXE, set Start = 0, End = 14 and Step = 1. Press EXIT and TABL to return to the table.

(b) From the calculator table, by the end of which year does your money double?

Question 3: If you invest \$5000 at 6% annual interest, compounded monthly, how long does it take to double your money?

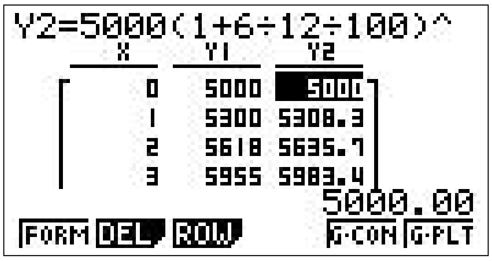


An annual interest rate of 6% compounded monthly gives a monthly interest rate $R = 6/12/100$, with the time now in months.

The amount of money at the end of year X, month 12X, is then $A = 5000(1+6/12/100)^{12\text{X}}$.

Set Y2 = 5000(1+6÷12÷100)∧(12X).

Now press $\boxed{\text{TABL}}$ and look at the values in the Y2 column.

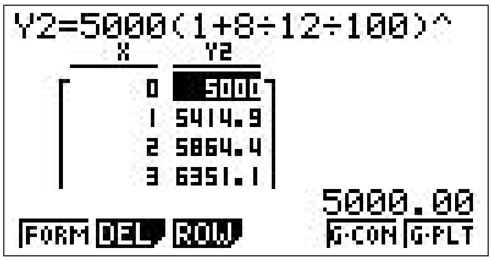


(a) In which year does the amount double now?

(b) Compare Y1 and Y2. What does each column represent? Which compounding method is better?

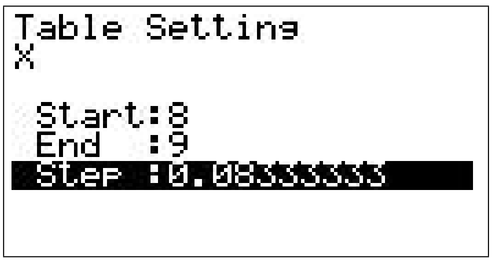
Note: From now on, we will just use Y2. Press $\boxed{\text{EXIT}}$, move the cursor over the = sign of Y1 and press $\boxed{\text{SEL}}$ (F1) to turn it off.

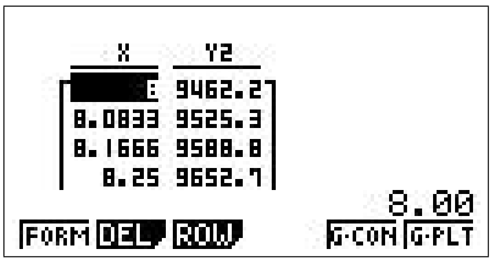
Question 4: If the annual interest rate is now 8% compounded monthly, in which year does the amount double?



In Y2, change the 6 to 8 and press $\boxed{\text{EXE}}$, then $\boxed{\text{TABL}}$. The table values will reflect the new interest rate.

Question 5: By the beginning of which month of the ninth year does the amount double if the annual interest rate is 8% compounded monthly?
Note that X = 0.00 is the beginning of the first year, so that X = 8.00 is the beginning of the ninth year.
Press $\boxed{\text{SET}}$, set Start = 8, End = 9 and Step = 1÷12 to give monthly increments.





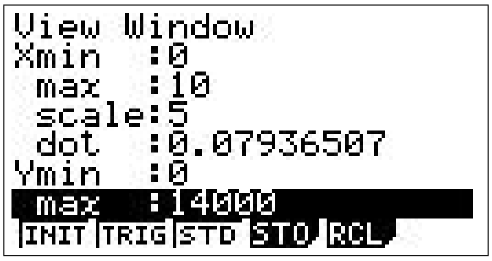
You will have to count down to find the month: 8.00 ≡ beginning of January; 8.08 ≡ beginning of February, etc.

(a) At the beginning of which month does 8.33 correspond to?

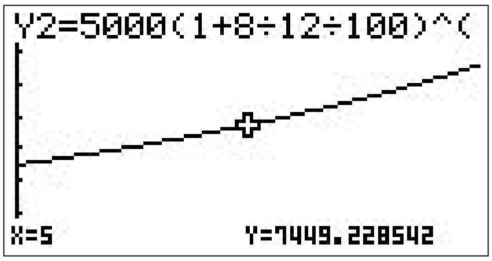
(b) Find the answer to the question from the table.

Method D: Using a graph

We already have the formula for the graph in Y2.



Press MENU 5, then V-Window. Here we have to tell the calculator how to set up the axes to view the graph. Put in the values shown. Yscale is 2000.



Press EXIT, then DRAW (F6) to see the graph of Y2.

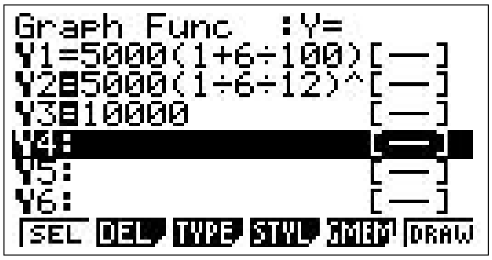
Press Trace and use the left- and right-arrow keys to move along the curve.

(c) In setting the View Window, what does X represent? Why choose 0 for Xmin? What is the smallest number we could choose for Xmax? What does Y represent? What is the smallest number we could choose for Ymax?

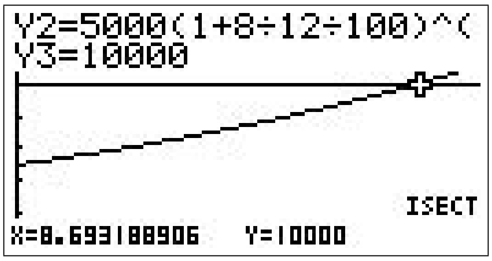
(d) When you use Trace, unless you are lucky you won't find a point at which Y is exactly 10,000. This is because the cursor jumps from pixel to pixel on the screen, rather than moving smoothly through all numbers. However, you can find points at which your money has at least doubled. Using the cursor, find the smallest value of X for which this is true. This is an approximation to the exact answer.

(e) If you move the cursor one pixel to the left (press the left-arrow key once) of the X value you found in (d), you can get some idea of the accuracy of your answer to the question. What are the X and Y values one pixel to the left of the X value you found in (e)? Between what times (in decimal years will do) does the exact answer then lie? You might like to think in terms like 'at this X, the Y value is just too large; at this X, the Y value is just too small'.

PTO



To find a more accurate answer, set Y3 = 10000, the amount of money we want to reach. Press DRAW to display this second curve. We will calculate the approximate intersection point of the two curves, i.e. solve the equation Y2 = Y3, to find when the original amount doubles.



Press G-Solv (SHIFT F5). Press F5 to select ISCT. The calculator will move to the first (and only) intersection point.

