Mathematics on a TI-84/CE
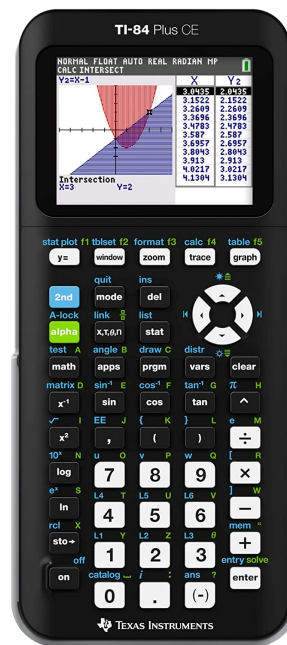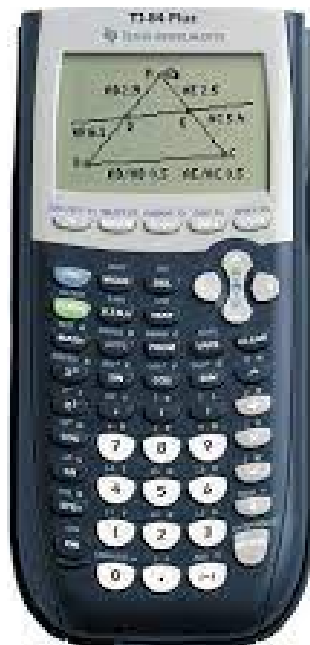
Volume 3: Advanced

Chapters 17 – 24

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Contents

Mathematics on a TI-84/CE

Volume 1 of this book contains the basic topics: *Graphics Calculators and Mathematics*; *Getting Started*; *Coordinate Geometry*; *Inequalities and Linear Programming*; *Fitting Curves to Data 1 – Calculator Functions*; *Population Modelling 1 – Exponential Growth*; *Financial Mathematics 1 – Compound Interest*; and *Probability and Statistics 1 – Descriptive Statistics*.

Volume 1 Supplement: Activities for Years 9 and 10 contains extra activities for *Coordinate Geometry* and *Probability and Statistics 1*.

Volume 2 of this book contains topics directly relevant to Calculus and its applications, although the first chapter, *Functions and their Graphs*, is of more general relevance and also contains details of how to capture screenshots from your calculator, crop them if desired and insert them into documents. The topics in Volume 2 are: *Functions and their Graphs*; *Graph and Calculus Operations*; *Numerical Integration*; *Taylor Series*; *Differential Equations*; *Population Modelling 2 – Logistic and Epidemic Models*; *Multivariable Calculus*; and *Program Information*.

Program Information gives a list of all the programs in the book, and full information on copying and using these programs.

Volume 4 contains 28 Mathematics labs or projects, together with a Lab Manual for teachers/instructors.

Calculator versions

Currently (early 2023), TI-84 calculators come in two versions: the TI-84Plus and the more recent TI-84CE. The main difference is that the CE screen has much higher resolution. It also has colour but I have done most of the screens in black and white to avoid the need for colour

printers or photocopiers. Calculations, screenshots and figures were done on a TI-84CE in CLASSIC mode.

Some programs have had to be changed for the CE because of the different screen: I usually append 'CE' to the program name to indicate this.

All the programs here are available at *canberramaths.org.au* under *Resources*.

17   Sequences and Series

17.1   Introduction

These notes provide a comprehensive review of generating, displaying and graphing sequences and series on a TI-84/CE graphics calculator. An arithmetic progression, a geometric progression and the Fibonacci numbers are used as examples. A number of questions (with solutions) illustrate the use of the calculator. Finally there are three topics that could be used as a basis for group investigation or a small project.

The TI-84/CE can generate sequences, sum series, and display sequence terms in a table or graph. However, we should first ask whether it makes sense to use a graphics calculator at all for sequences and series.

Certainly the first few lessons on sequences should be pencil and paper, until some of the concepts and calculations are understood, although a class activity such as that on page 9 can add variety to the early learning stages. However, having to work out terms of a sequence or series by hand eventually becomes tedious, especially those terms that are not simple and require a calculator anyway. This becomes an impediment to further learning and exploration.

The calculator automates the process of calculating terms in a sequence or series *once it is given an appropriate definition*. It is in finding an appropriate definition that most of the thought goes — the calculator can't do this. With automatic calculation comes the ability to explore particular sequences and series, to conjecture and test, and to look at ideas such as the convergence of an infinite sequence or series.[1]

Some of the questions and investigations in Sections 17.4 and 17.5 demonstrate this extra capability when using a graphics calculator.

At this stage, it is perhaps useful for the reader to review the use of the TI-84/CE graphics keys, the top row, by graphing, say $y = x^3$ and generating a table of function values.

- Set a suitable $\boxed{\text{window}}$ manually first and graph $Y_1 = X^3$.

- Then use ZoomFit $\left(\boxed{\text{zoom}}\ \boxed{0} \right)$ to carry out the process of finding a suitable Y scale automatically. Use both $\boxed{\text{graph}}$ and $\boxed{\text{trace}}$ to generate the graph.

- Review line types $\big($to the left of the function definition in $\boxed{\text{y=}}\big)$, particularly solid and dotted lines. On a TI-84CE, you can choose a colour for the graph here too.

- For the table, use $\boxed{\text{tblset}}$ to specify the starting value and increment for X. Generate a table with $\boxed{\text{table}}$.

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[1] These are not esoteric beasts — the humble AP and GP continue on indefinitely.

17.2   Sequences

A sequence is an ordered set of numbers, usually with the numbers or terms in the sequence determined by some sort of formula.[2] For example,

$$1, 3, 5, 7, 9, 11, \ldots \qquad 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots \qquad 1, 1, 2, 3, 5, 8, 13, \ldots$$

are sequences for which we can find a formula to determine each term.

In the usual notation, a general sequence is written as

$$u_1, u_2, u_3, \ldots, u_n, \ldots,$$

where each term u_1, u_2, $u_3, \ldots$ is a number. The subscript gives the position of the term in the sequence.

The TI-84/CE uses the notation $\mathrm{u}(n)$ for the nth term of the sequence, rather than u_n, so the general sequence is written

$$\mathrm{u}(1), \mathrm{u}(2), \mathrm{u}(3), \ldots, \mathrm{u}(n), \ldots.$$

This was presumably done for ease of display, but it reinforces an important fact about sequences: a sequence is really just a function with domain the positive integers or some subset of them.

There are two ways to give a formula for each term.

- **Recursively:** write the nth term in terms of the previous term or terms. Here we also need to know a value for the first term (or the first few terms) in the sequence.

  Examples

  1. $u_2 = u_1 + 2$, $u_3 = u_2 + 2$, $\ldots$ or in general, $u_n = u_{n-1} + 2$.
     With $u_1 = 1$, this recursive formula gives the first sequence above, an arithmetic progression (AP).

  2. $u_n = u_{n-1}/2$.
     With $u_1 = 1$, this gives the second sequence above, a geometric progression (GP).

  3. $u_n = u_{n-1} + u_{n-2}$.
     With $u_1 = 1$ and $u_2 = 1$, thus gives the third sequence above, the famous Fibonacci numbers.

- **Explicitly:** specify the nth term as a function of n, where n takes integer values.

  Examples

  1. $u_n = 2n - 1$, $n = 1, 2, 3, \ldots$, again giving the first sequence above.
  2. $u_n = 0.5^{n-1}$, $n = 1, 2, 3, \ldots$, again giving the second sequence above.

  The Fibonacci sequence can also be defined explicitly — see Section 17.5.

There are basically two ways on the TI-84/CE to generate and display terms of a sequence — *using the sequence grapher* and *using LIST commands*. Which method you use will depend on how you teach the topic. Here we'll look at both, using the above examples to illustrate the methods. There are questions to practise on in Section 17.4.

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[2]although we can have sequences of random numbers.

17.2.1   Using the sequence grapher

Sequences can be defined either recursively or explicitly, displayed and graphed by the built-in sequence grapher. To select this, press $\boxed{\text{mode}}$ and, with the cursor and $\boxed{\text{enter}}$, select SEQ as shown below.

TI-84Plus   TI-84CE

The graphing mode SEQ is just one of four possible ways to define a function on the TI-84/CE. Because a sequence is just a function as far as the calculator is concerned, all the calculator graphing keys (top row) are relevant.

Now press the function-definition key $\boxed{\text{y=}}$ to see the three sequences u, v, w available.

On a TI-84, the nth term u(n) can be written as any combination of n, u($n-1$), u($n-2$), v($n-1$), v($n-2$), w($n-1$) and w($n-2$). There is more flexibility on a CE.

Arithmetic progression

In an arithmetic progression, there is a constant difference between successive terms. In calculator notation, the recursive definition is

$$\mathrm{u}(n) = \mathrm{u}(n-1) + d,$$

where d is a constant called the common difference.

Example: $\mathrm{u}(n) = \mathrm{u}(n-1) + 2$, with $\mathrm{u}(1) = 1$.

Press $\boxed{\text{y=}}$ and set this sequence up on your calculator as shown below. The independent variable here, n, is produced by the $\boxed{\text{X,T,}\theta,n}$ key in SEQ mode. The sequence name u is $\boxed{\text{2nd}}$ $\boxed{7}$. The initial u value on a TI-84, u(nMin), must be in curly brackets (a list).

TI-84Plus   TI-84CE

While we are on this screen, check that you have dots to the left of the definition of u(n), indicating that only the sequence points will be plotted (no joining lines) when we graph the sequence. If not, move the cursor here and press $\boxed{\text{enter}}$.

On a TI-84, press $\boxed{\text{enter}}$ until three dots come up. On a CE, use the arrow keys to select bold dots; you can select a different colour here too.

Displaying the sequence

Here we use the table feature of the calculator.

First a table 'window': press ⏹tblset⏹ (⏹2nd⏹ ⏹window⏹) and set TblStart $=1$ and ΔTbl $=1$.

Press ⏹table⏹ (⏹2nd⏹ ⏹graph⏹). Scroll down in either column to see the terms of the sequence. If you want to scroll up past the top of the screen, you have to be in the n column.

Another way of displaying terms of a sequence u is to type on the Home screen a command of the form u $(start, end\,[, increment]\,)$.
For example, typing u $(1, 9, 2)$ and pressing ⏹enter⏹ will display u(1), u(3), u(5), u(7) and u(9).

increment is optional, a value of 1 assumed if it is not entered.

To display the value of just one term, just type u($n\,value$), for example u(6), and press ⏹enter⏹.

Graphing the sequence

There are several ways to graph sequences, as shown when you press ⏹format⏹ (⏹2nd⏹ ⏹zoom⏹): there's an extra line here (the top one) in SEQ mode.

TI-84Plus   TI-84CE

- *Time* plots u(n) against n, which is what we want here. Select it if necessary with the cursor and ⏹enter⏹.

- *Web* plots a cobweb plot, u(n) against u($n-1$). We'll look at this in Section 17.5.2.

- If you have more than one sequence, you can plot one sequence against another: *uv, vw, uw.* These are called *phase plots.*

Next set a graph window: press ⏹window⏹ and you will see a few more parameters than usual. In our *Time* plot, n is plotted along the X axis, so the n range should be contained within the X range.

Set up your window as shown below (not shown on the screen is Yscl $= 10$) and press $\boxed{\text{trace}}$.
Use the arrow keys to move along the points of the sequence.

```
WINDOW
 nMin=1
 nMax=20
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=20
 Xscl=5
 Ymin=0
↓Ymax=50
```

```
u=u(n-1)+2
                            . . .
                        . . ·
                     . ·
                  . .
              . ·
           . ·
        . ×
    . ·
  ·
n=5
X=5              Y=9
```
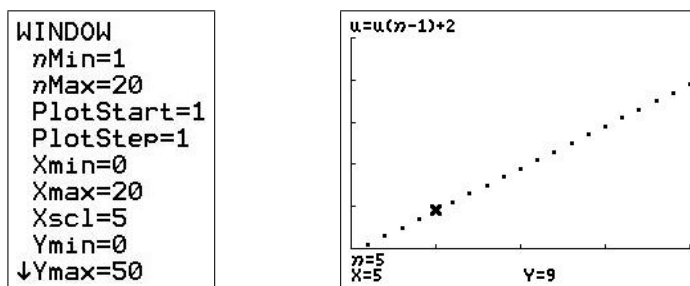
Shortcut: Specify nMax, then press $\boxed{\text{zoom}}$ $\boxed{0}$ (*ZoomFit*); this sets appropriate X and Y
scales. You may want to change these a bit in $\boxed{\text{window}}$ after the graph has been plotted.
Other zoom options, such as *Zoom In* and *Zoom Box*, work too.

Defining the AP explicitly

The nth term for a general arithmetic progression in calculator notation is

$$u(n) = a + (n-1)d,$$

where a is the first term $(n=1)$ and d is the common difference.

For our example, we have $a=1$ and $d=2$, so that

$$u(n) = 1 + 2(n-1).$$

We can compare the two definitions of the sequence by putting the nth term in sequence v,
as shown below. Note that v(nMin)/v(1) is not required when a sequence is defined explicitly
and should be cleared. Plot v as a solid line so you see which sequence is being plotted.

```
 nMin=1
\u(n)Bu(n-1)+2
 u(nMin)B{1}
\v(n)B1+2(n-1)
 v(nMin)B
\w(n)=
 w(nMin)=
```

```
TYPE: SEQ(n)  SEQ(n+1)  SEQ(n+2)
 nMin=1
■\u(n)Bu(n-1)+2
 u(1)B1
 u(2)=
■\v(n)B1+2(n-1)
 v(1)B
 v(2)=
■\w(n)=
```
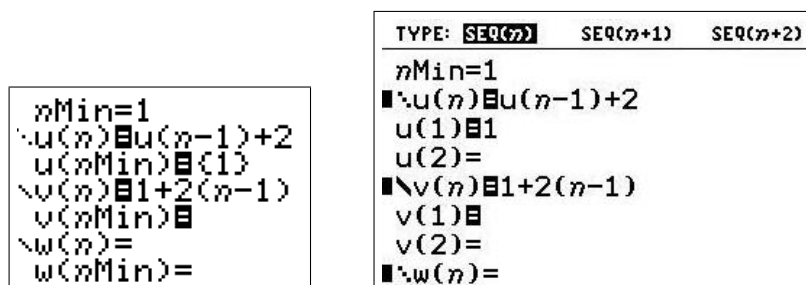
Now press $\boxed{\text{graph}}$ and compare u and v. $\boxed{\text{trace}}$ might be useful. You can also use $\boxed{\text{table}}$ to
compare values of u and v.

```
u=u(n-1)+2
                            ·
                       ·
                  ·
             ×
        ·
   ·
n=7
X=7              Y=13
```
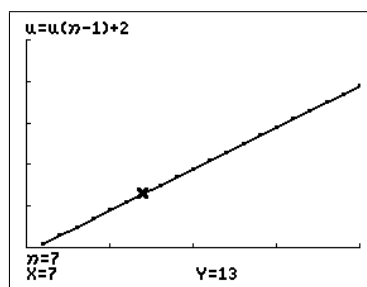
Using the APGP/APGPCE program

This program calculates the Nth term of an AP, T(N) $=$ A$+$(N-1)D, and the sum of N terms.
Useful if you need to do more than one or two of these.

Geometric progression

In a geometric progression, each term is a constant multiple of the previous term. In calculator notation, the recursive definition is

$$\mathrm{u}(n) = r\mathrm{u}(n-1),$$

where r is a constant called the common ratio or common multiplier.

Exercise: Display a table and graph the geometric sequence $\mathrm{u}(n) = 0.5\,\mathrm{u}(n-1)$, with $\mathrm{u}(1) = 1$. For the graph, use *ZoomFit* after specifying $n\mathrm{Max}$ in window .
Solutions to the exercises are in Section 17.6.

You may want to go back to window after the graph is plotted to adjust Xscl and Yscl so there aren't too many tick marks on the axes.

Press graph or trace to return to the graph after changing the window.

The nth term of a geometric progression is given *explicitly* by

$$\mathrm{u}(n) = ar^{n-1},$$

where a is the first term $(n=1)$ and r is the common multiplier.

For the sequence in the exercise above, the nth term is given by

$$\mathrm{u}(n) = 0.5^{n-1}.$$

Exercise: Put this explicit definition in sequence v and compare with the recursive definition.

Using the APGP/APGPCE program

This program calculates the Nth term of an GP, $\mathrm{T}(N) = \mathrm{AR}^{\wedge}(N-1)$, and the sum of N terms. Useful if you need to do more than one or two of these.

Fibonacci sequence

The Fibonacci sequence is defined, in calculator terms, by

$$\mathrm{u}(n) = \mathrm{u}(n-1) + \mathrm{u}(n-2) \quad \text{with} \quad \mathrm{u}(1) = \mathrm{u}(2) = 1.$$

Press y= and enter this formula (below left).