(f) The ISCT operation just gives us a better approximation to the exact answer. From ISCT, what is the answer to the question? Is it in dollars or years?

(g) How would you incorporate regular payments into Method A?

Use this to answer:

> If you invest \$1000 at an monthly rate of 1%, compounded monthly, and deposit \$100 into the account every month (starting at the beginning of the second month), how much money will you have after 1 year?

7.3 Using sequences

This section illustrates the use of RECUR mode for simple financial calculations, including regular payments, following on from the basic calculations in Section 7.2. The use of (discrete) sequences for these sorts of calculations is more intuitive than the usual treatment using continuous functions.

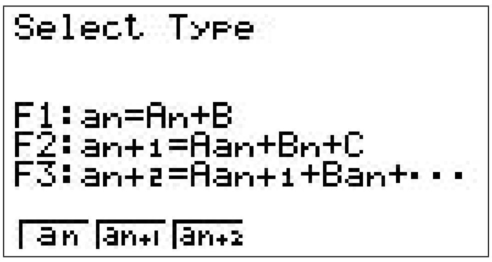
7.3.1 Sequence notation

In the usual notation, a general sequence is written as $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$, where each term a_i is a number. The subscript gives the position of the term in the sequence.

Casio calculators use the notation an for the nth term of the sequence, rather than a_n. Three sequences are available: an; bn; and cn.

7.3.2 Using sequence mode

Sequences can be defined either recursively or explicitly, displayed and graphed in RECUR mode. To select this, press MENU 8 (below left).



Press TYPE (F3) F2 (above right) to set the type of sequence used below.

Regular deposits

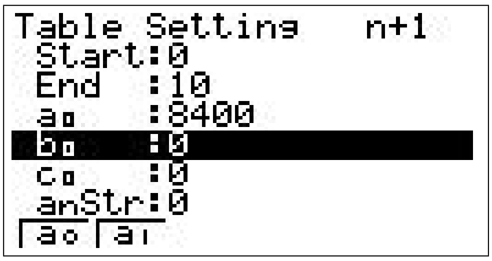
Example 1: Initial amount \$8400; deposit \$400 per time step (n increasing by 1; week, fortnight, month or year).

If a_n is the amount of money (\$) in the account at time (step) n: $a_0 = 8400$; $a_{n+1} = a_n + 400$.

The amount at time step $n+1$ is the amount at time step n plus 400.

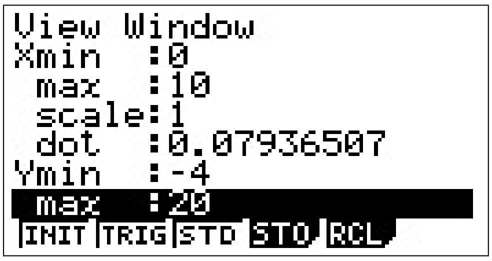
On the calculator: $\mathrm{a}_0 = 8400$ $\quad \mathrm{a}_{n+1} = \mathrm{a}_n + 400$.

Set this sequence up on your calculator as shown below left; pressing $\boxed{\text{F4}}$ gives access to a_n. Press $\boxed{\text{EXE}}$ to return to the main screen.



Press $\boxed{\text{EXIT}}$, then $\boxed{\text{SET}}$ to give the screen above right. Here you specify Start and End for the table (Step = 1 is assumed), and the initial value a_0.

Press $\boxed{\text{TABL}}$.



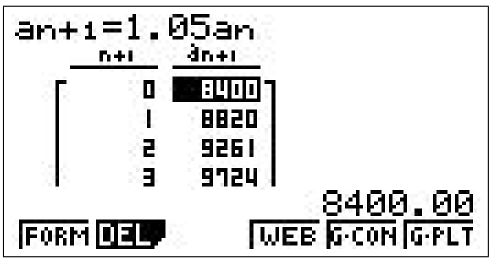
Compound interest

Example 2: Initial amount \$8400; compound interest 5% per time step.

If a_n is the amount of money (\$) in the account at time (step) n: $a_0 = 8400$; $a_{n+1} = 1.5a_n$.

The amount at time step n is the amount at time step $n-1$ plus 5% of that amount.

On the calculator: $\mathrm{a}_0 = 8400$ $\quad \mathrm{a}_{n+1} = 1.05\mathrm{a}_n$.



The SET screen is the same as in the previous example.

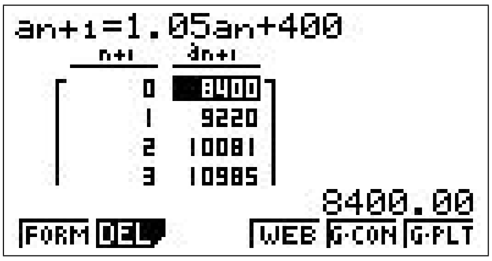
Compound interest and regular deposits

Example 3: Initial amount \$8400; compound interest 5% and deposit of \$400 per time step.

If a_n is the amount of money (\$) in the account at time n: $a_0 = 8400$; $a_{n+1} = 1.05a_n + 400$.

The amount at time step $n+1$ is the amount at time step n plus 5% of that amount plus 400.

On the calculator: $\mathrm{a}_0 = 8400$ $\quad \mathrm{a}_{n+1} = 1.05\mathrm{a}_n + 400$.



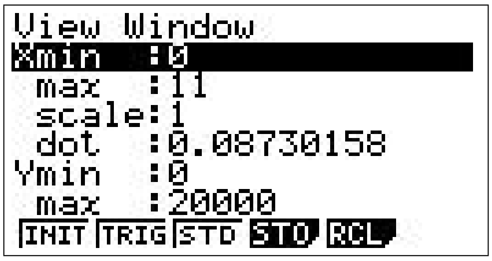
The SET screen is the same as in Example 1.

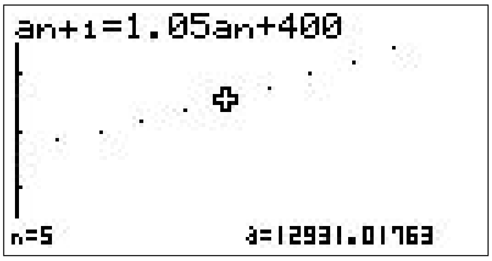
Exercise Question 5(g) but using a sequence

If you invest \$1000 at an monthly rate of 1%, compounded monthly, and deposit \$100 into the account every month (starting at the beginning of the second month), how much money will you have after 1 year? *Solution at the end of Section 7.4.*

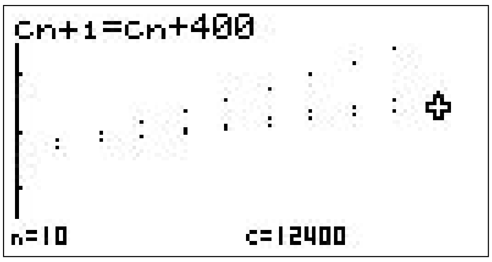
Graphing the sequence

First press V-Window and set up your window as shown below (Yscl is 5000). Then press EXIT TABL G-PLT (F6) to plot the sequence points. Press Trace and use the arrow keys to move along the points of the sequence. G-CON (F5) joins the points.





The plot below shows all three sequences: a_n, with both compound interest and regular deposits, at the top; b_n, with just interest in the middle; and c_n, with just regular deposits, at the bottom.



7.4 'The basics' solutions

Note for teachers

Before starting the calculations here, students should have done some basic compound-interest calculations by hand. We use the calculator to be able to answer questions about compound interest that would take a long time to do by hand.

Some of the questions are designed to make students think about their use of the calculator as a tool. This is important, but clearly the questions can be chosen/varied to match the ability of the class.

Question 1

(a) If you invest \$5000 at an annual rate of 6% compounded annually, how much money will you have after 5 years? after 10 years?

After 5 years, you will have \$6691.13, and after 10 years, you will have \$8954.24, both rounded to the nearest cent.

(b) What calculation does the calculator perform each time you press $\boxed{\text{EXE}}$ (except for the first time)?

The calculator multiplies the previous answer/result by 1.06.

(c) Write out the calculation steps as the calculator does them to find the amount of money after 5 years. Turn this into a formula involving a number raised to power 5 and hence do the calculation on the calculator the normal way to check your answer.

The calculation is

$$5000 \times 1.06 \times 1.06 \times 1.06 \times 1.06 \times 1.06 = 5000 \times 1.06^5 = 6691.13,$$

rounded to 2 decimal places.

The calculator should not be a 'black box': students need to understand what it is actually doing.

Question 2

How long does it take to double your money?

(a) Make up a table of the values of N you tried and the amount of money you found with each N. Identify which N answers the question and explain why it does.

Clearly values for N in a table will vary, but eventually they should find that $N = 12$ does the trick. $N = 12$ is the smallest (integer) value of N for which the amount of money is greater than or equal to 10,000.

(b) From the calculator table, by the end of which year does your money double?

The amount doubles by the end of the 12th year.

Question 3

(a) In what year does the amount double now?

The amount still only doubles by the end of the 12th year.

(b) Compare Y1 and Y2. What does each column represent? Which compounding option is better?

Y1 is the amount of money when the interest is compounded annually, Y2 the amount when the interest is compounded monthly.

Compounding monthly gives a greater amount than compounding yearly at any given time, and so is the better option.

Question 4

If the annual interest rate is 8% compounded monthly, in which year does the amount double?

The amount now doubles by the end of the 9th year.

Question 5

By the beginning of which <u>month</u> of the 9th year does the amount double if the annual interest rate is 8% compounded monthly?

(a) At the beginning of which month does 8.33 correspond to?

The beginning of May.

(b) Find the answer to the question from the table.

The amount has doubled when X = 8.75, corresponding to the beginning of October of the 9th year. The fact that the 9th year starts when X = 8.00 may cause some confusion here.

(c) In setting the $\boxed{\text{V-Window}}$, what does X represent? Why choose 0 for Xmin? What is the smallest number we could choose for Xmax? What does Y represent? What is the smallest number we could choose for Ymax?

X corresponds to time in years. Xmin is the starting time value, hence 0. Xmax must be some number larger than 9, because we know from previous work, doubling occurs in the 9th year.

Y corresponds to the amount of money in dollars. Ymax has to be some number greater than 10,000, because this is the amount we are aiming at. With some experimentation, we find that 14,000 leaves room at the top for the function formula.

(d) When you use $\boxed{\text{Trace}}$, unless you are lucky you won't find a point at which Y is exactly 10,000. This is because the cursor jumps from pixel to pixel on the screen, rather than moving smoothly through all numbers. However, you can find points at which your money has at least doubled. Using the cursor, find the smallest value of X for which this is true. This is an approximation to the exact answer.

Using $\boxed{\text{Trace}}$, the smallest value of X for which $Y \geqslant 10{,}000$ is $X = 8.73$.

(e) If you move the cursor one pixel to the left (press the left-arrow key once) of the X value you found in (e), you can get some idea of the accuracy of your answer to the question. What are the X and Y values one pixel to the left of the X value you found in (e)? Between what times (in decimal years will do) does the exact answer then lie? You might like to think in terms like 'at this X, the Y value is just too large; at this X, the Y value is just too small'.

For the pixel one to the left of the X value in (d), we have $X = 8.65$ and $Y = 9966.25$. The exact X value (time in years) therefore must lie between 8.65 ($Y < 10{,}000$) and 8.73 ($Y > 10{,}000$).

(f) The ISCT operation just gives us a better approximation to the exact answer. From ISCT, what is the answer to the question? Is it in dollars or years?

According to ISCT, $Y = 10{,}000$ (dollars) when $X = 8.69$ (rounded to 2 decimal places). This is in September, so the answer still remains 'by the beginning of October'.

(g) How would you incorporate regular payments into Method A?

Multiply by 1.01 and add 100.

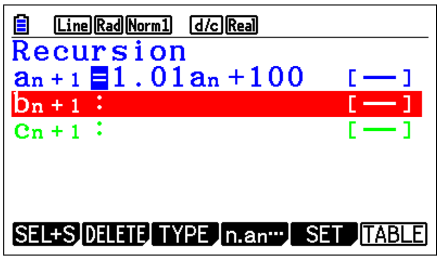
If you invest $1000 at an monthly rate of 1%, compounded monthly, and deposit $100 into the account every month (starting at the beginning of the second month), how much money will you have after 1 year?

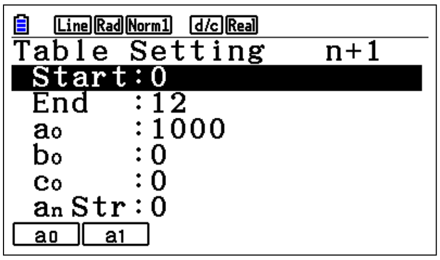
Month	**Type**	**See**	**Result**
0	1000 $\boxed{\text{EXE}}$	1000	1000.00
1	$\times 1.01 + 100$ $\boxed{\text{EXE}}$	Ans$\times 1.01 + 100$	1110.00
2	$\boxed{\text{EXE}}$	$\downarrow$	1221.10
3	$\boxed{\text{EXE}}$		1333.31
4	$\boxed{\text{EXE}}$		1446.64
5	$\boxed{\text{EXE}}$		1561.11
6	$\boxed{\text{EXE}}$		1676.72
7	$\boxed{\text{EXE}}$		1793.49
8	$\boxed{\text{EXE}}$		1911.42
9	$\boxed{\text{EXE}}$		2030.54
10	$\boxed{\text{EXE}}$		2150.84
11	$\boxed{\text{EXE}}$		2272.35
12	$\boxed{\text{EXE}}$		2395.08

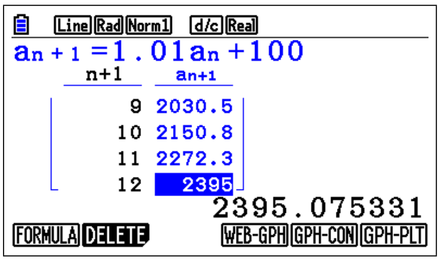
After 1 year, you would have $2395.08.