On a TI-84, the two starting values $\mathrm{u}(1)$ and $\mathrm{u}(2)$ are stored in $\mathrm{u}(n\mathrm{Min})$ as a list: $\{1, 1\}$.

The graph here is of the first 10 points of the sequence $(n\mathrm{Max} = 10)$ plotted with $\mathrm{Xmin} = 0$, $\mathrm{Xmax} = 10$, $\mathrm{Xscl} = 5$, $\mathrm{Ymin} = 0$, $\mathrm{Ymax} = 50$ and $\mathrm{Yscl} = 10$: i.e. a window $[0, 10, 5] \times [0, 50, 10]$.

17.2.2 Using LIST commands

The *seq* command in the $\boxed{\text{list}}$ ($\boxed{\text{2nd}}$ $\boxed{\text{stat}}$) OPS menu generates a sequence (list) specified by an explicit general term. The syntax is

$$\text{seq (general term, variable, start, end [, step]),}$$

where variable can be any letter and *step* is assumed 1 if not given.[3] The total number of terms calculated must be less than 1000.[4]

With STAT WIZARDS on (in $\boxed{\text{mode}}$), you get prompts for each of the entries in the *seq* command. Alternatively, CtlgHelp (scroll down to *seq* in the $\boxed{\text{list}}$ OPS menu and press $\boxed{+}$) tells you what to input in the *seq* command and in what order.

Arithmetic progression

In the following sections, there is an implicit $\boxed{\text{enter}}$ after each command.

To generate an AP: $\text{seq}\,(1+2(N-1), N, 1, 10)$

```
seq(1+2(N-1),N,1,10)
{1 3 5 7 9 11 13 15 17 19}
```

If the sequence goes off the right-hand side of the screen, you can scroll across using the right-arrow key.

A nice way to use the general expression for an AP is

$$1 \to A : 2 \to D : \text{seq}\,(A+(N-1)D, N, 1, 10) \qquad : \text{ is } \boxed{\text{alpha}}\ \boxed{\cdot}$$

```
1→A:2→D:seq(A+(N-1)D,N,1,1
0)
{1 3 5 7 9 11 13 15 17 19}
```

Press $\boxed{\text{entry}}$ ($\boxed{\text{2nd}}$ $\boxed{\text{enter}}$)[5] or use the up-arrow key and $\boxed{\text{enter}}$ to recall the command, change the values of A and D, and press $\boxed{\text{enter}}$ to re-execute the command.

Storing to a list

For analysis and graphing, it is often convenient to store a sequence in a named list, which is then stored in memory. The TI-84/CE has six built-in lists: $L_1 - L_6$; on the keyboard $\boxed{\text{2nd}}$ $\boxed{1}$ – $\boxed{\text{2nd}}$ $\boxed{6}$. Lists can also be given any name of up to five characters.

To store a sequence to list L_1,

$$\text{seq}\,(2N-1, N, 1, 10) \to L_1,$$

where $\to$ is the $\boxed{\text{sto}}$ key. If you have already generated the sequence, just press $\to L_1$.

Now press $\boxed{\text{stat}}$ $\boxed{1}$ (Edit. . .) to enter the list editor. You can scroll up and down, graph, edit and carry out other operations on any list here.

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[3]The variable in a *seq* command doesn't have to take integer values, e.g. seq $(\sin(X), X, 0, 1, 0.05)$ generates a sequence of values of the sine function.

[4]A list on the TI-84/CE can have a maximum of 999 entries. The same restriction applies to the *seq* command even if you are not storing the sequence in a list, because the TI-84/CE keeps the sequence in an internal list *Ans*.

[5]Pressing $\boxed{\text{entry}}$ once recalls the previous command, twice the command before that, and so on. There is limited memory for this, so you can't go back indefinitely.

Geometric progression

Try the following commands ($\boxed{\text{enter}}$ after each).

$\qquad$ seq $(0.5^{\wedge}(\text{N}-1), \text{N}, 1, 10)$

$\qquad$ $\boxed{\text{math}}$ $\boxed{1}$ $\qquad$ a useful command for sequences containing fractions

$\qquad$ $1 \rightarrow \text{A} : 0.5 \rightarrow \text{R} : \text{seq} (\text{AR}^{\wedge}(\text{N}-1), \text{N}, 1, 10)$

Exercise: Use $\boxed{\text{entry}}$ ($\boxed{\text{2nd}}$ $\boxed{\text{enter}}$) to produce terms 11 to 20 in the sequence. *Solutions to the exercises are in Section 17.6.*

A neat trick for sequences

What we'd really like is an endless display of the sequence terms. This is provided in $\boxed{\text{table}}$ by evaluating the continuous function corresponding to our sequence at integer values of its argument. This is a good way to start sequences — see the Class Activity below.

Select FUNC(TION) in $\boxed{\text{mode}}$ to return to the normal type of graph.

In $\boxed{\text{y=}}$: $\quad$ Y₁ $= 0.5^{\wedge}(\text{X}-1)$.

In $\boxed{\text{tblset}}$: $\quad$ TblStart $= 1$ $\quad$ ΔTbl $= 1$.

Press $\boxed{\text{table}}$.

Scroll down in the Y₁ column (the sequence terms) to see more digits for each entry at the bottom of the screen.

Provided TblStart and ΔTbl are integers, we get sequence values in the table. This is an alternative to using the sequence grapher to generate a table, and is usually faster.

17.2.3 Using the SEQUENCE/SEQNCECE program

Given an infinite sequence $\{A_N\}$, what is the limit of the sequence, i.e. what is $\lim\limits_{N\to\infty} A_N$?

The program evaluates A_N for $N = 1, 2, 4, 8, 16, 32, \ldots$ as an aid to *estimating* the limit.

Use: Store the Nth term A_N in Y₁ (in terms of N).

Run the program. Press $\boxed{\text{enter}}$ to see successive values. Be wary of round-off error for large N. Stop the program at any time by pressing $\boxed{\text{on}}$ Quit.

We could do the calculation using Sequence mode on the calculator, but this is much slower.

Example: $A_N = (1 + 1/N)^N \rightarrow$ Y₁ $= (1 + \text{N}^{-1})^{\wedge}\text{N}$

$A_1 = 2$
$A_2 = 2.25$
$A_4 = 2.4414065$
$A_8 = 2.565784514$
$A_{16} = 2.637928497$
$\qquad \vdots$
$\lim\limits_{N\to\infty} A_N = \qquad ?$

```
N, A(N), A(N)▶Frac
                  1
                  2
                  2
N, A(N), A(N)▶Frac
                  2
               2.25
                9/4
```

$\boxed{\blacksquare \backslash Y_1 \boxminus (1 + N^{-1})^{\wedge}N}$

You may see the effect of roundoff error if you let N become large enough.

Class Activity

Give students one calculator between two. Have them press $\boxed{\text{tblset}}$ ($\boxed{\text{2nd}}$ $\boxed{\text{window}}$) and set both TblStart and ΔTbl to 1.

On the viewscreen calculator, with the OHP turned off, set $Y_1 = 2X+1$ and press $\boxed{\text{table}}$. Turn the OHP on and ask the students: *What's the rule?* If you don't have a viewscreen (or a TI-84/CE emulator on a computer), just write the numbers up on the board.

When they have worked out the rule as a class, press $\boxed{\text{y=}}$ to show them (or just tell them) how the rule is entered. Have them enter the rule and generate the table. Write on the board that the rules to follow are all of the form __X + __.

Now, again with the OHP off, enter a different rule and show them the table. Ask them to make the table on their calculators the same.

Give them various rules to find, moving eventually to negative numbers for the coefficients. Ask them to summarise their findings regarding the two numbers in the rules.

17.3 Series

A series is a sequence with + signs between successive terms. The sum of a series is the actual value when we carry out the additions. To calculate the sum of a series, we need to find the terms in the corresponding sequence, then add them up.

If a series has a finite number of terms, we just add them up to give the sum. All of what we do below can be applied to finite series. However, the more interesting series are infinite — we can't calculate the sum for these by carrying out the additions because there is an infinite number of them.[6]

However, we can work out the sum of a finite number of terms, called a *partial sum* — the nth partial sum is the sum of the first n terms of the series. The behaviour of the partial sums as n gets bigger tells us something about the convergence of the series — whether the sum may be a finite number or infinite. The sum of an infinite series is defined as the limit of its partial sums as $n \to \infty$.

Both the sequence grapher and the $\boxed{\text{list}}$ commands can be used to sum a series, but, unless we are quite clever (see Section 17.3.3 below), both methods require an explicit nth term.

Because of this, the $\boxed{\text{list}}$ commands are probably simpler, and the 'neat trick' above, suitably modified (see Section 17.3.2), allows us to use $\boxed{\text{table}}$. However, the sequence grapher is easier to use if we want a graph of the partial sums.

The SERIES/SERIESCE program (Section 17.3.3) combines the speed of the $\boxed{\text{list}}$ commands with the flexibility of the sequence grapher. It is available at *canberramaths.org.au* under *Resources*.

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[6]There are algebraic methods for summing some infinite series, one of the triumphs of Calculus.

17.3.1   Using the sequence grapher

The nth partial sum S_n of a series is the sum of the first n terms:

$$S_n \;=\; u_1 + u_2 + u_3 + \cdots + u_n \;=\; \sum_{i=1}^{n} u_i.$$

Recursively, S_n is the sum of the previous n terms, S_{n-1}, plus the nth term:

$$S_n = S_{n-1} + u_n \qquad S_1 = u_1.$$

We'd like to generate the sequence of partial sums

$$S_1,\ S_2,\ S_3,\ S_4,\ S_5,\ \ldots.$$

We'll use the sequence v in the sequence grapher for the partial sums: $\mathrm{v}(n)$ will be the nth partial sum of the sequence we put in $\mathrm{u}(n)$.

Re-select SEQ in $\boxed{\text{mode}}$ to return to the sequence grapher if necessary.

Arithmetic progression

For our AP, with nth term $\mathrm{u}(n) = 1{+}2(n{-}1)$, we have for the nth partial sum

$$\mathrm{v}(n) = \mathrm{v}(n{-}1) + \mathrm{u}(n), \quad \text{with} \quad \mathrm{v}(1) = 1,$$

as shown below centre.

Note: On a TI-84, the sequence grapher only lets us use the $(n{-}1)$th and $(n{-}2)$th terms in the definition of an nth term, so we have to use the following for v (left-hand figure below):

$$\mathrm{v}(n) = \mathrm{v}(n{-}1) + 1{+}2(n{-}1), \quad \text{with} \quad \mathrm{v}(1) = 1.$$

The table of partial sums can now be displayed as usual (above right); the terms of the sequence are shown in the $\mathrm{u}(n)$ column and the corresponding partial sums in the $\mathrm{v}(n)$ column.

Geometric progression

Exercise: Set up the table of partial sums for our GP with $\mathrm{u}(n){=}0.5^{n-1}$.

To what value does the (infinite) series appear to converge?

Confirm your answer algebraically (using the usual formula for the infinite sum of a GP).

Solutions to the exercises are in Section 17.6.

17.3.2 Using LIST commands

The *sum* command in the ⎡list⎤ MATH menu sums a sequence (list). The syntax is

sum $(list)$,

where *list* can also be a *seq* command.

The *cumSum* command in the ⎡list⎤ OPS menu generates a sequence of partial sums. The syntax is

cumSum $(list)$.

Arithmetic progression

sum $(\text{seq} (1+2(\text{N}-1), \text{N}, 1, 20))$ finds the sum of the first 20 terms in our AP.

cumSum $(\text{seq} (1+2(\text{N}-1), \text{N}, 1, 20))$ generates the first 20 partial sums of our AP: the ith entry is the sum of the first i terms.

```
sum(seq(1+2(N-1),N,1,20)
                     400
cumSum(seq(1+2(N-1),N,1,10
)
{1 4 9 16 25 36 49 64 81 …
```

The sum of an infinite series is defined as the limit as $n \to \infty$ of the nth partial sum. Scrolling across a list like the one above can give an idea of what that limit might be. Here, of course, there is no limit — the nth partial sum $\to \infty$ as $n \to \infty$.

If you prefer scrolling up and down, rather than side to side, store the partial sums to a list (⎡sto⎤ ⎡L1⎤ ⎡enter⎤) and use ⎡stat⎤ Edit .

Geometric progression

sum $(\text{seq} (0.5^{\wedge}(\text{N}-1), \text{N}, 1, 20))$

cumSum $(\text{seq} (0.5^{\wedge}(\text{N}-1), \text{N}, 1, 20))$

```
sum(seq(0.5^(N-1),N,1,20))
               1.999998093
cumSum(seq(05^(N-1),N,1,20
))
{1 6 31 156 781 3906 1953…
```

Exercise: What's the sum of the GP $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \ldots$?

You might want to use ⎡entry⎤ to change the end value for N in the first command above to be (almost) sure.

Solutions to the exercises are in Section 17.6.

A neat trick for partial sums

Again we'd really like an endless sequence of partial sums. Try this.

Select FUNC(TION) in $\boxed{\text{mode}}$.

In $\boxed{\text{y=}}$: $Y_1 = \text{sum}\,(\text{seq}\,(0.5^{\wedge}(N-1), N, 1, X))$.

In $\boxed{\text{tblset}}$: $\text{TblStart} = 1$ $\Delta\text{Tbl} = 1$.

Press $\boxed{\text{table}}$.   Scroll down in the Y_1 column (the partial sums) to see more digits at the bottom. *What's your best estimate for the sum?*

Why does this work? In generating the table, the calculator takes a given value of X, an integer in this case generated by the values of TblStart and ΔTbl, and puts it in the formula for Y_1, giving the partial sum of X terms. It then displays the result in the Y_1 column of the table. TblStart and ΔTbl must be integers.

17.3.3   Using the SERIES/SERIESCE program

We can get around the problem of not being able to sum recursively defined sequences, such as the Fibonacci sequence, by a combination of methods. If we define the sequence as usual in u(n), then use the 'neat-trick' idea by setting $Y_2 = \text{sum}(\text{seq}(u(n), n, 1, X))$, we end up with the Y_2 column in the table giving the partial sums of the sequence/series. If we also set $Y_1 = u(X)$, we have the sequence values in the Y_1 column.

The SERIES/SERIESCE program sums series defined either recursively (using the above method) or explicitly. Run the program, select how the nth term is specified, enter the appropriate definition and:

- generate a table of sequence values and corresponding partial sums. Once a table has been generated, you can change TblStart to look at the partial sums starting at different values of N, and increase ΔTbl to move through the partial sums more quickly, for example to examine the convergence or otherwise of the series.

- graph the partial sums. Somtimes a graph gives a better idea of where the partial sums are heading than a table.

- use FAST SUM to calculate a particular partial sum faster.

The SERIES/SERIESCE program is a useful tool for investigating infinite series.

Exercise: Use the SERIES/SERIESCE program to generate a table and a plot of the first 20 partial sums of:

1. the GP with nth term $u(n) = 0.5^{n-1}$;

2. the Fibonacci sequence.

Solutions to the exercises are in Section 17.6.

17.4   Questions

Solutions to the questions are in Section 17.6.

1. Generate a sequence of the cubes of first 10 positive integers using a LIST command. Store the sequence in a list. Use the *sum* command to evaluate $1^3 + 2^3 + \cdots + 10^3$.

2. Find the sum of the first 10 terms of the series $1 + \dfrac{1}{2^3} + \dfrac{1}{3^3} + \dfrac{1}{4^3} + \cdots$.

3.   (a) Use a single command to find the sum of the first 100 positive integers. Store the answer in memory S for use in (c).

    (b) Edit your command in (a) to find the sum of the *cubes* of the first 100 positive integers.

    (c) Which is bigger: the sum of the cubes of the first 100 positive integers or the cube of the sum of the first 100 positive integers?

4. The half-life of a certain radioactive substance is 1 week. This means that, of the amount present at a particular time, only half will be left a week later. Suppose 1000 grams of the substance exists today, the beginning of Week 1.

    (a) Write down the amount left at the beginning of Week 2, Week 3, . . . , Week 10.

    (b) Determine an infinite geometric sequence (recursive or explicit) that is a model of the amount of the substance at the beginning of Week n, where $n = 1, 2, 3, \ldots$. What is the common ratio of this sequence?

    (c) When will there be only 0.005 grams remaining?

    (d) How much of the substance was there a week ago (beginning of Week 0)?

    (e) When will the substance be reduced to nothing *according to this model*?

5. The height of a particular fast-growing plant increases at the rate of 2.5% per month. Assume the plant is 30 cm high today and that it dies after 12 months.

    (a) Determine a finite geometric sequence that is a model of the height of the plant after n months. Write out all the terms of the sequence. What is the common ratio?

    (b) How long would the plant have to live to double in height?

6. Sue had $1250 in a savings account 3 years ago. How much will be in her account 2 years from now, assuming no deposits or withdrawals are made and the account earns 6.5% interest compounded annually?