Exercise Question 5(g) but using a sequence.

If you invest \$1000 at an monthly rate of 1%, compounded monthly, and deposit \$100 into the account every month (starting at the beginning of the second month), how much money will you have after 1 year?







You will have \$2395.08.

8 Probability and Statistics 1: Descriptive Statistics

8.1 Introduction

The most comprehensivel book I have seen on this topic for graphics calculators is *Probability and Statistics with the TI-83 Plus: For A-Level Mathematics* by Peter Jones and Chris Barling.[16] Sadly, this is no longer readily available; if you have a copy, treasure it. They explain everything in great detail, with large numbers of screen shots. The exposition here is somewhat briefer but I'll follow their general outline, and even borrow some of their examples (labelled JB), all translated for the Casio 9860. Most of the exercises are also from there.

Calculating with a CASIO fx-9860G AU PLUS[17] has a chapter on using STAT, which is a good introduction. *Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS*[18] by Barry Kissane also has material relevant to this topic. Some of the examples and exercises are from the course notes for a first-year introductory Statistics course by Dr Leesa Sidhu of UNSW Canberra.

What follows is not a textbook on Probability and Statistics (you'll need one) but how to do the various operations in such a course on a Casio 9860 graphics calculator. This replaces the various tables usually used in such courses, and makes many of the calculations quite simple. As always, students should do all these operations by hand first, so that they understand the process but, ultimately, always having to do the calculations manually stands in the way of doing any meaningful modelling and interesting applied problems.

Three associated calculator-based activities for Years 9 and 10 are in the supplementary volume *Supplementary Material: Activities for Years 9 and 10* available at *canberramaths.org.au* under *Resources.* Year levels and subject matter are indicated with each summary. Solutions and teachers' notes are provided with each activity.

- *Probably Finding π*: An experimental-probability method for finding π.
- *Reaction Times and Statistics*: Programs are used to measure reaction times in various scenarios including simulated braking in a car. The data are displayed as box-and-whisker plots for subsequent analysis.
- *Statistics from Birthdays*: Class data on day and month of birth are used to provide an introduction to data presentation on a graphics calculator.

8.1.1 Australian Curriculum

References are given here to the texts *Nelson Senior Maths Methods 11 for the Australian Curriculum* (NSM11) and *Nelson Senior Maths Methods 12 for the Australian Curriculum* (NSM12) used in the ACT.

The material here is relevant to the topics *Discrete random variables* (Chapter 2 in NSM12), *Binomial distributions* (Chapter 5 in NSM12), *Continuous random samples and the normal distribution* (Chapter 8 in NSM12), *Random samples and proportions* (Chapter 9 in NSM12), and *Confidence intervals* (Chapter 10 in NSM12).

[16] Cambridge University Press, 2002, ISBN 0 521 52531 4; I have a copy.
[17] StepsInLogic, 2009; ISBN 978-0-646-51473-4; I have a copy.
[18] The Mathematical Association of Western Australia, 2003, ISBN 1-876583-24-X; I have a copy.

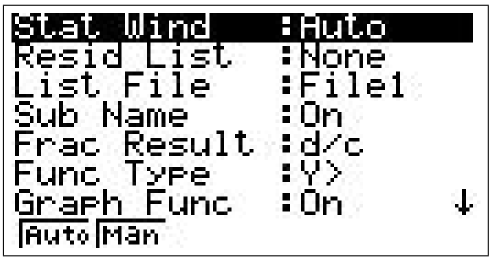
8.2 Setting up

Launch the STAT application MENU 2. This takes you to the List Editor.

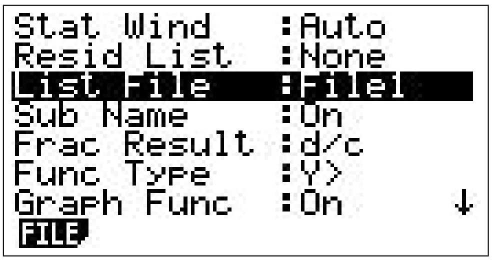
Enter the SET UP menu (SHIFT MENU).

Most of the following options will already be set.

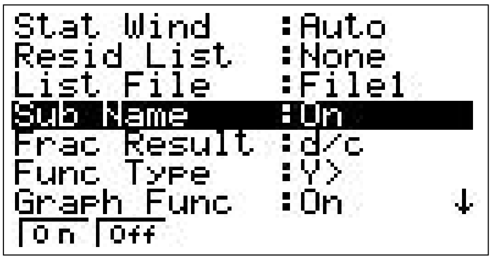
When *Stat Wind* is set to *Auto*, the calculator will set axis scales on a plot that are appropriate for the data. You can always change these later. Set on *Auto*.



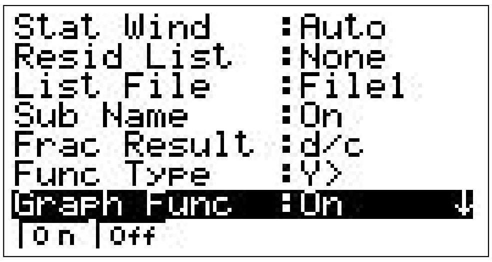
With *List File* set to *File1*, the List Editor displays the 26 lists that are collectively called File1. The first six of these exist. Choose FILE to change the set of lists displayed.



Sub Name ON lets you give lists a text name. Choose ON.



With *Graph Func* ON, the formula for a graph will be displayed when it is being plotted or Traced.



The defaults for the other options in SET UP are usually the ones you want.

Press EXIT to return to the List Editor.

8.3 Descriptive statistics

8.3.1 Data management

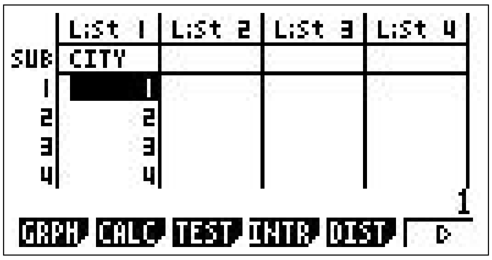
Example: To do any statistical analysis on the calculator, you need some data. These are stored in lists, entered manually using the STAT List Editor Here's an example of some data, pollution levels measured in six cities (JB).

City number	1	2	3	4	5	6
Pollution level (mg/m^3)	12	43	27	38	21	24

The lists on the 9860 are List 1 – List 26 in the List Editor. There is the option of giving lists names (SUB row). However, the calculator only knows a list by its number not by this name.

Entering data into a list

Enter the City numbers into List 1 in the List Editor and give it the name CITY. Type in the data, pressing EXE after each value.

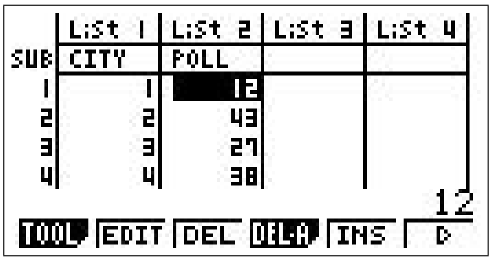


Enter the pollution data into List 2, calling it POLL.

Make sure you enter these values in the same order as those in CITY.

Here F6 has been pressed to show the remaining list options at the bottom of the STAT screen.

To delete the contents of a list from the List Editor, move the cursor to the list and press DEL-A .

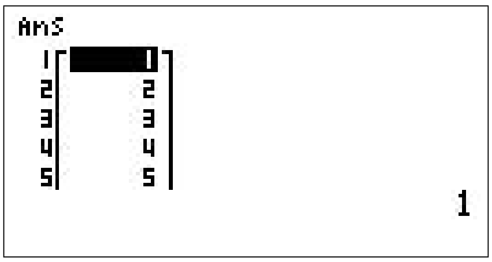


Displaying a list on the RUN screen

Press MENU 1 to take you to the RUN screen.

To refer to a list, use 'List' in the OPTN LIST menu, followed by the appropriate number.

Press EXE to display the contents of the list, which you can scroll.



8.3.2 Univariate data

Displaying the data

1. Histograms

Example: The following set of data are marks in a first-year Mathematics course.

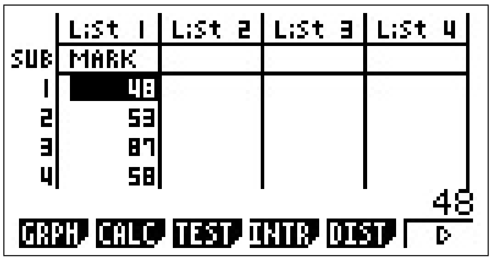
48 53 87 58 63 39 51 75 57 77 72 62 47 68 72 38 73 49 74 80 50 86 73 77

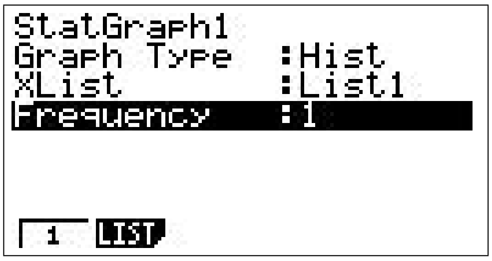
Delete the contents of List 1 and List 2.

Enter the data here into List 1 and call it MARK (below left).

Press GRPH (F1), then SET. The StatGraph1 screen should appear. If not choose it at the bottom of the screen.

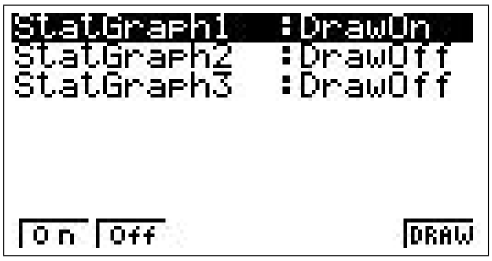
Set up StatGraph1, as shown in the figure below right.

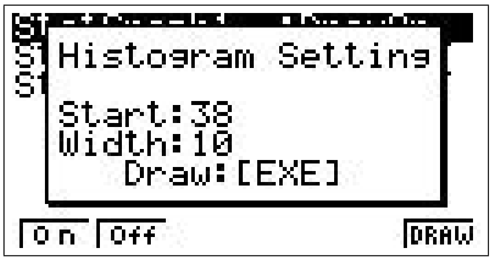


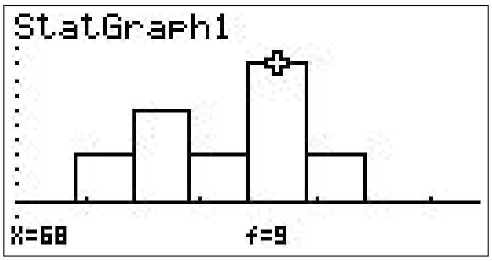


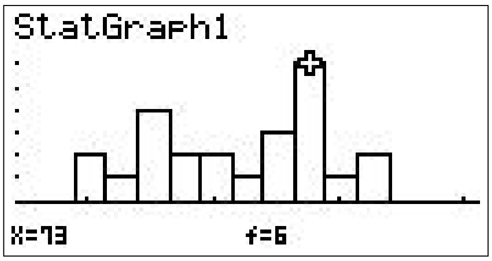
Press EXIT, then SEL; turn on StatGraph1 and make sure the other two are turned off (below left).

Press DRAW: you will see a screen like that below right. Put in the appropriate numbers and press EXE to display the histogram. The View Window is set automatically.









In the histogram plots, Trace has been pressed and the cursor moved to the interval with the highest frequency.

For the right-hand figure above, the (bin) Width has been changed to 5 (EXIT SELECT DRAW and change Width).

Exercises *solutions in Section 8.4*

1. The life expectancies (in years) of people in 25 countries are listed below.

   58 65 68 74 73 73 75 71 72 61 67 66 37 50 66 64 72 74 48 41 44 44 49 48 48

   Construct histograms on your graphics calculator with the first class interval starting at 35 and the last class interval ending at 80 using interval widths of:
   (a) 2; **(b)** 5; **(c)** 15.

2. The data below give the wrist circumference (in cm) of 15 men.

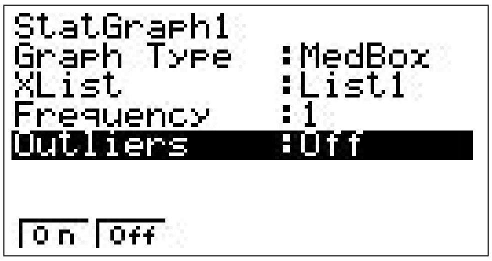
   16.9 17.3 19.3 18.5 18.2 18.4 19.9 16.7 17.1 17.6 17.7 16.5 17.0 17.2 17.6

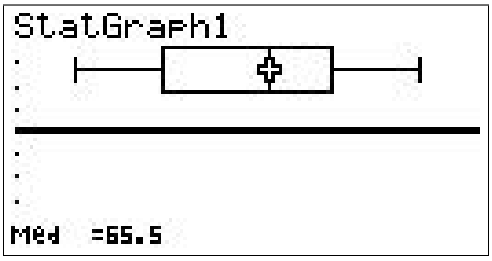
   Construct histograms on your graphics calculator with the first class interval starting at 16.5 and the last class interval ending at 20.5 using interval widths of:
   (a) 0.5; **(b)** 1.0; **(c)** 2.0.

2. Boxplots

Set up a boxplot for the MARK data (List 1) of the previous example. From the List Editor, press $\boxed{\text{GRPH}}$ ($\boxed{\text{F1}}$), then $\boxed{\text{SET}}$. The StatGraph1 screen should appear. If not choose it at the bottom of the screen.

Set up StatGraph1, as shown in the figure below left using the options at the bottom of the screen.