7. Frank has $12,876 in a savings account today. He made no deposits or withdrawals during the last 6 years. What was the value of his account 6 years ago? Assume that the account earned 5.75% interest compounded monthly.

PTO

8. Generating sequences recursively is equivalent to another mathematical process called *iteration*, in which we do the same operation over and over. Try some of the following sequences/iterations.

   Generate some terms in each sequence. What happens to u(n) in each sequence as n becomes large? Be careful with brackets.

   (a) $u(n) = \big(u(n-1)\big)^2$.   Try $u(1) < -1$; $u(1) = \pm 1$; $-1 < u(1) < 1$; $u(1) > 1$.

   (b) $u(n) = (u(n-1))^2 - 1$ with $-2 < u(1) < 2$.
       What happens if we set
       $$u(1) = \tfrac{1}{2}\big(1 + \sqrt{5}\big) = 1.618\ldots ?$$
       $$u(1) = \tfrac{1}{2}\big(1 - \sqrt{5}\big) = 0.618\ldots ?$$

   (c) $u(n) = \sqrt{u(n-1)}$.

   (d) $u(n) = \cos\big(u(n-1)\big)$.

   (e) $u(n) = \tan\big(u(n-1)\big)$.

9. What value does the sequence $\left(1 + \dfrac{1}{n}\right)^n$ approach as n gets larger and larger?

   Hint: Use a sequence command with a step of at least 1000. Or apply the neat-trick method for sequences: set $Y_1 = (1+1/X)^X$ and use a suitably large ΔTbl, say 1000.

10. What is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$?   factorial ! is in the $\boxed{\text{math}}$ PRB menu.

    The *cumSum(seq* combination works well here, as we don't need too many terms to see the convergence of the partial sums. Store 20 or so terms in a list to make it easier to see the partial sums or use the neat trick for series. Note that the sum starts at $n=0$.

    What is $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$? $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, where x is any number?

    The following questions are from Chapter 9 of *Intermediate Algebra: Functions and Graphs* by K. Yoshiwara and B. Yoshiwara, Thomson Brooks/Cole, 2004.

11. Generate a table of values for each of the following recursive sequences. What happens to the terms as n gets larger? Do you recognise the number?

    Hint: The answers for the first two sequences are square roots of particular numbers. *Can you make and test a conjecture here? Write down a sequence whose limit is 2; $\sqrt{5}$.*

    The answers for the last two sequences should be obvious.
    Can you make and test a conjecture here? Write down a sequence whose limit is 4; 5.

    (a) $a_1 = 1$ $a_n = \dfrac{1}{1+a_{n-1}} + 1$ (b) $b_1 = 1$ $b_n = \dfrac{2}{1+b_{n-1}} + 1$

    (c) $s_1 = 1$ $s_n = \dfrac{1}{2}\left(s_{n-1} + \dfrac{4}{s_{n-1}}\right)$ (d) $t_1 = 1$ $t_n = \dfrac{1}{2}\left(t_{n-1} + \dfrac{9}{t_{n-1}}\right)$

12. A rubber ball is dropped from a height of 8 metres and returns to three-quarters of its previous height on each bounce.

   (a) How high does the ball bounce after hitting the floor for the 3rd time? for the 10th time? for the nth time?

   (b) How far has the ball travelled vertically when it hits the floor for the 4th time? for the 20th time? for the bth time?

13. According to legend, a man who had pleased the Persian king asked for the following reward. The man was to receive a single grain of wheat for the first square of the chessboard, 2 grains for the second square, 4 grains for the third square, and so on, doubling the amount for each square up to the 64th square. How many grains would he receive in all. (Fortunately the king had a good sense of humour.)

14. Find the sum of all integral multiples of 6 between

   (a) 10 and 100.

   (b) between 1 and 10,000.

<div align="right">

PTO

</div>

17.5   Further investigations

17.5.1   Fibonacci numbers

Fibonacci (fib–on–archie), real name Leonardo Pisano (Leonardo of Pisa), was born in about 1170. He too thought about populations, but much earlier than Malthus.[7] One of his problems concerning a rabbit population led to the famous Fibonacci numbers or Fibonacci sequence

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad \ldots .$$

The Rule: Add the previous two numbers to get the next number (after starting with 1 1) .

Exercise: Write down the first twenty Fibonacci numbers.
Solutions to the exercises in this section are in Section 17.6.

Here is Fibonacci's rabbit problem. See if you can understand why the Fibonacci numbers give the number of *pairs* of rabbits each month and answer the question. The diagram might help.

Exercise: A pair of new-born rabbits is put in a pen. How many pairs of rabbits are there after a year if, every month, each adult pair produces a new pair? The rabbits become adult one month after birth.
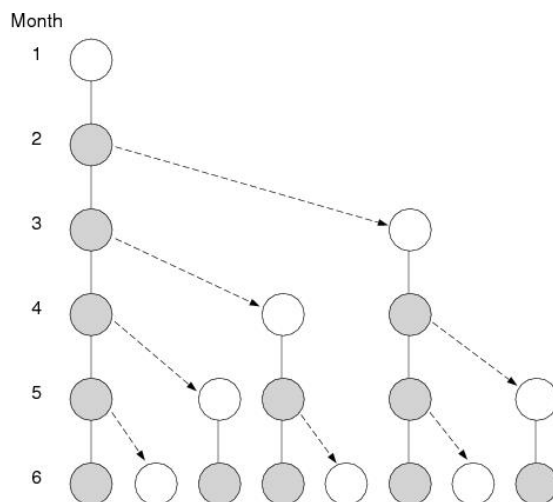
In the diagram, an open circle represents a *pair* of immature rabbits (too young to breed) and a shaded circle a *pair* of mature breeding rabbits. The arrows lead to offspring. Adding the number of circles for each month gives the number of pairs of rabbits — the Fibonacci numbers.

Note that the growth here is not exponential (we are not multiplying by a constant to obtain the next number), but the number of rabbits still increases rapidly.

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[7]see *Population Modelling 1* in Volume 1 of this book

The Fibonacci sequence and the Golden Ratio

Here is one illustration of the many interesting properties of the Fibonacci sequence.

1. Generate a table of values of the sequence (be very careful with brackets)

$$u_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Do you recognise this sequence?

2. One of the many interesting things about the Fibonacci sequence becomes apparent when we look at ratios of successive terms. Using the sequence grapher, set

$$u(n) = u(n-1) + u(n-2) \qquad u(1) = 1, \quad u(2) = 1 \ [8]$$

to generate the Fibonacci sequence, and

$$v(n) = u(n-1)/u(n-2)$$

to calculate the ratios of successive terms.

Put TblStart $= 3$, ΔTbl $= 1$ and generate the table of values.

Scroll down and look at the $v(n)$ column containing a sequence of the ratios of successive terms of $u(n)$. Does this sequence appear to be approaching a particular value? What value? Find this value accurate to six decimal places.

Now find a decimal value for the Golden Ratio or Golden Section, $\dfrac{1+\sqrt{5}}{2}$.

What conclusion do you reach?

The Golden Number/Ratio/Section

The Golden Ratio (also called the Divine Proportion) is denoted by the Greek letter ϕ or τ. If a length is divided in the Golden Ratio $1:\phi$, the ratio of the longer part to the whole, $\phi:1+\phi$, is also the Golden Ratio. When used to construct a rectangle, this ratio was thought to make the rectangle pleasing to the eye. Hence the Golden Ratio occurs everywhere, and has supposedly been used to design buildings from the Parthenon to the United Nations building in New York, as well as by artists and musicians. It is closely connected with the Fibonacci series, and has a value of $(1+\sqrt{5})/2 = 1.61803\ldots$.

Fibonacci numbers are evident in particular (equi-angular or logarithmic) spirals which appear frequently throughout the natural world — in things as small as the double helix and other microscopic twisting structures, to the galaxies that move in equi-angular spirals. These spirals also account for gastropods growing while maintaining their shape. In order for all leaves on a stem to catch sunlight, they are arranged in equiangular spirals that incorporate the Fibonacci numbers. Other examples of this natural phenomenon include pine-cone seeds, flower petals, sunflower seeds, the horns of mountain goats, elephant tusks, lions' claws, scales in pineapples, and so on. Find out more at *www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html*.

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[8]On a TI-84, set u(nMin) $= \{1, 1\}$.

17.5.2   The logistic sequence

The logistic sequence or logistic map has become famous because it is one of the simplest sequences that exhibits chaotic behaviour. It also turns up in a number of areas such as population modelling.[9]

The logistic sequence is defined by (be very careful with brackets)

$$u(n) = Au(n-1)\big(1-u(n-1)\big),$$

where A is a constant.

The following figures show the set up for *time plots*, u(n) vs n, on the left and for *cobweb plots*, u(n) vs u($n-1$), on the right.

The top line of the [format] screen is where you choose which plot you want.

- Note that nMin$=0$ here and we have taken u(0)$=0.5$.[10] The value for u(0) can be changed in subsequent plots.

- Store a value for A $\big(0<A<4\big)$ in memory A and press [trace].

- For a cobweb plot, keep pressing the right arrow to generate the plot.

  The curves $y = Ax(1-x)$ and $y = x$ are also plotted; the cobweb lines move between these two curves.

  [2nd][quit] returns you to the Home screen.

- The TI-84/CE program[11] CHAOS/CHAOSCE makes the process of graphing the logistic sequence a bit easier, allowing you to concentrate on the effects of changing A and u(0).

    format

PTO

---

[9]For examples, see *Population Modelling 2* in Volume 2 of this book.

[10]u(nMin)$=\{0.5\}$ on an 84

[11]available at *canberramaths.org.au* under *Resources*

Time   **Web**

```
WINDOW
 nMin=0
 nMax=20
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=20
 Xscl=5
 Ymin=0
↓Ymax=1
```
                              window
```
WINDOW
 nMin=0
 nMax=20
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=1
 Xscl=0.5
 Ymin=0
↓Ymax=1.15
```



```
u=Au(n-1)(1-u(n-1))

n=15
X=15          Y=0.668819
```
                              $A = 2.9$
```
u=Au(n-1)(1-u(n-1))

n=15
X=0.639186    Y=0.668819
```



```
u=Au(n-1)(1-u(n-1))

n=15
X=15          Y=0.7994555
```
                              $A = 3.2$
```
u=Au(n-1)(1-u(n-1))

n=15
X=0.5130445   Y=0.7994555
```



```
u=Au(n-1)(1-u(n-1))

n=15
X=15          Y=0.8982439
```
                              $A = 3.7$
```
u=Au(n-1)(1-u(n-1))

n=15
X=0.5850375   Y=0.8982439
```
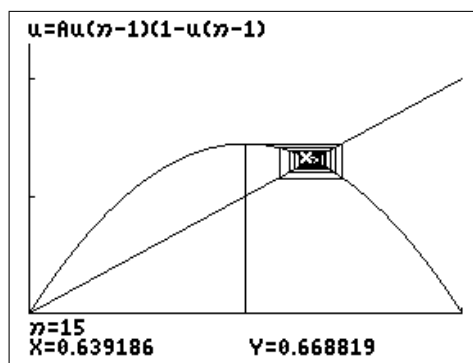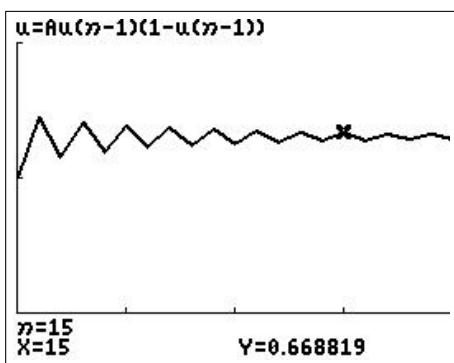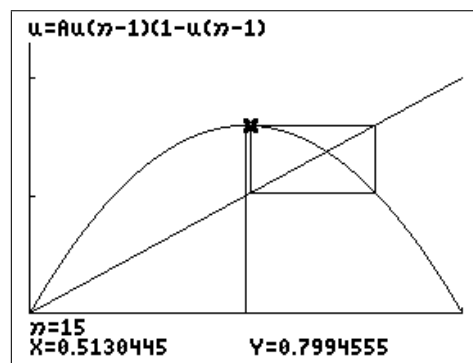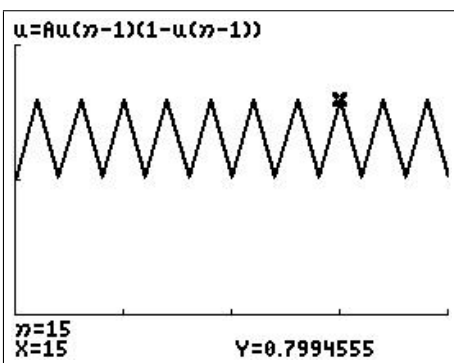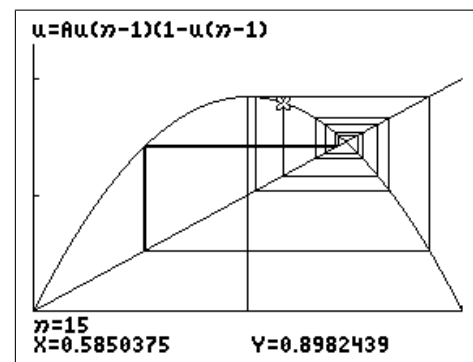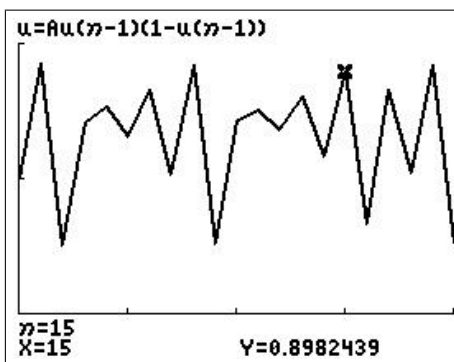
Exercise: Confirm graphically (by choosing appropriate values for A) that the logistic sequence:

- converges to 0 for $0 < A < 1$;

- converges to $1 - 1/A$ for $1 < A < 3$;

- oscillates between 2 values, then 4 values, 8 values and so on, as A is increased above 3;

- becomes chaotic for values of A greater than about 3.568;

- diverges rapidly to $\pm\infty$ for values of A greater than 4 or values of a_1 less than 0 or greater than 1

Try both Time and Web plots.

Solutions are in Section 17.6.

PTO

17.5.3 The harmonic series

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

has many interesting properties.[12]

1. Use the neat-trick method to generate the partial sums, starting with the first.

    Then press $\boxed{\text{tblset}}$ and change *Indpnt* to *Ask* using the cursor and $\boxed{\text{enter}}$. Press $\boxed{\text{table}}$.

    Now you can input the number of terms (X) in the partial sum, up to $X = 999$, and the calculator will calculate the corresponding partial sum (Y) .
    Does the series appear to be converging?

2. The *seq* command can only generate up to 999 terms. To sum more terms than this, we need a program. Use FAST SUM in the SERIES/SERIESCE program (Section 17.3.3) to find the sum of the harmonic series to 2000 and 4000 terms. Record the number of terms and the partial sum in a table, continuing on from your calculator table in Q1. Continue the table for as long as you have patience. Doing something else while the program runs is a good idea. The CE is actually quite quick.
    Does the series appear to be converging?

3. Graph the data using the SERIES/SERIESCE program.

    What can you conclude about the harmonic series?

4. There is an approximation to the partial sums of the harmonic series,

    $$\sum_{n=1}^{m} \frac{1}{n} \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{m} \;\approx\; \frac{1}{2m} + \ln(m) + \gamma,$$

    where $\gamma = 0.5772\dots$ is the Euler or Euler-Mascheroni constant.

    Check this approximation against the values you found in Q1 and Q2. To compare with your values in 1, set $Y_2 = (2X)^{-1} + \ln(X) + 0.5772$ and use the table.
    What do you conclude?

5. Find the next few decimal places in the Euler-Mascheroni constant.

6. Find out what you can about the Euler-Mascheroni constant.

Solutions are in Section 17.6.

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[12]The series does not converge — the more more terms you take, the larger the sum, despite the fact that the terms tend to 0 as $n \to \infty$.

17.6   Solutions to exercises, questions and activities

Exercises

Exercise   page 6

Display a table and graph the geometric sequence $u(n) = 0.5\,u(n-1)$, with $u(1) = 1$.
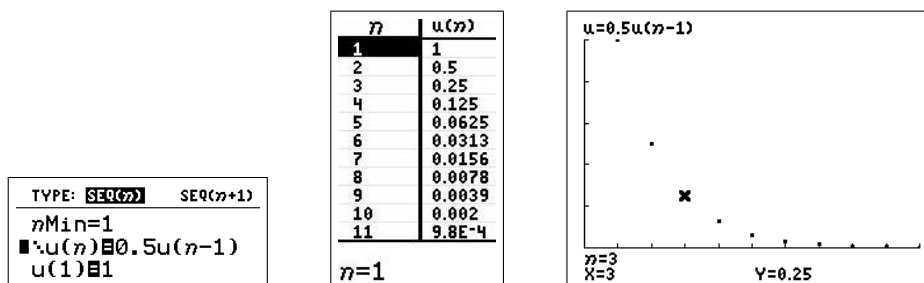
Exercise   page 6

Put this explicit definition in sequence v and compare with the recursive definition.
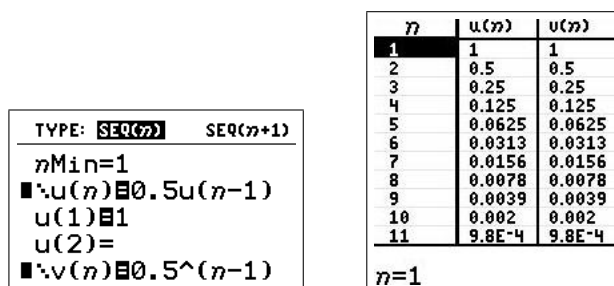
The two sequences are identical.

Exercise   page 10

Set up the table of partial sums for our GP with $u(n) = 0.5^{n-1}$.
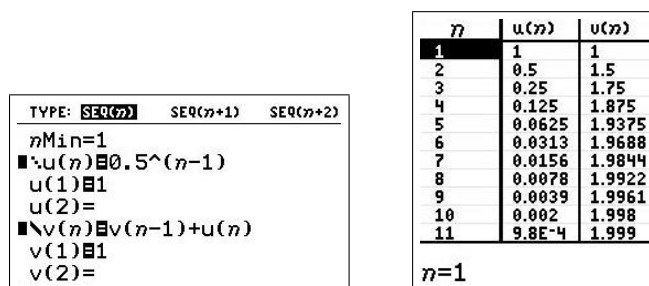
To what value does the (infinite) series appear to converge: 2.

Confirm your answer algebraically (using the usual formula for a GP).

Here $a = 1$ and $r = 0.5$, so that

$$S_\infty \;=\; \frac{a}{1-r} \;=\; \frac{1}{1-0.5} \;=\; 2.$$

Exercise   page 11

What's the sum of the GP $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \ldots$?

You might want to use $\boxed{\text{entry}}$ to change the end value for N to be (almost) sure.

```
sum(seq(0.5^(N-1),N,1,20))
                   1.999998093
sum(seq(0.5^(N-1),N,1,50))
                            2
sum(seq(0.5^(N-1),N,1,100)
)
                            2
```

From the figure, we are pretty sure the sum of the GP is 2.

Exercise   page 12

Use the SERIES/SERIESCE program to generate a table and a plot of the first twenty partial sums of:

1. the GP with nth term $u(n) = 0.5^{n-1}$
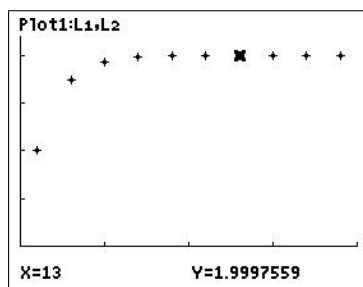
2. the Fibonacci sequence: $u(n) = u(n-1) + u(n-2)$; $u(1) = u(2) = 1$.
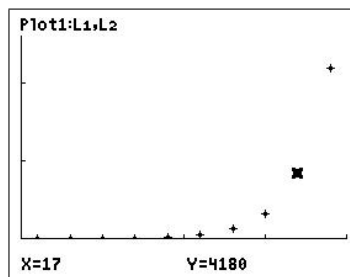
Questions   page 13

1. The command is seq $(X^3, X, 1, 10)$ [sto] L1. You can use any letter instead of X: X is easiest to use because it requires only one key press. If you just execute the *seq* command, the combination [sto] L1 on the next line produces the output Ans → L1 and stores the sequence. To calculate the sum, use sum (L1) to produce the answer 3025.

```
seq(X³,X,1,10,1)→L1
{1 8 27 64 125 216 343 51…
sum(L1)
                       3025
```

2. The command is sum $(\text{seq}\,(1/X^3, X, 1, 10))$, giving 1.197531986.

```
sum(seq(1/X³,X,1,10)
              1.197531986
```

3. **(a)** The sum of the first 100 positive integers is
   sum (seq $(X, X, 1, 100) = 5050$.

   (b) The sum of the *cubes* of the first 100 positive integers is
   sum (seq $(X^3, X, 1, 100) = 25,502,500 = 2.55 \times 10^7$ to 3 significant digits.

   (c) The cube of the sum of the first 100 positive integers is $5050^3 \approx 1.29 \times 10^{11}$, larger than the sum of the cubes of the first 100 positive integers.

```
sum(seq(X,X,1,100)
                       5050
Ans³
              1.28787625E11
sum(seq(X³,X,1,100)
                  25502500
```

4. **(a)** The amount of radioactive substance in grams at the beginning of successive weeks, starting at Week 1, is (to two decimal places)

   $$1000,\ 500,\ 250,\ 125,\ 62.5,\ 31.25,\ 15.63,\ 7.81,\ 3.91,\ 1.95.$$

   (a) Explicitly, the amount at the beginning of Week n is $u_n = 1000(0.5)^{n-1}$. Recursively, $\text{u}(n) = 0.5\text{u}(n-1)$, with $\text{u}(1) = 1000$. The common ratio here is 0.5.

   (b) Using a calculator table with $Y_1 = 1000(0.5)^{X-1}$ or a table generated from the recursive definition using the sequence grapher, there is 0.005 g remaining sometime in Week 18, i.e. between $n = 18$ and $n = 19$.

X	Y1		n	u(n)
9	3.9063		9	3.9063
10	1.9531		10	1.9531
11	0.9766		11	0.9766
12	0.4883		12	0.4883
13	0.2441		13	0.2441
14	0.1221		14	0.1221
15	0.061		15	0.061
16	0.0305		16	0.0305
17	0.0153		17	0.0153
18	0.0076		**18**	0.0076
19	0.0038		19	0.0038

(c) A week ago (beginning of Week 0), there was twice as much as there is now (beginning of Week 1), that is $2000\,\text{g}$.

(d) According to this model, there will always be some of the substance left, although the amount becomes small very rapidly. You can't reduce any number to 0 by dividing it by 2 or raising it to a power.

5. (a) The height n months from now is $u_n = 30\left(1+\frac{2.5}{100}\right)^n = 30\left(1.025\right)^n$, $n=1,2,\dots,12$. The table below shows months (top row) and corresponding heights, rounded to one decimal place.

0	1	2	3	4	5	6	7	8	9	10	11	12
30	30.8	31.5	32.3	33.1	33.9	34.8	35.7	36.6	37.5	38.4	39.4	40.4

The common ratio is 1.025.

(b) Using the calculator table (remember to use X instead of n if you don't use the sequence grapher), the plant would double in height in the 29th month.

6. Let S_n be the amount of money in Sue's account at the start of year n, with $n=1$ corresponding to 3 years ago. Then,

$$S_n = 1250\left(1+\frac{6.5}{100}\right)^{n-1} = 1250\left(1.065\right)^{n-1}.$$

In 2 years' time, $n=1+5=6$, so the amount of money in her account will be

$$S_5 = 1250\left(1.065\right)^5 = \$1712.61 \quad \text{to the nearest cent.}$$

You could also use the calculator table to reach this answer.

7. The amount in Frank's account is given by

$$S_n = 12876\left(1+\frac{5.75}{12\times100}\right)^{12(n-1)} = 12876\left(1+\frac{0.0575}{12}\right)^{12(n-1)},$$

where $n=1$ corresponds to now.

Six years ago, $n=-5$, and the amount in Frank's account was

$$S_{-5} = 12876\left(1+\frac{0.0575}{12}\right)^{-72} = \$9126.56 \quad \text{to the nearest cent.}$$

You could also use the calculator table (scroll up) to reach this answer.

8. (a) Set $u(n) = \left(u(n-1)\right)^2$ in the sequence grapher, set $u\left(n\text{Min}\right)/u(1)$ to an appropriate starting value and use the table to see what happens as n becomes large, the limit of the sequence.

Initial value	Limit of sequence
$u(1) < 1$	∞
$u(1) = -1$	1
$-1 < u(1) < 1$	0
$u(1) = 1$	1
$u(1) > 1$	∞

(b) Set $u(n) = \big(u(n-1)\big)^2 - 1$.

Initial value	Limit of sequence
$u(1) < -1.618\ldots$	∞
$u(1) = -1.618\ldots$	$-1.618\ldots$ (constant)
$-1.618\ldots < u(1) < -0.618\ldots$	oscillates between 0 and -1
$u(1) = -0.618\ldots$	$-0.618\ldots$ (constant)
$-0.618\ldots < u(1) < 0.618\ldots$	oscillates between 0 and -1
$u(1) = 0.618\ldots$	$-0.618\ldots$ (constant)
$0.618\ldots < u(1) < 1.618\ldots$	oscillates between 0 and -1
$u(1) = 1.618\ldots$	$1.618\ldots$ (constant)
$u(1) > 1.618\ldots$	∞

(c) $u(n) = \sqrt{u(n-1)}$ gives 0 if $u(1) = 0$, and tends to 1 otherwise.

(d) $u(n) = \cos\big(u(n-1)\big)$ tends to $0.7391\ldots$ in radian mode and $0.99985\ldots$ in degree mode.

(e) $u(n) = \tan\big(u(n-1)\big)$ looks several times as though it is going to settle down to a limit, but never does.

9. The sequence $\left(1 + \dfrac{1}{n}\right)^n$ approaches the value $e = 2.71828\ldots$ as n goes to ∞.

10. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e; \quad \sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2; \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,$ the Taylor-MacLaurin series for e^x.

11. (a) approaches $\sqrt{2}$; **(b)** approaches $\sqrt{3}$; **(c)** approaches 2; **(d)** approaches 3.

12. (a) The following table gives the rebound height after bounce n.

n	1	2	3	4	$\ldots$
Height (m)	6	$\frac{9}{2}$	$\frac{27}{8}$	$\frac{81}{32}$	$\ldots$

The rebound height after the 3rd bounce is therefore $27/8$ m.

The rebound height after bounce n is $6\big(0.75\big)^{n-1}$ m. Therefore, after the 10th bounce, the rebound height is $6\big(0.75\big)^9 \approx 0.45$ m.

Alternatively, put $Y_1 = 6\big(0.75\big)^{X-1}$ and use the table.

X	Y$_1$
1	6
2	4.5
3	3.375
4	2.5313
5	1.8984
6	1.4238
7	1.0679
8	0.8009
9	0.6007
10	0.4505
11	0.3379

(b) The total distance travelled is $8 + 2 \times$ each rebound height, i.e.

$$8 + 2\left(6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \ldots\right).$$

When the ball hits the floor for the 4th time, it has travelled

$$8 + 2\left(6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32}\right) = 35.75\,\text{m}.$$

Therefore, the distance travelled when it hits the floor for the bth time ($b \geqslant 2$) is

$$8 + 2\sum_{n=1}^{b-1} 6(0.75)^{n-1} = 8 + 12\sum_{n=1}^{b-1} 0.75^{n-1}.$$

Therefore, when the ball hits the floor for the 20th time, it has travelled

$$8 + 12\sum_{n=1}^{19} 0.75^{n-1} \approx 55.8\,\text{m}.$$

Here, use the command $8 + 12\,\text{sum}\,(\text{seq}\,(0.75 \wedge (\text{B–1}), \text{B}, 1, 19))$ to do the calculation.

13. Number of grains $= 1 + 2 + 4 + 8 + 16 + \ldots = \sum_{n=1}^{64} 2^{n-1} \approx 1.89 \times 10^{19}$.

Use $\text{sum}\,(\text{seq}\,(2\wedge(\text{N}-1), \text{N}, 1, 64))$ to calculate this.

To put this number into context, 10^{18} grains would be about the same volume as the Great Wall of China and the five Great Pyramids combined.[13]

14. We need to evaluate $\sum_{n=a}^{b} 6n$, where

(a) $6a$ is the smallest multiple of 6 greater than 10 and $6b$ is the largest multiple of 6 less than 100. Clearly, $a=2$ and $b=16$, so that the required sum is

$$\sum_{n=2}^{16} 6n = 810,$$

where we used the command $\text{sum}\,(\text{seq}\,(6\text{N}, \text{N}, 2, 16))$.

PTO

---

[13]From *Large Numbers* by Victor Scharaschkin, Australian Senior Mathematics Journal 4 (2), 111–125 (1990).

(b) $6a$ is the smallest multiple of 6 greater than 1 and $6b$ is the largest multiple of 6 less than 10,000. Clearly, $a=1$ and b is the integer part of $10,000/6$, that is 1666. The required sum is

$$\sum_{n=1}^{1666} 6n = 8,331,666.$$

We can't use the single command $\mathrm{sum\,(seq\,(6N,N,1,1666))}$ here because there are more than 999 terms in the series. Either break the sum up into two parts, $\mathrm{sum\,(seq\,(6N,N,1,999))}$ and $\mathrm{sum\,(seq\,(6N,N,1000,1666))}$, and add the two answers, or use the SERIES/SERIESCE program.

Activities

Rabbits and Fibonacci Numbers

The first twenty Fibonacci numbers are

1   1   2   3   5   8   13   21   34   55   89   144   233   377   610   987   1597   2584   4181   6765.

How many pairs will there be in one year?

1. There is only 1 pair of immature rabbits during the first month.

2. At the beginning of the 2nd month they mate, but there is still only 1 pair during the month.

3. At the beginning of the 3rd month the female produces a new pair, so now there are 2 pairs of rabbits in the pen.

4. At the beginning of the 4th month, the original female produces a second pair, making 3 pairs in all in the pen.

5. At the beginning of the 5th month, the original female has produced yet another new pair, while the female born 2 months ago produces her first pair, making 5 pairs in all.

   and so on.

The number of pairs of rabbits after one year is the 12th Fibonacci number, 144 pairs.

Why do the Fibonacci numbers appear as the number of rabbits in the pen each month?

If we let $f(n)$ be the number of pairs of rabbits in the pen at the start of the nth month, we will show that $f(1)=1$, $f(2)=1$ and $f(n)=f(n-1)+f(n-2)$, which is exactly the definition of the Fibonacci numbers.

We start in month 1 with one newly born pair, so $f(1)=1$.

There is also only 1 pair during month 2, since although the adults mate at the start of month 2, babies are not born until the start of month 3: $f(2)=1$.

Now look at the nth month.

All the rabbits from the previous month ($f(n-1)$ pairs of them) survive. Any rabbit (pair) that was alive 2 months ago is now able to produce a new pair; we assume they produce 1 and only 1 new *pair* per month. Thus, the number of newly born pairs is the same as the number of pairs alive 2 months ago, $f(n-2)$. The total number of rabbits this month is the sum of all the rabbits alive last month and those that are newly born this month, that is $f(n)=f(n-1)+f(n-2)$ — the definition of the Fibonacci numbers.

The Fibonacci sequence and the Golden Ratio   page 17

1. Generate a table of values of the sequence (be very careful with brackets)

$$u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

TYPE: SEQ(*n*)   SEQ(*n*+1)   SEQ(*n*+2)
*n*Min=1
∎∵u(*n*)⊟∎((1+√(5))/2)^*n*-((1
-√(5))/2)^*n*)/√(5)
u(1)⊟1
u(2)⊟1

n	u(*n*)
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89
n=1	

Do you recognise this sequence? It's the Fibonacci sequence.

2. One of the many interesting things about the Fibonacci sequence becomes apparent when we look at ratios of successive terms. Using the sequence grapher, set

$$u(n) = u(n-1) + u(n-2) \qquad u(1)=1, \quad u(2)=1 \text{ [14]}$$

to generate the Fibonacci sequence, and

$$v(n) = u(n-1)/u(n-2)$$

to calculate the ratios of successive terms.

Put TblStart $=3$, ΔTbl $=1$ and generate the table of values.

n	u(*n*)	v(*n*)
3	2	1
4	3	2
5	5	1.5
6	8	1.6667
7	13	1.6
8	21	1.625
9	34	1.6154
10	55	1.619
11	89	1.6176
12	144	1.6182
13	233	1.618
n=3		

Scroll down and look at the v(*n*) column containing a sequence of the ratios of successive terms of u(*n*).

n	u(*n*)	v(*n*)
9	34	1.6154
10	55	1.619
11	89	1.6176
12	144	1.6182
13	233	1.618
14	377	1.6181
15	610	1.618
16	987	1.618
17	1597	1.618
18	2584	**1.618**
19	4181	1.618
v(18)=1.6180344478217		

---

[14] On a TI-84, set u(*n*Min) $=\{1,1\}$.

Does this sequence appear to be approaching a particular value? What value? Find this value accurate to six decimal places.