Press $\boxed{\text{EXIT}}$, then $\boxed{\text{SEL}}$; turn on StatGraph1 and make sure the other two are turned off.

Press $\boxed{\text{DRAW}}$ to give you the figure above right. Here $\boxed{\text{Trace}}$ has also been pressed and the cursor moved to the median.

Exercises

Construct boxplots for the two histogram exercises above.

PTO

Summarising the data

1. Measures of centre

The calculator can calculate the **median** and **mean** of data in a list. The relevant operations are accessed from the RUN screen in the $\boxed{\text{OPTN}}$ $\boxed{\text{LIST}}$ menu.

For the MARK data, use the relevant operations from the RUN screen in the $\boxed{\text{OPTN}}$ $\boxed{\text{LIST}}$ menu to calculate the median and mean. *List* comes from the $\boxed{\text{OPTN}}$ $\boxed{\text{LIST}}$ menu.

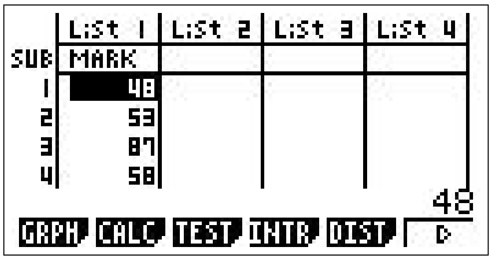
2. Variability

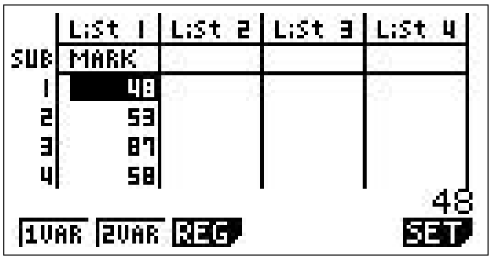
The relevant commands for **variance** and **standard deviation** are in the $\boxed{\text{OPTN}}$ $\boxed{\text{STAT}}$ menu of the RUN screen.

PTO

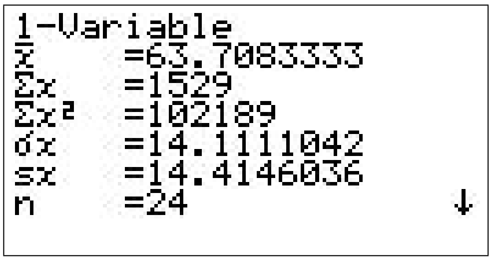
3. 1VAR command

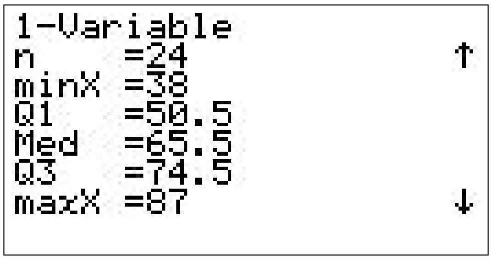
You can also calculate the various statistics for a set of data in a list from the STAT screen using the 1VAR command in the CALC menu (below left). Press SET (below right) to set up the calculation.

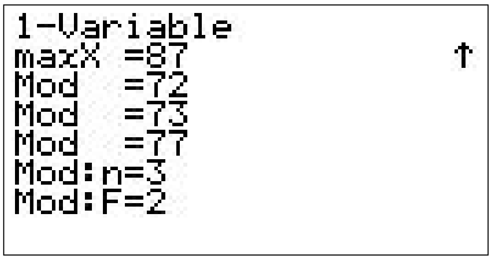




Make sure List 1 is set as the X-List (below left); if not, change it using LIST (F1). Press EXIT, then 1VAR. The remaining three screens below show the result.







Exercise *solutions in Section 8.4*

The weekly amounts (in \$) spent on food by 10 households were as follows:

170 123 87 98 112 150 98 134 106 114.

Find:

(a) the mean amount spent on food each week (to the nearest dollar);

(b) the standard deviation of the mean σ_x;

(c) the unbiased estimate of the population variance s_x^2;

(d) the median amount spent on food each week;

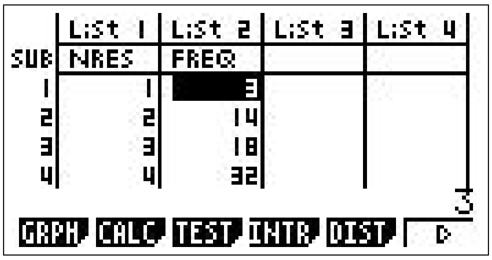
(e) the interquartile range $Q_3 - Q_1$.

4. Grouped data

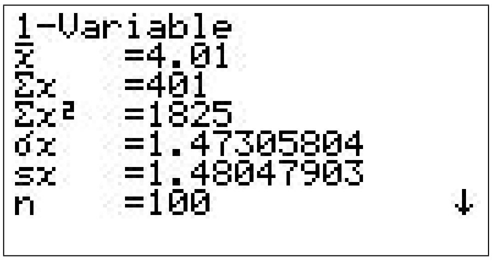
Example: One hundred households were surveyed, and the number of persons normally in residence recorded in tabular form as shown in the table (JB).

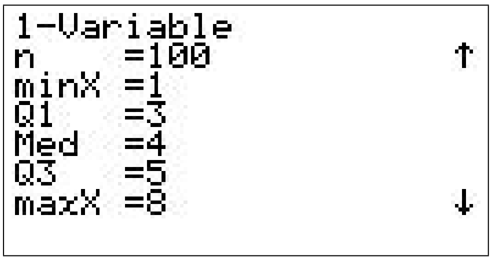
Number of residents	Frequency
1	3
2	14
3	18
4	32
5	16
6	12
7	4
8	1

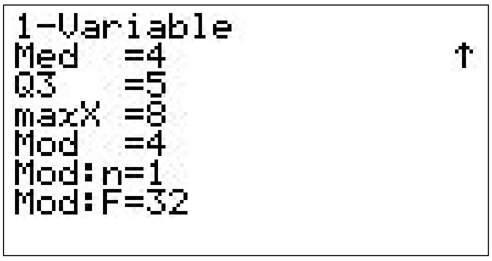
The statistics here are calculated using the 1-VAR command. Enter the number of residents into List 1, called NRES, and the frequency into List 2, called FREQ.



Press CALC SET and specify the frequency list.
Press EXIT.
Execute the 1-VAR command. The three screens of results are shown below.







PTO

Exercises *solutions in Section 8.4*

1. One hundred households were surveyed, and the number of children normally in residence recorded in tabular form as shown in the table (JB).

Number of children	Frequency
0	14
1	23
2	31
3	18
4	12
5	0
6	2

Find:

(a) the mean number of children per residence;

(b) the standard deviation of the mean σ_x;

(c) the unbiased estimate of the population variance s_x^2;

(d) the median number of children per residence;

(e) the interquartile range $Q_3 - Q_1$.

2. The life (in hours) of 11 batteries is: 30 31 38 35 36 60 40 31 33 62 43. Generate a boxplot with outliers (the boxplot in stat plot with a dot on the right) of the data and write down all the associated values (using Trace).