From the table, the ratio is approaching 1.618034 (to 6 decimal places); all values after v(17) round to this value.

Now find a value for the Golden Ratio or Golden Section, $\dfrac{1+\sqrt{5}}{2}$.

$\dfrac{1+\sqrt{5}}{2} = 1.618034$ to 6 decimal places.

What conclusion do you reach?

The ratio of successive terms in the Fibonacci sequence tends to the Golden Ratio as $n \to \infty$.

The harmonic series   page 21

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

has many interesting properties.

1. Use the neat-trick method to generate the partial sums, starting with the first.

$$\blacksquare \text{\\Y}_1\text{\Bsum(seq(1/N,N,1,X))}$$

Then press $\boxed{\text{tblset}}$ and change *Indpnt* to *Ask* using the cursor and $\boxed{\text{enter}}$. Press $\boxed{\text{table}}$. Now you can input the number of terms (X) in the partial sum, up to X = 999, and the calculator will calculate the corresponding partial sum (Y).

X	Y₁		
100	5.1874		
200	5.878		
300	6.2827		
400	6.5699		
500	6.7928		
600	6.975		
700	7.129		
800	7.2625		
900	7.3802		
999	7.4845		

Y₁=7.4844708605505

Does the series appear to be converging?

Not obvious one way or the other.

PTO

2. The *seq* command can only generate up to 999 terms. To sum more terms than this, we need a program. Use FAST SUM in the SERIES/SERIESCE program (Section 17.3.3) to find the sum of the harmonic series to 2000 and 4000 terms. Record the number of terms M and the partial sum in a table, continuing on from your calculator table in Question 1. Continue the table for as long as you have patience.

Table values rounded to 3 decimal places. *Approx* refers to Question 4.

M	Sum	Approx
100	5.187	5.187
200	5.878	5.878
300	6.283	6.283
400	6.570	6.570
500	6.793	6.793
600	6.975	6.975
700	7.129	7.129
800	7.263	7.263
900	7.380	7.380
999	7.484	7.484
2000	8.178	8.178
4000	8.871	8.871
6000	9.277	9.277
8000	9.564	9.564
10,000	9.788	9.788
20,000	10.481	10.481
30,000	10.886	10.886
40,000	11.174	11.174

Does the series appear to be converging?

Still not clear.

3. Graph the data using the SERIES/SERIESCE program.
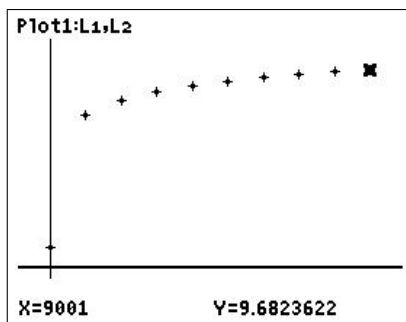
What can you conclude about the harmonic series?

Still not clear.

PTO

4. There is an approximation to the partial sums of the harmonic series,

$$\sum_{n=1}^{M} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{M} \approx \frac{1}{2M} + \ln(M) + \gamma,$$

where $\gamma = 0.5772\ldots$ is the Euler or Euler-Mascheroni constant.

Check this approximation against the values you found in Questions 1 and 2. To compare with your values in 1, set $Y_2 = (2X)^{-1} + \ln(X) + 0.5772$ and use the table.

See the table in Question 2.

What do you conclude?

The approximation is accurate to 3 decimal places for M up to at least $40,000$.

5. Find the next few decimal places in the Euler-Mascheroni constant.

Rewriting the equation above for the approximation to the partial sums of the harmonic series,

$$\gamma = \sum_{n=1}^{M} \frac{1}{n} - \left(\frac{1}{2M} + \ln(M) \right).$$

Setting $Y_1 = \text{sum}(\text{seq}(1/N, N, 1, X)) - ((2X)^{-1} + \ln(X))$ and using the table gives, from comparing the values for $X = 900$ and $X = 999$, $\gamma = 0.5772156$.

To go further, we need to use the program.

M $= 10,000$ $\gamma = 0.577\,215\,664\,1$

M $= 20,000$ $\gamma = 0.577\,215\,664\,7$

M $= 30,000$ $\gamma = 0.577\,215\,664\,8$

M $= 40,000$ $\gamma = 0.577\,215\,664\,8$

From this process, we obtain $\gamma = 0.577\,215\,664\,8$, probably accurate to this number of decimal places ± 1 in the last place.

Wolfram Mathworld: $\gamma = 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\,402\,431\,042\ldots$

Evidently, the larger M, the better the approximation to the partial sums of the harmonic series, and hence the more accurate the value we obtain for γ.

As a corollary, $\ln(M) \to \infty$ as $M \to \infty$, so the harmonic series must diverge.

18   Probability and Statistics 2
Probability Distributions and Hypothesis Testing

18.1   Introduction

Probability and Statistics 1 in the first volume of this book dealt with descriptive statistics, that is using the calculator to store, process and display statistical data. Here, we deal with discrete and continuous probability distributions, and hypothesis testing. We end with a section on gambling systems as a 'practical' example.

The most comprehensive calculator book I have seen on this topic is *Probability and Statistics with the TI-83 Plus: For A-Level Mathematics* by Peter Jones and Chris Barling.[15] Sadly, this is no longer readily available; if you have a copy, treasure it. They explain everything in great detail, with large numbers of screen shots. The exposition here is somewhat briefer but I'll follow their general outline, and even borrow some of their examples (labelled JB). Most of the exercises are also from there.

Some of the examples and exercises are from the course notes for a first-year introductory Statistics course by Dr Leesa Sidhu of UNSW Canberra.

What follows is not a textbook on Probability and Statistics (you'll need one) but how to do the various operations in such a course on a TI-84/CE graphics calculator. This replaces the various tables usually used in such courses, and makes many of the calculations quite simple. As always, students should do all these operations by hand first, so that they understand the process but, ultimately, always having to do the calculations manually stands in the way of doing any meaningful modelling and interesting applied problems.

18.1.1   Australian Curriculum

References are given here to the texts *Nelson Senior Maths Methods 11 for the Australian Curriculum* (NSM11) and *Nelson Senior Maths Methods 12 for the Australian Curriculum* (NSM12) used in the ACT.

The material here is relevant to the topics *Discrete random variables* (Chapter 2 in NSM12), *Binomial distributions* (Chapter 5 in NSM12), *Continuous random samples and the normal distribution* (Chapter 8 in NSM12), *Random samples and proportions* (Chapter 9 in NSM12), and *Confidence intervals* (Chapter 10 in NSM12).

18.1.2   Calculator note

There are four options on the calculator that can help with the calculations here.

MATHPRINT

In CLASSIC mode, commands are typed on one line, with arguments in brackets. In MATH-PRINT mode, the calculator tries to display commands in mathematical notation, with small boxes in the relevant positions for the inputs. This really only applies to the powers and roots (blue) commands on the keyboard and to many of the commands in the $\boxed{\text{math}}$ NUM menu, and so is of limited use in the Probability and Statistics operations here; where it applies, it is illustrated with a screen shot. Set CLASSIC or MATHPRINT in $\boxed{\text{mode}}$.

---

[15]Cambridge University Press, 2002, ISBN 0 521 52531 4

Catalog Help

An app on the TI-84Plus,[16] built in to the operating system of a TI-84CE, this tells you what arguments are required for most of the commands on the calculator in CLASSIC mode, allows you to input the arguments and paste the whole command onto the Home screen. Activated by scrolling down to the particular command and pressing $\boxed{+}$.

STAT WIZARDS

An alternative to *Catalog Help*, and much more useful here than MATHPRINT, this provides a menu for inputs to most of the commands and tests in $\boxed{\text{stat}}$ CALC and $\boxed{\text{stat}}$ TESTS, rather than having to type these into a one-line command in the correct order. It also works for some operations in the $\boxed{\text{math}}$ PROB menu.

Turn STAT WIZARDS on in $\boxed{\text{mode}}$ and leave on. It is used in most of the examples here.
Turn on STAT DIAGNOSTICS too while you are here.

```
MATHPRINT  CLASSIC
NORMAL  SCI   ENG
FLOAT   0 1 2 3 4 5 6 7 8 9
RADIAN  DEGREE
FUNCTION  PARAMETRIC  POLAR   SEQ
THICK DOT-THICK THIN DOT-THIN
SEQUENTIAL   SIMUL
REAL   a+bi   re^(θi)
FULL   HORIZONTAL   GRAPH-TABLE
FRACTION TYPE: n/d   Un/d
ANSWERS: AUTO   DEC
STAT DIAGNOSTICS: OFF   ON
STAT WIZARDS: ON   OFF
SET CLOCK  17/09/21 13:10
LANGUAGE:        ENGLISH
```
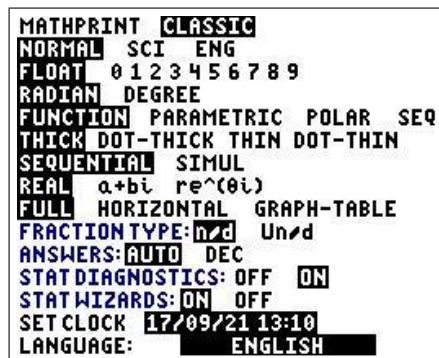
Prob Sim app

This app[17] simulates a number of different ways of obtaining random data normally done with physical materials (see the figure).

You can simulate single or multiple occurrences, with the outcomes summarised in on-screen histograms. *Random Numbers* can be used to simulate Lotto draws. Lots of data can be generated and stored in lists on the calculator for subsequent analysis. You can even change the probability of an event occurring.

Highly recommended.

```
PROB SIM APP
      Simulation
1: Toss Coins
2: Roll Dice
3: Pick Marbles
4: Spin Spinner
5: Draw Cards
6: Random Numbers

  OK        SEED ABOUT QUIT
```

PTO

---

[16]download from *education.ti.com*

18.2 Discrete probability distributions

18.2.1 Counting methods

The relevant commands here are all in the
math PROB menu.

```
MATH NUM CMPLX PROB FRAC
1:rand
2:nPr
3:nCr
4:!
5:randInt(
6:randNorm(
7:randBin(
8:randIntNoRep(
```

To evaluate 6! $\boxed{6}$ $\boxed{\text{math}}$ PROB $\boxed{4}$

To evaluate $^{10}P_2$ $\boxed{10}$ $\boxed{\text{math}}$ PROB $\boxed{2}$ $\boxed{2}$

To evaluate $^{10}C_2$ $\boxed{10}$ $\boxed{\text{math}}$ PROB $\boxed{3}$ $\boxed{2}$

```
6!
                    720
10 nPr 2
                     90
10 nCr 2
                     45
```

Both the CLASSIC and MATHPRINT versions
are shown here. In MATHPRINT, select the
commands before inserting any numbers; how-
ever, note what happens here if you do this
with the factorial command !.

```
6!
                    720
10P2
                     90
10C2
                     45
```

18.2.2 Random numbers

A random number between 0 and 1 is generated by the *rand* command in the math PROB
menu; an integer in brackets after the *rand* command generates a list of that many random
numbers. You can store this list to *listname* using sto *listname* either with the *rand* command
or after the list of random numbers has been generated.

Random integers are generated by the *randInt* command: to generate three random integers
between 1 and 17, use the command randInt(1, 17, 3). If you leave off the 3, you get a single
random integer.

With STAT WIZARDS on, you get a menu for the input. Enter the respective numbers, then
move the cursor to paste the command to the Home screen and press enter to execute it.

```
    randInt
lower:1
upper:17
n:3
Paste
```

```
randInt(1,17,3)
            {11 10 12}
```

Note the commands in the math PROB menu (top of the page) generating random numbers
that are normally or binomially distributed. The last command in the menu generates random
integers with no repeats, useful, for example, if you are simulating a raffle draw.

Exercises   *Solutions are in Section 18.6*

1. Evaluate each of the following by hand, then check with your calculator.

    (a) 6!   (a) $^6P_4$ (a) $^8C_5$.

2. Find: (a) 13!   (b) $^{15}P_9$ (c) $^{35}C_{22}$.

3. Generate 100 random integers between 1 and 6, simulating throwing a die. Store them
    in an appropriately named list.[17] Find: (a) the mean; (b) the standard deviation of
    the mean; (c) the median. Comment on your answers.

---

[17]see *Probability and Statistics 1* in Volume 1

18.2.3   Expected value and variance

Theory

The *probability distribution function* (pdf) of a discrete random variable X is

$$p(x) \;=\; P(X\!=\!x),$$

for each possible value of x; P stands for *(the) probability (that)*.[18]

The *cumulative distribution function* (cdf) is

$$F(x) \;=\; P(X\!\leqslant\!x).$$

The *mean* (μ or μ_X) or *expected value* $\big(E[X]\big)$ of a discrete random variable X is

$$\mu = \mu_X = E[X] \;=\; \sum_{\text{all }x} x p(x).$$

The *variance* of a discrete random variable X with mean μ is

$$V[X] \;=\; \sum_{\text{all }x}(x-\mu)^2 p(x) \;=\; \sum_{\text{all }x} x^2 p(x) - \mu^2.$$

The *standard deviation* of a random variable is the square root of the variance.

List operations

To calculate the expected value and variance of a discrete distribution, we must first enter the distribution values into lists.

Example: Consider the following distribution of a discrete random variable X.

x	0	1	2	3	4
$p(x) = P(X\!=\!x)$	0.2	0.3	0.1	0.3	0.1

Put the two sets of numbers in lists X and PX (see Probability and Statistics 1 in Volume 1 of this book).

X	PX
0	0.2
1	0.3
2	0.1
3	0.3
4	0.1

Then, the cdf $P(X\!\leqslant\!2)$ is given by $p(0) + p(1) + p(2) = 0.2 + 0.3 + 0.1 = 0.6$.

On the calculator: $\text{sum}\big(\text{\tiny L}\text{PX}, 1, 3\big)$.
The calculator command *sum* is in the list MATH menu. The 1 and 3 indicate the first and third terms of ʟPX.

```
sum(ʟPX,1,3)
                0.6
```

---

[18]Pr in JB

Mean or expected value: On the calculator, the product ʟX ʟPX is a new list in which each element is the product of the corresponding elements of the individual lists, i.e. a list containing the values $xp(x)$.

If we now sum this list, sum$($ʟX ʟPX$)$, we obtain the mean value μ or the expected value $E[X]$ of x, 1.8 in this case (figure below).

Variance: Similarly, the variance $V[X]$ is sum$($ʟX^2 ʟPX$) -$ sum$($ʟX ʟPX$)^2$, 1.76 here.

Standard deviation: The standard deviation, the square root of the variance, is 1.33 here.

```
sum( LX LPX)
                    1.8
sum(LX² LPX)-sum(LX LPX)²
                    1.76
√(Ans
             1.326649916
```

18.2.4   The Binomial Distribution

1. Calculating binomial probabilities

If a discrete random variable X has a binomial distribution, $X \sim \text{bin}(n, p)$, the *probability distribution function* (pdf) is given by

$$P(X\!=\!x) \;\; = \;\; {}^nC_x\, p^x\, (1\!-\!p)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

The Binomial Distribution is defined by two parameters, the number of independent trials n and the probability p of success at one trial.

If $X \sim \text{bin}(n, p)$, the calculator command to calculate $P(X\!=\!x)$ is binompdf (n, p, x).

Example: If $X \sim \text{bin}(20, 0.6)$, calculate $P(X\!=\!8)$.

Press $\boxed{\text{distr}}$ $\left(\boxed{\text{2nd}}\ \boxed{\text{vars}} \right)$ and scroll down to *binompdf*.
Press $\boxed{\text{enter}}$ to generate an input menu.

Put in the three arguments, move the cursor to *Paste*, press $\boxed{\text{enter}}$ to paste the command to the Home screen and $\boxed{\text{enter}}$ again to execute it.

```
DISTR DRAW
4↑invT(
5:tpdf(
6:tcdf(
7:χ²pdf(
8:χ²cdf(
9:Fpdf(
0:Fcdf(
A:binompdf(
B↓binomcdf(
```

```
        binompdf
trials:20
p:0.6
x value:8
Paste
```

```
binompdf(20,0.6,8)
        0.0354974396
```

The binomial *cumulative distribution function* (cdf), denoted $F(x)$, is $F(x)\!=\!P(X\!\leqslant\!x)$.

To calculate $P(X\!\leqslant\!x)$, we use the binomial cdf command binomcdf (n, p, x) (DISTR menu).