8.3.3 Bivariate data

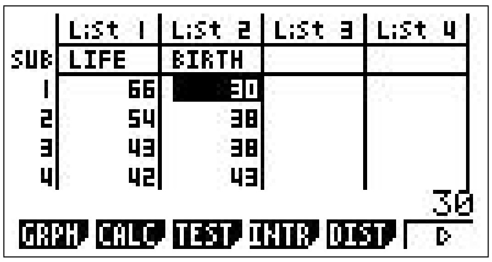
Displaying the data

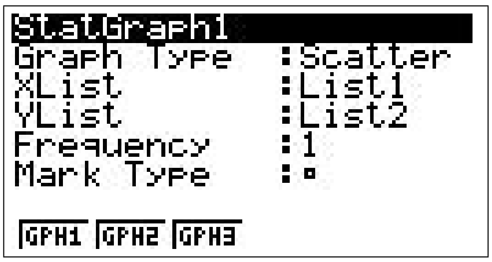
Scatterplots

Example: The table shows the life expectancy (in years) and birthrate (per thousand) of people living in ten countries (JB).

Life expectancy (years)	Birthrate (per 1000)
66	30
54	38
43	38
42	43
49	34
45	42
64	31
61	32
61	26
66	34

Enter the data into List 1 LIFE and List 2 BIRTH, respectively.

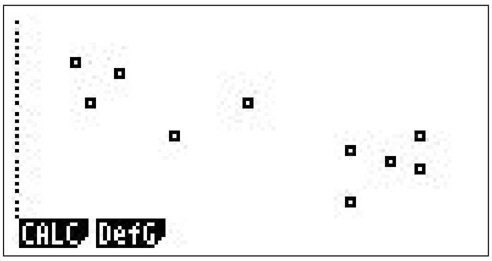




Press GRPH SET, set a scatterplot, and specify the two lists to plot. Press EXIT.

Press SEL to make sure StatGraph1 is on and the other two off.

Press DRAW to generate the scatterplot. The View Window is set automatically.

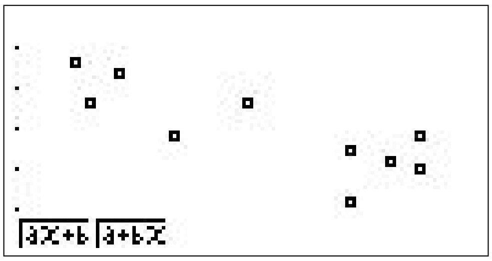


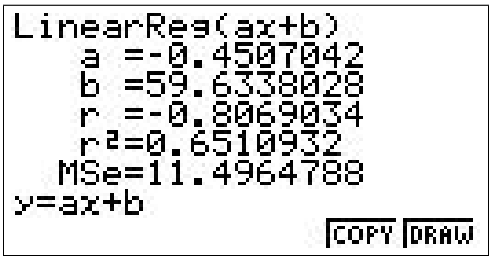
Press SHIFT F1 (Trace) and use the arrow keys to move around the data points.

Summarising the data

1. Determining the equation of the least-squares line

Press CALC on the scatterplot and X for the least-squares menu (below left). Press F1 for $ax+b$. The equation for the regression line, and values for the Pearson correlation coefficient r and the coefficient of determination r^2 are given.

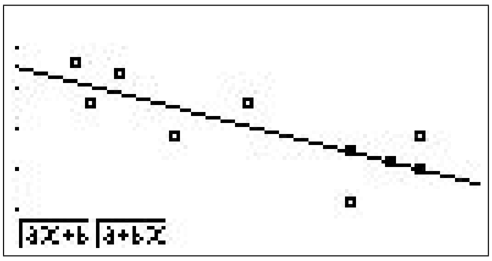




Here, the line of best fit is $\text{BIRTH} = 59.6 - 0.451\,\text{LIFE}$ (to 3 significant digits), with the Pearson correlation coefficient $r = -0.81$ and the coefficient of determination $r^2 = 0.65$.

2. Plotting the least-squares line on a scatterplot

Just press DRAW to plot the data and the least-squares line of best fit. If you then press Trace, you can read off values from both the line and the data.



If you press COPY, you can paste the equation of the line into one of the Y functions (MENU 5).

Exercise *solutions in Section 8.4*

The table shows the life expectancies of males and females in the period 1900 to 1990.

Year	**1900**	**1910**	**1920**	**1930**	**1940**	**1950**	**1960**	**1970**	**1980**	**1990**
Males	52	54	56	60	65	66	67	67	70	72
Females	55	57	63	67	69	73	74	74	76	78

(a) Construct a scatterplot with female life expectancy on the vertical axis and male life expectancy on the horizontal axis. Determine

  (i) the equation of the least-squares line for these data;

  (ii) The Pearson correlation coefficient r and the coefficient of determination r^2.

(b) Plot the least-squares line on the scatterplot.

(c) Predict female life expectancy when the male life expectancy reaches 80.

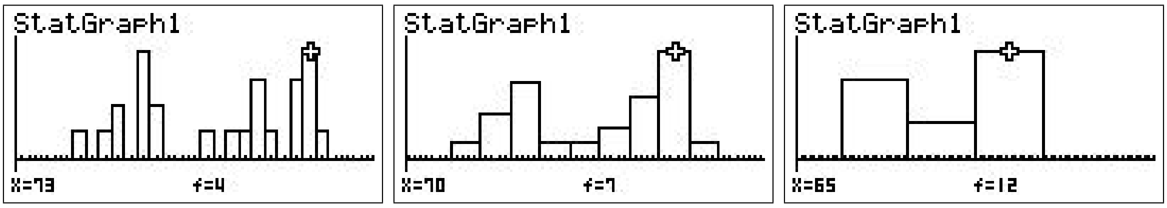
8.4 Solutions to exercises

Exercises page 100

1. The life expectancies (in years) of people in 25 countries are listed below.

58 65 68 74 73 73 75 71 72 61 67 66 37 50 66 64 72 74 48 41 44 44 49 48 48

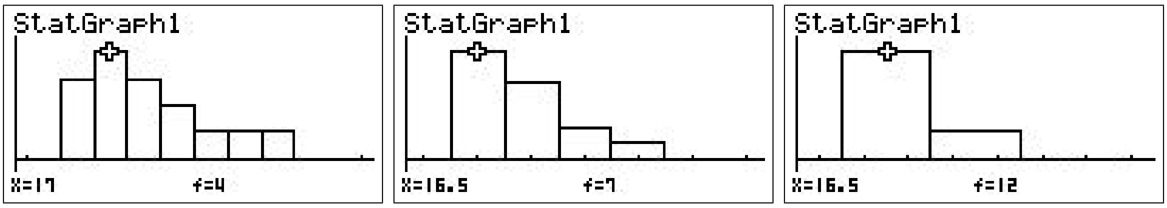
Construct histograms on your graphics calculator with the first class interval starting at 35 and using interval widths of: **(a)** 2; **(b)** 5; **(c)** 15.