Example: If $X \sim \text{bin}(20, 0.6)$, find $P(X\!\leqslant\!11)$.

```
binomcdf(20,0.6,11)
        0.4044012768
```

Example: If $X \sim \text{bin}(20, 0.6)$, find $P(X\!\geqslant\!11)$:
$\quad P(X\!\geqslant\!11) = 1 - P(X\!\leqslant\!10)$.

```
1-binomcdf(20,0.6,10)
        0.7553372032
```

Example: If $X \sim \text{bin}(20, 0.6)$, find $P(2 \leqslant X \leqslant 14)$:

$P(2 \leqslant X \leqslant 14) = P(X \leqslant 14) - P(X \leqslant 1)$.

```
binomcdf(20,0.6,14)-binomc
df(20,0.6,1)
                0.8744006864
```

Exercises

1. If $X \sim \text{bin}(8, 0.4)$, find
    (a) $P(X = 4)$
    (b) $P(X \leqslant 1)$
    (c) $P(X \geqslant 7)$
    (d) $P(1 \leqslant X \leqslant 3)$
    (e) $P(X < 6) = P(X \leqslant 5)$

2. If $X \sim \text{bin}(25, 0.2)$, find
    (a) $P(X = 4)$
    (b) $P(X \leqslant 10)$
    (c) $P(X \geqslant 7)$
    (d) $P(5 \leqslant X \leqslant 8)$
    (e) $P(X > 4) = 1 - P(X \leqslant 4)$

Solutions are in Section 18.6.

2. Graphing a Binomial Distribution

Example: If $X \sim \text{bin}(20, 0.6)$, graph the distribution of X for $0 \leqslant X \leqslant 20$.

To do this, we need to store values of x in the X list and the corresponding values of $P(X = x)$ in the PX list, as we did in Section 18.2.3.

A quick way to fill the X list is to move the cursor to the list name, press $\boxed{\text{enter}}$ and type in the command $\text{seq}(X, X, 0, 20)$ (below left); *seq* is in the $\boxed{\text{list}}$ OPS menu.

Press $\boxed{\text{enter}}$ again to fill the list (below right).

Similarly, move the cursor to the header of the PX list, press $\boxed{\text{enter}}$ and insert the *binomcdf* command as shown below left. ʟX comes from the $\boxed{\text{list}}$ NAMES menu.

Press $\boxed{\text{enter}}$ again to fill the list (below right).

Set up the $\boxed{\text{stat plot}}$ as shown below left and press $\boxed{\text{zoom}}$ $\boxed{9}$ (ZoomStat) to plot the graph (below right). Adjust Xscl and Yscl if necessary and press $\boxed{\text{graph}}$ to regraph. $\boxed{\text{trace}}$ has been pressed here and the cursor moved to $X = 12$.
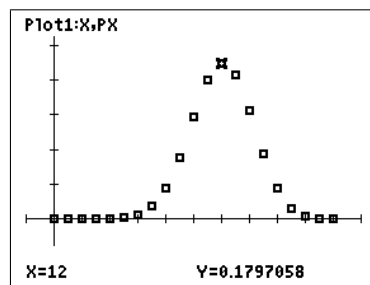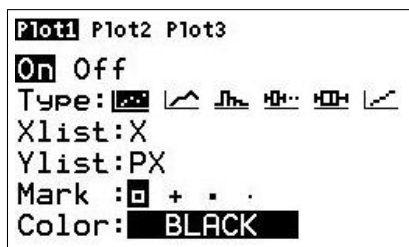
Exercises

1. If $X \sim \text{bin}(10, 0.4)$, display the distribution of X graphically.

2. On the one graph, plot the distribution of: [19]

   $X_1 \sim \text{bin}(10, 0.1)$ use Plot1 with a square marker;
   $X_2 \sim \text{bin}(10, 0.5)$ use Plot2 with a + marker;
   $X_3 \sim \text{bin}(10, 0.9)$ use Plot3 with a dot marker.

   Comment on how changing the value of p changes the shape of the distribution.

Solutions are in Section 18.6.

18.2.5   The Geometric Distribution

1. Calculating geometric probabilities

If a discrete random variable X has a Geometric Distribution, $X \sim \text{geom}(p)$, the *probability distribution function* (pdf)

$$P(X\!=\!x) \;=\; p\,(1\!-\!p)^{x-1} \quad \text{for } x = 1, 2, \ldots.$$

The Geometric Distribution is defined by a single parameter p.

If $X \sim \text{geom}(p)$, the calculator command to calculate $P(X\!=\!x)$ is geometpdf(p, x).

Example: If $X \sim \text{geom}(0.5)$, calculate $P(X\!=\!4)$.

Press $\boxed{\text{distr}}$ ($\boxed{\text{2nd}}$ $\boxed{\text{vars}}$) and go to *geometpdf.*
Press $\boxed{\text{enter}}$ to generate an input menu.[a]

Put in the two arguments, move the cursor to *Paste*, press $\boxed{\text{enter}}$ to paste the command to the Home screen and $\boxed{\text{enter}}$ again to execute it.

---
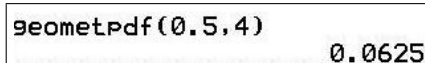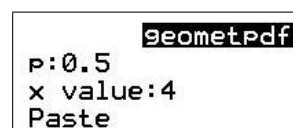
[a]STAT WIZARDS on

---

[19]You might try line graphs, set in $\boxed{\text{stat plot}}$, rather than scatterplots to show the three distributions better.

To calculate $P(X \leqslant x)$, we use the *geometric cdf* command geometcdf(p, x).

Example: If $X \sim \text{geom}(0.5)$, calculate $P(X \leqslant 6)$.

```
geometcdf(0.5,6)
                    0.984375
```

Example: If $X \sim \text{geom}(0.5)$, calculate $P(X \geqslant 3)$:
  $P(X \geqslant 3) = 1 - P(X \leqslant 2)$.

```
1-geometcdf(0.5,2)
                        0.25
```

Example: If $X \sim \text{geom}(0.5)$, find $P(2 \leqslant X \leqslant 5)$:
  $P(2 \leqslant X \leqslant 5) = P(X \leqslant 5) - P(X \leqslant 1)$.

```
geometcdf(0.5,5)-geometcdf
(0.5,1)
                     0.46875
```

Exercises

If $X \sim \text{geom}(0.2)$, find
(a) $P(X = 5)$ **(b)** $P(X \leqslant 8)$ **(c)** $P(X \geqslant 7)$ **(d)** $P(3 \leqslant X \leqslant 9)$.

Solutions are in Section 18.6.

2. Graphing a Geometric Distribution

Example: If $X \sim \text{geom}(0.2)$, graph the distribution of X for $x = 1, 2, \ldots, 10$.

To do this, we need to store values of x in the X list and the corresponding values of $P(X = x)$ in the PX list, as we did in Section 18.2.4.

Here, to fill the X list, execute the command seq$(X, X, 1, 10)$.

Move the cursor to the header of the PX list, press enter and insert the *geometpdf* command as shown. Press enter again to fill the list.

Set up the stat plot as shown above right and press zoom ZoomStat to plot the graph below.

trace has been pressed and the cursor moved to X = 4.
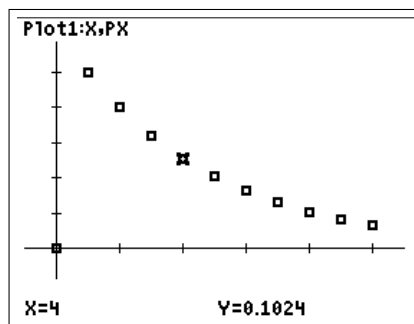
Exercises

  1. If $X \sim \mathrm{geom}(0.4)$, display the distribution of X graphically for $1 \leqslant x \leqslant 8$.

  2. On the one graph, plot the distribution of:

      $X_1 \sim \mathrm{geom}(0.1)$ use Plot1 with a square marker;

      $X_2 \sim \mathrm{geom}(0.5)$ use Plot2 with a + marker;

    Comment on how changing the value of p changes the shape of the distribution.

Solutions are in Section 18.6.

18.2.6   The Hypergeometric Distribution

1. Calculating hypergeometric probabilities

If a discrete random variable X has a Hypergeometric Distribution, $X \sim \mathrm{hypg}(N, k, n)$, the *probability distribution function* (pdf)

$$P(X = x) \;\; = \;\; \frac{{}^{k}C_x \, {}^{N-k}C_{n-x}}{{}^{N}C_n} \quad \text{with } 0 \leqslant x \leqslant n \text{ and } x \leqslant k \quad \text{(see Section 18.2.1)}.$$

The Hypergeometric Distribution is defined by three parameters, the population size N, the number of successes in the population k and the sample size n.

There is no command for the pdf or cdf of the Hypergeometric Distribution but the calculations can be done both numerically and graphically by putting the pdf in Y1.

M has been used for n.

```
∎\Y₁◻(K nCr X)((N−K) nCr (
M−X))/(N nCr M)
```

Example: If $X \sim \mathrm{hypg}(50, 10, 8)$, calculate $P(X = 2)$.

Store values of N, k and M in the relevant memories, then evaluate Y1(2).

```
50→N
                          50
10→M
                          10
8→K
                           8
Y₁(2)
                 0.3217246579
```

To calculate the cdf, $P(X \leqslant x)$, we have to add up the probabilities from 0 up to and including x.

Example: If $X \sim \mathrm{hypg}(50, 10, 8)$, calculate $P(X \leqslant 1)$.

$P(X \leqslant 1) = P(X = 0) + P(X = 1)$.

Following on from the previous example, calculate Y1(0) + Y1(1).

```
Y₁(0)+Y₁(1)
                 0.4905024348
```

Example: If $X \sim \text{hypg}(50, 10, 8)$, calculate $P(X \geqslant 2)$.

$P(X \geqslant 2) = 1 - P(X \leqslant 1) = 1 - 0.4905 = 0.509$ from the previous example.

Exercises

If $X \sim \text{hypg}(100, 15, 5)$, find

(a) $P(X=3)$ **(b)** $P(X \leqslant 3)$ **(c)** $P(X>3)$ **(d)** $P(1 \leqslant X \leqslant 2)$.

Solutions are in Section 18.6.

2. Graphing a Hypergeometric Distribution

Example: If $X \sim \text{hypg}(50, 10, 8)$, graph the distribution of X for $x = 0, 2, \ldots, 8$.

To do this, we need to store values of x in the X list and the corresponding values of $P(X=x)$ in the PX list, as we did in Section 18.2.4.

Here, to fill the X list, execute the command $\text{seq}(X, X, 0, 8)$.

Store values of N, k (K) and n (M) in the relevant memories. Move the cursor to the header of the PX list, press enter and insert the command as shown below left.
Press enter again to fill the list.

Set up the stat plot as shown above right and press zoom ZoomStat to plot the graph. trace has been pressed in the figure below and the cursor moved to X = 4.

Exercises

1. If $X \sim \text{hypg}(100, 15, 8)$, display the distribution of X graphically for $0 \leqslant x \leqslant 8$.

2. On the one graph, plot the distribution of:

$\qquad X \sim \text{hypg}(100, 15, 8)$ use Plot1 with a square marker;

$\qquad X \sim \text{hypg}(100, 30, 8)$ use Plot2 with a + marker;

Comment on how changing the value of k changes the shape of the distribution.

Solutions are in Section 18.6.

18.2.7   The Poisson Distribution

1. Calculating Poisson probabilities

If a discrete random variable X has a Poisson Distribution, $X \sim \text{pois}(\mu)$, the *probability distribution function* (pdf)

$$P(X\!=\!x) \;\; = \;\; \frac{\mu^x}{x!}\, e^{-\mu} \quad \text{for } x = 0, 1, 2, \ldots.$$

The Poisson Distribution is defined by a single parameter μ.

If $X \sim \text{pois}(\mu)$, the calculator command to calculate $P(X\!=\!x)$ is poissonpdf(μ, x).

```
DISTR DRAW
9↑Fpdf(
0:Fcdf(
A:binompdf(
B:binomcdf(
C:invBinom(
D:poissonpdf(
E:poissoncdf(
F:geometpdf(
G:geometcdf(
```

Example: If $X \sim \text{pois}(3)$, calculate $P(X\!=\!5)$.

Press `distr` (`2nd` `vars`), scroll down to *poissonpdf* and press `enter` to generate an input menu.

Put in the two arguments, move the cursor to *Paste*, press `enter` to paste the command to the Home screen and `enter` again to execute it.

```
          poissonpdf
μ:3
x value:5
Paste
```

```
poissonpdf(3,5)
          0.1008188134
```

To calculate $P(X\!\leqslant\!x)$, we use the *Poisson cdf* command poissoncdf(μ, x).

Example: If $X \sim \text{pois}(3)$, calculate $P(X\!\leqslant\!6)$.

```
poissoncdf(3,6)
          0.9664914647
```

Example: If $X \sim \text{pois}(3)$, calculate $P(X\!\geqslant\!2)$:
   $P(X\!\geqslant\!2) = 1 - P(X\!\leqslant\!1)$.

```
1-poissoncdf(3,1)
          0.8008517265
```

Example: If $X \sim \text{pois}(3)$, calculate $P(3\!\leqslant\!X\!\leqslant\!9)$:
   $P(3\!\leqslant\!X\!\leqslant\!9) = P(X\!\leqslant\!9) - P(X\!\leqslant\!2)$.

```
poissoncdf(3,9)-poissoncdf
(3,2)
          0.5757074307
```

Exercises

If $X \sim \text{pois}(0.5)$, find

(a) $P(X\!=\!0)$ **(c)** $P(X\!\geqslant\!1)$

(b) $P(X\!\leqslant\!3)$ **(d)** $P(1\!\leqslant\!X\!\leqslant\!4)$.

Solutions are in Section 18.6.

2. Graphing a Poisson Distribution

Example: If $X \sim \text{pois}(4)$, graph the distribution of X for $x = 0, 1, \ldots, 10$.

To do this, we need to store values of x in the X list and the corresponding values of $P(X = x)$ in the PX list (Section 18.2.4).

Here, to fill the X list, execute the command $\text{seq}(X, X, 0, 10)$.

Move the cursor to the header of the PX list, press enter and insert the *poissonpdf* command as shown. Press enter again to fill the list.

Set up the stat plot as shown and press zoom 9 (ZoomStat) to plot the graph. trace has been pressed in the figure and the cursor moved to X = 4.
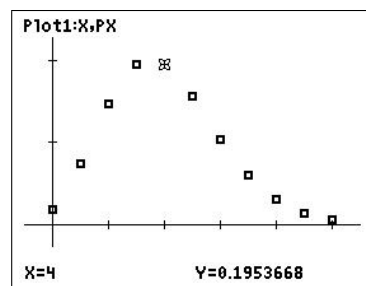
Exercises

1. If $X \sim \text{pois}(1.5)$, display the distribution of X graphically for $0 \leqslant x \leqslant 6$.

2. On the one graph, for $0 \leqslant x \leqslant 10$, plot the distribution of:

$X_1 \sim \text{pois}(2)$ use Plot1 with a square marker;

$X_2 \sim \text{pois}(5)$ use Plot2 with a + marker;

$X_2 \sim \text{pois}(9)$ use Plot2 with a dot marker.

Comment on how changing the value of λ changes the shape of the distribution.

Solutions are in Section 18.6.

18.2.8   Discrete-distribution spinner

A spinner,[20] here a program SPDISPDS/SPDSCDCE,[21] allows you to vary the parameters on screen of each of the four types of discrete probability distributions (pdf and cdf) here, by pressing a key, and see the distribution replotted immediately. The program contains the necessary information (in HELP) on how to vary the parameters.

---

[20]From a feature in an earlier version of Excel used by Ed Staples and Margie Smith in SpinnaMaths to explore the effects of varying parameters in a variety of applications; sadly, SpinnaMaths is no longer available because of subsequent 'upgrades' to Excel.

[21]available at *canberramaths.org.au* under *Resources*

18.3   Continuous probability distributions

For continuous random variables, we do not specify the probability for every value in the range. In fact, $P(X=a)=0$ for any value of a but we can work out the probability that X lies in a range of values. For example, the probability of finding a person whose height is exactly 2 metres is zero, but the probability of finding someone with a height between 2 and 2.01 metres might be 0.05.

For a continous random variable X, we define a *probability density function* (pdf) $f(x)$.

The continuous analogue of a summation is an integral, so the *cumulative distribution function* (cdf), $F(x)=P(X\leqslant x)$ is naturally defined as

$$F(x) \;=\; \int_{-\infty}^{x} f(u)\,du.$$

The probability that X lies in an interval is then the area under the (positive) pdf curve over that interval.

18.3.1   The Normal Distribution

A continuous random variable X with a Normal Distribution has an exponential pdf

$$f(z) \;=\; \frac{1}{\sigma\sqrt{2\pi}}\,\exp\left(\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right) \qquad -\infty < z < \infty.$$

The Normal Distribution is defined by two parameters, the mean μ and variance σ^2; we write $X\sim\mathrm{N}(\mu,\sigma^2)$.

1. Calculations with the Normal Distribution

The basic (cdf) calculation is

$$P(a\leqslant X\leqslant b) \;=\; \int_a^b f(z)\,dz.$$

This is achieved by the calculator command (with appropriate numbers inserted)

$$\text{normalcdf}\,(a,b,\mu,\sigma).$$

The *normalcdf* command is found in the $\boxed{\text{distr}}$ menu.