2. The data below give the wrist circumference (in cm) of 15 men.

16.9 17.3 19.3 18.5 18.2 18.4 19.9 16.7 17.1 17.6 17.7 16.5 17.0 17.2 17.6

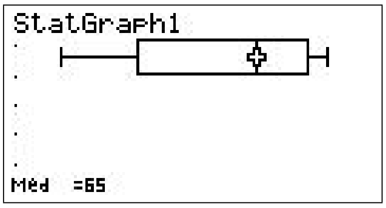
Construct histograms on your graphics calculator with the first class interval starting at 16.5 and the last class interval ending at 20.5 using interval widths of: **(a)** 0.5; **(b)** 1.0; **(c)** 2.0.



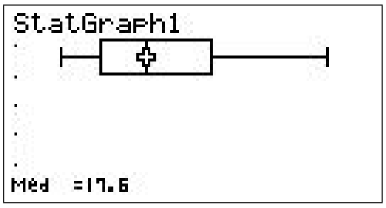
Exercises page 100

Construct boxplots for the two histogram exercises above.

1.



2.



PTO

Exercise page 102

The weekly amounts (in \$) spent on food by 10 households were as follows:

170 123 87 98 112 150 98 134 106 114.

Using 1-Var Stats:

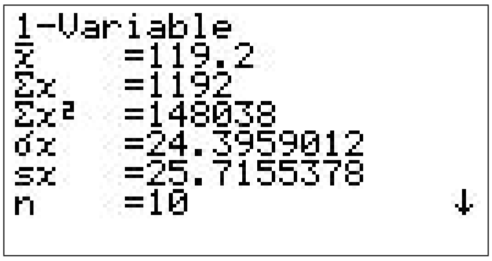
(a) the mean amount spent on food each week (to the nearest dollar) is \$119;

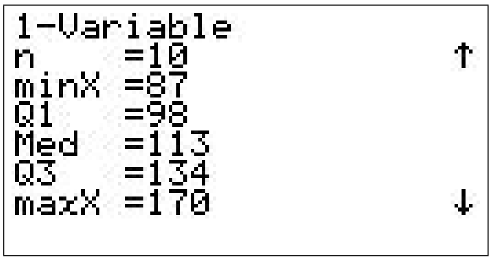
(b) the standard deviation of the mean σ is \$24;

(c) the unbiased estimate of the population variance s_x^2 is $25.716^2 = 661$;

(d) the median amount spent on food each week is \$113;

(e) the interquartile range $Q_3 - Q_1$ is \$36.





Exercises page 104

1. One hundred households were surveyed, and the number of children normally in residence recorded in tabular form as shown in the table (JB).

Number of children	Frequency
0	14
1	23
2	31
3	18
4	12
5	0
6	2

Using 1-Var Stats:

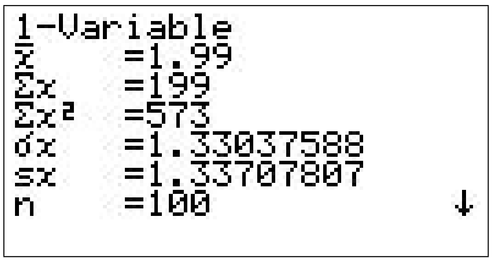
(a) the mean number of children per residence is 2.0;

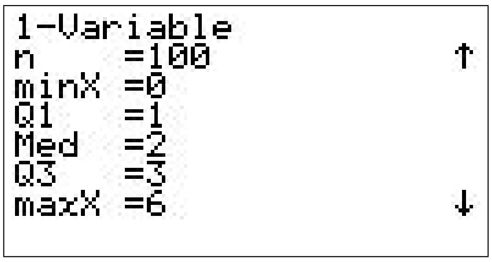
(b) the standard deviation of the mean σ is 1.3;

(c) the unbiased estimate of the population variance s_x^2 is $1.34^2 = 1.8$;

(d) the median number of children per residence is 2;

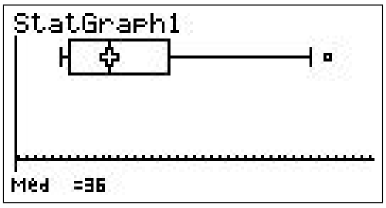
(e) the interquartile range $Q_3 - Q_1$ is 2.

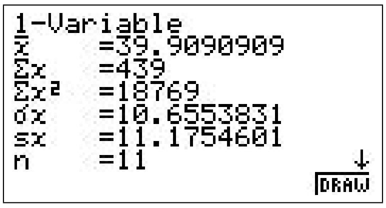


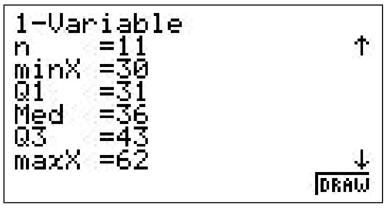


2. The life (in hours) of 11 batteries is: 30 31 38 35 36 60 40 31 33 62 43.

Generate a boxplot with outliers and write down all the associated values (using $\boxed{\text{Trace}}$).





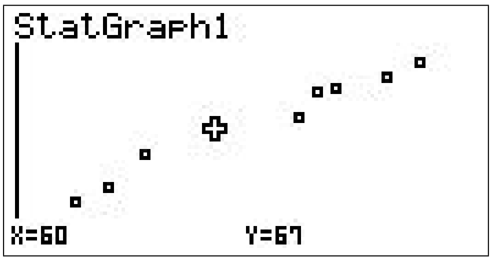


Exercise page 106

The table shows the life expectancies of males and females in the period 1900 to 1990.

Year	**1900**	**1910**	**1920**	**1930**	**1940**	**1950**	**1960**	**1970**	**1980**	**1990**
Males	52	54	56	60	65	66	67	67	70	72
Females	55	57	63	67	69	73	74	74	76	78

(a) Construct a scatterplot with female life expectancy on the vertical axis and male life expectancy on the horizontal axis.

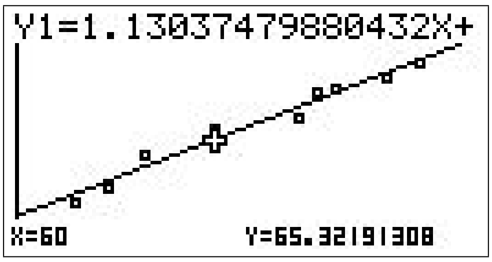


Determine

(i) the equation of the least-squares line for these data; $y = 1.13x - 2.5$

(ii) The Pearson correlation coefficient r and the coefficient of determination r^2.
$r = 0.98$, $r^2 = 0.97$

(b) Plot the least-squares line on the scatterplot.



(c) Predict female life expectancy when the male life expectancy reaches 80.

From the equation of the liine of best fit, the female life expectancy when the male life expectancy reaches 80 is given by $y(80) = 1.13\times 80 - 2.5 = 87.9$, i.e. 87.9 years.