If μ and σ are omitted, the standard values $\mu=0$ and $\sigma=1$ are assumed. *It's best to always put them in.*

You can also use commands in the $\boxed{\text{distr}}$ DRAW menu to show geometrically the calculation you have done. Both are shown below.

If the lower limit a is $-\infty$, set $a=-\mathrm{E}99=-1\times10^{99}$, the smallest number on the calculator.[22] Similarly, if the upper limit b is ∞, set $b=\mathrm{E}99=1\times10^{99}$, the largest number on the calculator.

Always think geometrically here: What area under the normal-distribution curve (pdf) am I calculating? The graphical representation below should help. You need to do a certain number of examples by hand first to understand this.

---

[22]The symbol E is $\boxed{\text{EE}}$ (on the comma key).

Example: Find $P(Z \leqslant 0.7) = P(-\infty < Z \leqslant 0.7)$,

where $Z \sim \mathrm{N}(0,1)$, the Standard Normal Distribution (STAT WIZARDS on).

```
DISTR DRAW
1:normalpdf(
2:normalcdf(
3:invNorm(
4:invT(
5:tpdf(
6:tcdf(
7:χ²pdf(
8:χ²cdf(
9↓Fpdf(
```

```
      normalcdf
lower:-1E99
upper:0.7
μ:0
σ:1
Paste
```

```
normalcdf(-E99,0.7,0,1)
             0.7580364218
```

Example: Find $P(168 \leqslant X \leqslant 196)$, where $X \sim \mathrm{N}(180, 8^2)$.

Normally you would rewrite this problem in terms of the standard Normal Distribution, then use tables to find the answer. However, because we can specify the mean and standard deviation in the relevant calculator command, we can do this calculation in one go.

Learn how to do both.

```
normalcdf(168,196,180,8)
             0.9104427093
```

Example: Find $P(X \geqslant 13.5)$, where $X \sim \mathrm{N}(10, 2^2)$.

Two ways to do this.

(a) $P(X \geqslant 13.5) = P(13.5 \leqslant X < \infty)$ (top line below).

```
normalcdf(13.5,E99,10,2)
             0.0400591135
1-normalcdf(-E99,13.5,10,2
)
             0.0400591135
```

(b) $P(X \geqslant 13.5) = 1 - P(X \leqslant 13.5)$ (second line above).

Example: *An inverse-normal problem*

Find c such that $P(Z \leqslant c) = 0.975$, where $Z \sim \mathrm{N}(0,1)$

Here we could generate a table of normal-distribution cdf values using ⎡table⎤ on the calculator and search for the appropriate value (figures below). This requires some experimentation with the values in ⎡tblset⎤ (⎡2nd⎤ ⎡window⎤).

X	Y₁
1.9	0.9713
1.91	0.9719
1.92	0.9726
1.93	0.9732
1.94	0.9738
1.95	0.9744
1.96	0.975
1.97	0.9756
1.98	0.9761
1.99	0.9767
2	0.9772

X=1.96

```
TABLE SETUP
 TblStart=1.9
 ⌂Tbl=0.01
Indpnt: Auto Ask
Depend: Auto Ask
```

```
■\Y₁∎normalcdf(-E99,X,0,1)
```

However, there is an easier way. The command invNorm(M, μ, σ) (in the $\boxed{\text{distr}}$ menu) finds c such that $P(Z \leqslant c) = M$, the inverse of the normal-distribution function.

There are actually three options: the one above, find c such that $P(Z \leqslant c) = M$ (LEFT); find c such that $P(-c \leqslant Z \leqslant c) = M$ (CENTER); and find c such that $P(c \leqslant Z) = M$ (RIGHT).

Selecting *invNorm* from the $\boxed{\text{distr}}$ menu gives an input menu (below left). Entering area $M = 0.975$, selecting LEFT, CENTER and RIGHT successively and executing the commands gives the screen below right.

```
      invNorm
area:0.975
μ:0
σ:1
Tail: LEFT CENTER RIGHT
Paste
```

```
invNorm(0.975,0,1,LEFT)
                1.959963986
invNorm(0.975,0,1,CENTER)
{-2.241402725 2.241402725}
invNorm(0.975,0,1,RIGHT)
               -1.959963986
```

Another way to find c such that $P(Z \leqslant c) = M$ is to solve the equation (in calculator terms)
$$\text{normalcdf}(-\text{E}99, X, 0, 1) - 0.975 = 0$$

for X. This can be done using the Solver (last item in the $\boxed{\text{math}}$ menu) but this requires a reasonable guess, especially if there is more than one solution (there isn't here).

Much better is to solve the equation by graphing the left-hand side and using *zero* in the $\boxed{\text{calc}}$ menu, as in the figure.
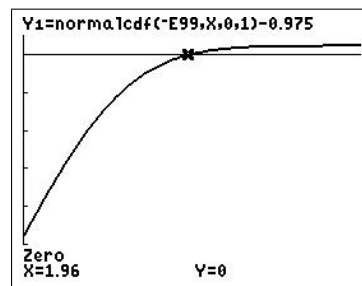
```
Y₁=normalcdf(-E99,X,0,1)-0.975




Zero
X=1.96            Y=0
```

Exercises

1. If $Z \sim \text{N}(0, 1)$, find
   (a) $P(-1.5 \leqslant Z \leqslant 2)$ (c) $P(Z \geqslant 1.6)$
   (b) $P(Z \leqslant -0.8)$ (d) $P(-1 \leqslant Z \leqslant 1)$.

2. If $X \sim \text{N}(100, 20)$, find
   (a) $P(88 \leqslant X \leqslant 112)$ (c) $P(X \leqslant 107)$
   (b) $P(100 \leqslant X \leqslant 105)$ (d) $P(X \geqslant 97)$.

3. If $Z \sim \text{N}(0, 1)$, find the value of z to 2 decimal places if
   (a) $P(Z \leqslant z) = 0.8413$ (c) $P(Z \geqslant z) = 0.9772$
   (b) $P(Z \leqslant z) = 0.95$ (d) $P(|Z| \leqslant 0.95)$.
   You'll need to think geometrically here.

4. If $X \sim \text{N}(10, 4)$, find the value of x to 1 decimal place if
   (a) $P(X \leqslant x) = 0.05$
   (b) $P(X \geqslant x) = 0.90$ (c) $P(X \geqslant x) = 0.025$
   Here too.

Solutions are in Section 18.6.

2. Graphical representation

Example: Graph $P(Z \leqslant 0.7) = P(-\infty < Z \leqslant 0.7)$, where $Z \sim N(0, 1)$.

```
WINDOW
 Xmin=-3
 Xmax=3
 Xscl=1
 Ymin=0
 Ymax=0.4
 Yscl=0.1
```

```
        ShadeNorm
lower: -1E99
upper:0.7
μ:0
σ:1
Color:  BLACK
Draw
```

```
ShadeNorm(-E99,0.7,0,1,BLA
CK)
```
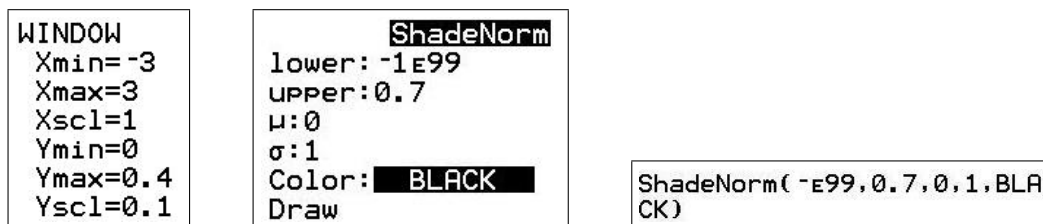
ShadeNorm is in the $\boxed{\text{distr}}$ DRAW menu. *BLACK* (CE only) is in the $\boxed{\text{vars}}$ COLOR menu. If you omit it, the shading comes out blue, the default.


```
Area=0.75804
low=-1E99         up=0.7
```
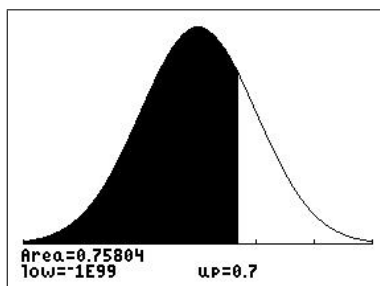
Exercises

If $X \sim N(60, 25)$, display each of the following probabilities as an area under the normal pdf curve: **(a)** $P(54 \leqslant X \leqslant 58)$;   **(b)** $P(X \leqslant 70)$;   **(c)** $P(X \geqslant 62.5)$

Solutions are in Section 18.6.

3. Using the Normal Distribution to approximate a Binomial Distribution

When n is large, the Binomial Distribution $\text{bin}(n, p)$ can be approximated by the Normal Distribution $N\big(np, \, np(1-p)\big)$. This can be seen by plotting the two distributions on the one graph.

Example: Compare $\text{bin}(5, 0.3)$ with $N(1.5, 1.05)$.

Construct a scatterplot of $\text{bin}(5, 0.3)$
(Section 18.2.4).


```
Plot1:X,PX




X=4          Y=0.02835
```
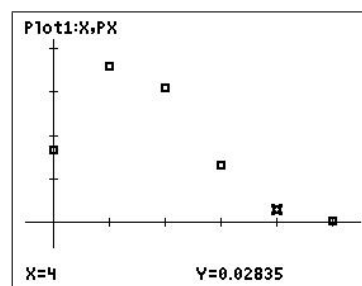
Put $N(1.5, 1.05)$ into Y₁ as shown in the figure.
Note that $\sigma = \sqrt{1.05}$.

```
∎\Y₁=normalpdf(X,1.5,√(1.0
5))
```

Press ⌷graph⌷ to display both graphs. A reason-
able but not good approximation with $n = 5$.
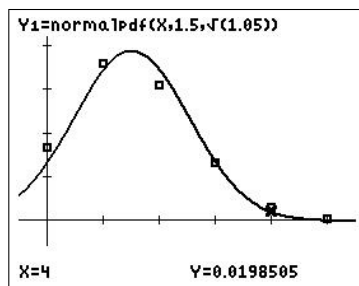
Example: Compare bin$(25, 0.3)$ with N$(7.5, 5.25)$.

Construct a scatterplot of bin$(25, 0.3)$
(Section 18.2.4).
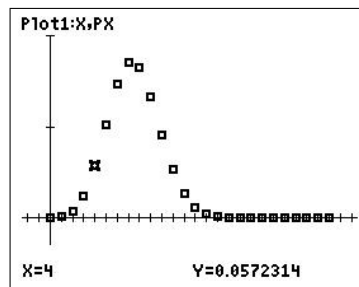
Put N$(7.5, 5.25)$ into Y₁ as shown in the figure.
Note that $\sigma = \sqrt{5.25}$.

Press ⌷graph⌷ to display both graphs.
A much better approximation with $n = 25$.
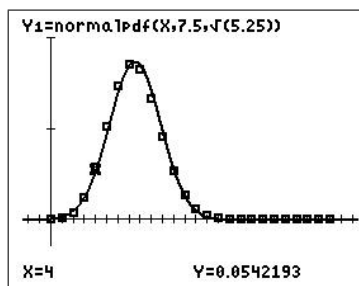
Exercise

Graphically compare the normal approximation to the binomial for $p = 0.5$ when $n = 5$ and
$n = 20$.

Solutions are in Section 18.6.

18.3.2 The Exponential Distribution

A continuous random variable X with an Exponential Distribution has pdf

$$p(x) = \lambda e^{-\lambda x} \qquad 0 < x < \infty.$$

The Exponential Distribution is defined by one parameter, λ. We write $X \sim \mathrm{Exp}(\lambda)$.

The cdf $P(a \leqslant X \leqslant b)$ is given by

$$P(a \leqslant X \leqslant b) = \int_a^b \lambda e^{-\lambda x}\, dx = e^{-\lambda a} - e^{-\lambda b}.$$

Example: If $X \sim \mathrm{Exp}(0.5)$, find $P(1 \leqslant X \leqslant 4)$.

$P(1 \leqslant X \leqslant 4) = e^{-0.5 \times 1} - e^{-0.5 \times 4} = e^{-0.5} - e^{-2} = 0.471$ to 3 significant digits.

To obtain a visual representation of this calculation, use $\int$f(x)dx in the $\boxed{\text{calc}}$ menu.

First set a suitable window and graph Y1. Then, select the command and specify the integration limits. This shades the area between the function $Y = 0$ and Y1 for X between 1 and 4; it also gives you the integral.

```
                    Y1=0.5e^(-0.5X)                        CALCULATE
WINDOW                                                     1:value
  Xmin=0                                                   2:zero
  Xmax=5                                                   3:minimum
  Xscl=1                                                   4:maximum
  Ymin=0                                                   5:intersect
  Ymax=0.6                                                 6:dy/dx
  Yscl=0.1    X=1              Y=0.3032653                 7:∫f(x)dx    ∫f(x)dx=0.4711954
                                                                       [1,4]
```
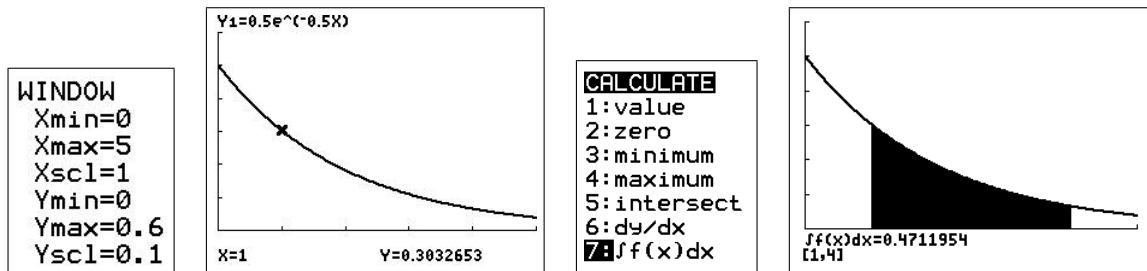
Exercises

If $X \sim \mathrm{Exp}(0.2)$, find to 4 decimal places

(a) $P(3 \leqslant X \leqslant 5)$ **(b)** $P(X \leqslant 5)$ **(c)** $P(X \geqslant 2)$

Solutions are in Section 18.6.

Continuous-distribution spinner

A spinner,[23] here a program SPCTSPDS/SPCTSDCE,[24] allows you to vary the parameters on screen of each of the two types of continuous probability distributions (both their pdf and cdf) here (more may be added), by pressing a key, and see the distribution replotted immediately.

The program contains the necessary information (in HELP) on how to vary the parameters.

---

[23]From a feature in an earlier version of Excel used by Ed Staples and Margie Smith in SpinnaMaths to explore the effects of varying parameters in a variety of applications; sadly, SpinnaMaths is no longer available because of subsequent 'upgrades' of Excel.

[24]available at *canberramaths.org.au* under *Resources*

18.4 Statistical inference

18.4.1 Hypothesis testing and confidence intervals for means

1. Single-sample mean drawn from a normal population: σ known

Testing the significance of the difference of a sample mean from a hypothesised population mean: Z test

Example: (JB) A sample of 25 workers in a factory spent, on average, 2.2 minutes to assemble the co-processor of a personal computer. Assuming that the assembly times are normally distributed with a standard deviation $\sigma = 0.4$ seconds, can we conclude that the average assembly time exceeds 2 minutes? Test at the 5% level of significance.

Steps

H$_0$: $\mu = 2$.

H$_1$: $\mu > 2$.

Significance level: 5%.

Test statistic: $Z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$.

Given: $\mu = 2$; $\sigma = 0.4$; $\bar{x} = 2.2$; and $n = 25$.

Then: $z = 2.5$ the observed value of Z, from the calculator (see below); and

$p = P(Z \geqslant 2.5) = 0.0062$ from the calculator.

As $p < 0.05$, reject H$_0$ and conclude that, on average, assembly times exceed 2 minutes.

Calculator operations

Go to the $\boxed{\text{stat}}$ TESTS menu and select Z-Test (below left). Press $\boxed{\text{enter}}$.

```
EDIT CALC TESTS
1:Z-Test…
2:T-Test…
3:2-SampZTest…
4:2-SampTTest…
5:1-PropZTest…
6:2-PropZTest…
7:ZInterval…
8:TInterval…
9↓2-SampZInt…
```

```
                Z-Test
Inpt:Data Stats
μ₀:2
σ:0.4
x̄:2.2
n:25
μ:≠μ₀  <μ₀  >μ₀
Color:  BLACK
Calculate Draw
```

Complete the menu items as shown (above right): *Stats* because we are given the sample mean rather than the raw data; $>\mu_0$ because H$_1$: $\mu > 2$. Put the cursor on *Calculate* and press $\boxed{\text{enter}}$ to obtain the screen below left.

If you select *Draw* instead of *Calculate*, the result appears graphically (below right): shading at the right-hand end of the curve.

```
            Z-Test
μ>2
z=2.5
p=0.0062096799
x̄=2.2
n=25
```

```
Z-Test
z=2.5          p=0.0062
```
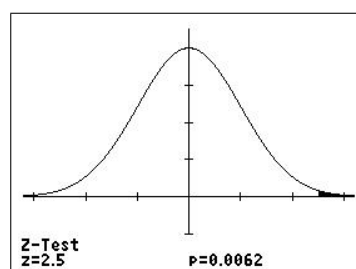
Estimate the population mean from the sample mean

Use a 95% confidence interval to form an interval estimate for the population mean μ.

Steps

Given: $\sigma = 0.4$; $\bar{x} = 2.2$; $n = 25$; and C(onfidence)-Level 0.95.

From the calculator (see below), we are 95% confident that the interval $2.04 < \mu < 2.36$ captures the population mean.

Calculator operations

Go to the $\boxed{\text{stat}}$ TESTS menu and select ZInterval (below left). Press $\boxed{\text{enter}}$.

```
EDIT CALC TESTS
1:Z-Test…
2:T-Test…
3:2-SampZTest…
4:2-SampTTest…
5:1-PropZTest…
6:2-PropZTest…
7:ZInterval…
8:TInterval…
9↓2-SampZInt…
```

```
          ZInterval
Inpt:Data Stats
σ:0.4
x̄:2.2
n:25
C-Level:0.95
Calculate
```

```
            ZInterval
(2.0432,2.3568)
x̄=2.2
n=25
```

Complete the menu items as shown (above centre) if necessary.

Put the cursor on *Calculate* and press $\boxed{\text{enter}}$ to obtain the screen above right giving the confidence interval.

2. Single-sample mean drawn from a normal population: σ unknown

Testing the significance of the difference of a sample mean from a hypothesised population mean: T test

Example: (JB) A study was conducted to see whether the ascorbic-acid (Vitamin C) content (mg/100g) of a certain brand of frozen vegetables was less than the recommended level of 18.6 mg/100g. Eight packets of frozen vegetables were randomly selected from the production line and were found to have the following ascorbic-acid levels:

14.3   19.1   16.3   19.2   15.8   16.2   18.7   14.7.

Assuming that assembly times are normally distributed, can we conclude that the average ascorbic-acid level of this brand of vegetables is less than 18.6 mg/100g?

Steps

> H_0: $\mu = 18.6$.
>
> H_1: $\mu < 18.6$.
>
> Significance level: 5%.
>
> Test statistic: $T = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$,   with $n-1$ degrees of freedom.
>
> Then, from the calculator (see below), $\bar{x} = 16.79$; $s = 1.961$; and $n = 8$, giving
>
> $t = -2.614$,   the observed value of T with df $= n - 1 = 7$, and
>
> $p = P(T \leqslant 2.614) = 0.017$.

As $p < 0.05$, reject H_0 and conclude that the mean ascorbic-acid level of this brand of vegetables is less than 18.6 mg/100g.

Calculator operations

Enter the data into a list called ACID (below left).

Go to the $\boxed{\text{stat}}$ TESTS menu and select T-Test (above right). Press $\boxed{\text{enter}}$.

Complete the menu items (below left): *Data* because we are given the raw data; $<\mu_0$ because H_1: $\mu < 2$. Put the cursor on *Calculate* and press $\boxed{\text{enter}}$ to obtain the screen below center. If you select *Draw* instead of *Calculate*, the result appears graphically (below right).

Estimate the population mean from the sample mean

Example: Use a 95% confidence interval to form an interval estimate for the mean ascorbic-acid level μ of the brand of vegetables.

Steps

Given: C(onfidence)-Level 0.95.

From the calculator (see below), we are 95% confident that the interval $15.1 < \mu < 18.4$ captures the population mean.

Calculator operations

Go to the stat TESTS menu and select TInterval (below left). Press enter.

```
EDIT CALC TESTS
1:Z-Test…
2:T-Test…
3:2-SampZTest…
4:2-SampTTest…
5:1-PropZTest…
6:2-PropZTest…
7:ZInterval…
8:TInterval…
9↓2-SampZInt…
```
```
        TInterval
Inpt:Data Stats
List:ACID
Freq:1
C-Level:0.95
Calculate
```
```
        TInterval
(15.148,18.427)
x̄=16.7875
Sx=1.961367963
n=8
```

Complete the menu items as shown (above centre) if necessary.

Put the cursor on *Calculate* and press enter to obtain the screen above right giving the confidence interval.

18.4.2 Hypothesis testing and confidence intervals for population proportion

Single-sample proportion

1. Significance test using the binomial distribution (exact test)

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 15 components contains 4 faulty ones. Does this challenge the manufacturer's claim? Test at the 5% level of significance.

Steps

H_0: $\pi = 0.10$.

H_1: $\pi > 0.10$.

Significance level: 5%.

Let X be the number of faulty components.

Then, under the null hypothesis, $X \sim \text{bin}(15, 0.10)$.

Then, from the calculator (see below), $p = P(X \geqslant 4) = 1 - P(X \leqslant 3) = 0.056$.

As $p > 0.05$, do not reject H_0. There is not sufficient evidence that the percentage of faulty components exceeds 10%.

Calculator operation:
```
1-binomcdf(15,0.1,3)
          0.0555556298
```

2. Significance test using the normal approximation to the binomial distribution:
Z test with no continuity correction

For n large, $n\pi > 10$ and $n(1-\pi) > 10$, the binomial distribution $\text{bin}(n, \pi)$ can be approximated by a normal distribution with $\mu = n\pi$ and $\sigma^2 = n\pi(1-\pi)$ (Section 18.3.1).

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 120 components contains 23 faulty ones. Does this challenge the manufacturer's claim? Test at the 5% level of significance.

Steps

H_0: $\pi = 0.10$.

H_1: $\pi > 0.10$.

Significance level: 5%.

Given: $x = 23$; $n = 120$.

Test statistic: $Z = \dfrac{x - n\pi}{s\sqrt{n\pi(1-\pi)}}$.

Then, from the calculator (see below), the observed value of Z, $z = 3.347$, and

$p = P(Z > 3.347) = 0.0004$ to 4 decimal places.

As $p < 0.05$, reject H_0, and conclude that the percentage of faulty components exceeds 10%.

Calculator operations

Go to the ⟨stat⟩ TESTS menu and select 1-PropZTest (below left). Press ⟨enter⟩.

Fill in the resulting screen (below centre), put the cursor on *Calculate* and press ⟨enter⟩ to obtain the screen below right.

```
EDIT CALC TESTS
1:Z-Test…
2:T-Test…
3:2-SampZTest…
4:2-SampTTest…
5▮1-PropZTest…
6:2-PropZTest…
7:ZInterval…
8:TInterval…
9↓2-SampZInt…
```

```
      1-PropZTest
P₀:0.1
x:23
n:120
Prop:≠P₀ <P₀ >P₀
Color:▮ BLACK ▮
Calculate Draw
```

```
        1-PropZTest
Prop>0.1
z=3.347193407
p=4.082264262ε⁻4
p̂=0.1916666667
n=120
```

*3. Significance test using the normal approximation to the binomial distribution:
Z test with continuity correction*

For n large, $n\pi > 10$ and $n(1-\pi) > 10$, the binomial distribution can be approximated by a normal distribution with $\mu = n\pi$ and $\sigma^2 = n\pi(1-\pi)$ (Section 18.3.1). The approximation can be improved by using the continuity correction.

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 120 components contains 23 faulty ones. Does this challenge the manufacturer's claim? Test at the 5% level of significance.

Steps

H_0: $\pi = 0.10$.

H_1: $\pi > 0.10$.

Significance level: 5%.

Given: $x = 23$; $n = 120$.

Test statistic: $Z = \dfrac{x - n\pi}{\sqrt{n\pi(1-\pi)}}$.

Given $x = 22.5$ (taking into account the continuity correction), $\pi = 0.10$ and $n = 120$, we find from the calculator (see below) that the observed value of Z,

$$z = \frac{22.5 - 120 \times 0.1}{\sqrt{120 \times 0.1(1-0.1)}} = 3.195, \quad \text{and}$$

$p = P(Z > 3.195) = 1 - P(Z \leqslant 3.195) = 0.0007$ to 4 decimal places.

As $p < 0.05$, reject H_0, and conclude that the percentage of faulty components exceeds 10%.

Calculator operations

```
(22.5-120*0.1)/√(120*0.1(1
-0.1))
.............................3.195048252
1-normalcdf(-E99,3.1950,0,
1)
                6.992185905E-4
```

4. Estimating the population proportion from a sample proportion: Normal approximation

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 120 components contains 23 faulty ones. Use a 95% confidence interval to form an interval estimate for the true proportion of components that have manufacturing faults.

Steps

> Given: C(onfidence)-Level 0.95.

From the calculator (see below), we are 95% confident that the interval $0.12 < \pi < 0.26$ captures the true population proportion.

Calculator operations

Go to the ⎡stat⎤ TESTS menu and select 1-PropZInt (below left). Press ⎡enter⎤.

```
EDIT CALC TESTS
4↑2-SampTTest…
5:1-PropZTest…
6:2-PropZTest…
7:ZInterval…
8:TInterval…
9:2-SampZInt…
0:2-SampTInt…
A:1-PropZInt…
B↓2-PropZInt…
```

```
      1-PropZInt
x:23
n:120
C-Level:0.95
Calculate
```

```
         1-PropZInt
(0.12124,0.26209)
p̂=0.1916666667
n=120
```

Complete the menu items as shown (above centre) if necessary.

Put the cursor on *Calculate* and press ⎡enter⎤ to obtain the screen above right giving the confidence interval.

18.5   Probability and gambling schemes

These are programs I wrote for a first-year university course on Probability taught against a background of the probabilities involved in various modes of gambling.[25] The programs are available at *canberramaths.org.au* under *Resources*.

18.5.1   Throwing dice

Throwing dice is at the heart of many games and methods of gambling. The DICE/DICECE program simulates rolling any number of six-sided dice.

```
       ROLLING SIX-SIDED DICE

   PRESS [ENTER] FOR ANOTHER
   ROLL, [ON] [1] TO STOP

   HOW MANY DICE? 8
```

```
HOW MANY DICE? 8
          {6 1 2 4 5 4 1 6}
          {1 2 1 5 4 5 1 2}
          {5 2 5 3 6 1 4 3}
          {5 2 6 6 6 1 1 1}
          {1 5 6 4 2 5 4 2}
          {1 2 2 5 5 2 5 1}
          {3 3 2 5 2 6 4 4}
```

You can also do this in the ProbSim app, and store the accumulated data for later analysis.

18.5.2   Random walks

In the WALK/WALKCE program, two players, A and B, start with a specified bank each. You then specify the probability that A wins at each go. If A wins a go, they take $1 from B's bank and add it to theirs; if A loses, B receives $1 from A.

The program then runs until either A or B has no money left.

With graphics, the amount that each player has at each go is plotted so you can see how the game is progressing (and cheer on your favourite).

```
        RANDOM WALK

   STARTING BANKS
   A: 50
   B: 50
   P(A WINS): 0.5
```
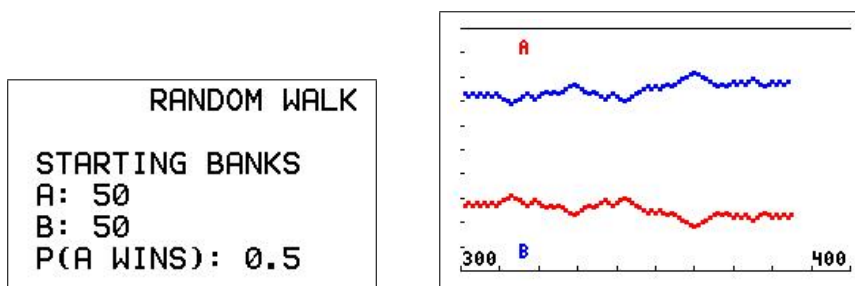

For a faster simulation, run the program without the graphics.

---

[25]Students were not required to demonstrate their understanding of the course by making money at their chosen form of gambling. In fact, it soon became obvious during the course that gambling is not a way to make money; it's an often costly entertainment.

18.5.3   Basic betting

In the WINBET/WINBETCE program, you specify the amount won for a $1 bet, the probability of winning and the number of trials. The program then simulates the trials, tells you how much you won (+) or lost (−), and shows the mean win per trial $\bar{x}$ and the variance σ_x^2.

```
        WIN PROGRAM
   AMOUNT WON FOR A
     1 DOLLAR BET: 2
   PROBABILITY OF
     WINNING: 0.51
   NO OF TRIALS (≤999): 100
   TOTAL WON/LOST: 80
   I=100

                       [ENTER]
```

```
   AMOUNT WON FOR A
     1 DOLLAR BET: 2
   PROBABILITY OF
     WINNING: 0.51
   NO OF TRIALS (≤999): 100
   TOTAL WON/LOST: 80
   x̄: 0.8
   (σx)²: 2.16
```

18.5.4   Gambling systems

The GAMBLSYS/GBLSYSCE program simulates five different gambling systems.

```
      SYSTEM BETTING


   FIRST BET 1 DOLLAR
   STARTING BANK: 100
   PROBABILITY OF
     A WIN: 0.4929
   NUMBER OF BETS: 100
```
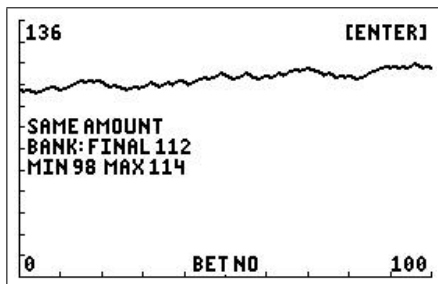
```
   █SYSTEM█
   1:BET SAME AMOUNT
   2:MARTINGALE
   3:REVERSE MARTINGALE
   4:LABOUCHERE
   5:REVERSE LABOUCHERE
   6:CHANGE PARAMETERS
   7:QUIT
```
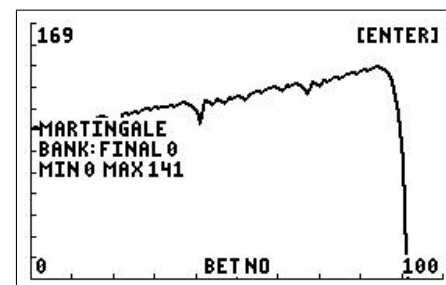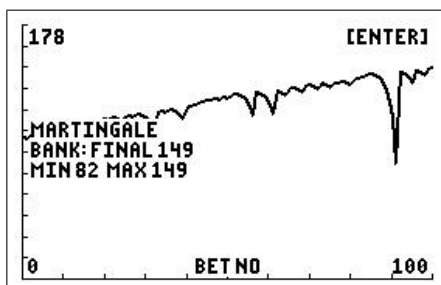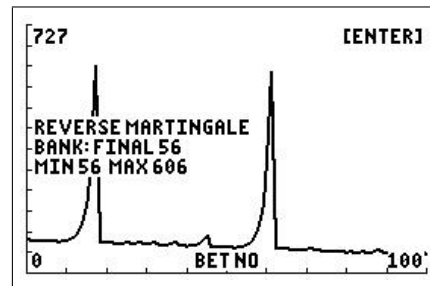
Two runs of each system are shown below.

(a) **bet the same amount each time**.
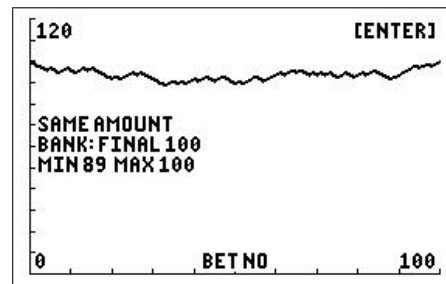
(b) **Martingale:** double your bet after a win; keep the same bet after a loss.

(c) Reverse Martingale: double your bet after a loss; keep the same bet after a win.
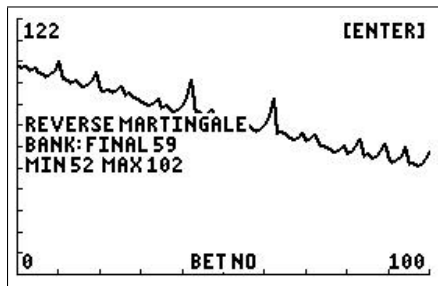
(d) Labouchère   (sequence 1,2,3,4)

The user of this strategy decides before playing how much money they want to win each bet. He or she then writes down a list or sequence of positive numbers that sum to this amount. With each bet, the player stakes an amount equal to the sum of the first and last numbers on the list (if only one number remains, that number is the amount of the stake).

If the bet is successful, the two amounts are removed from the list. If the bet is unsuccessful, the amount lost is appended to the end of the list. This process continues until either the list is completely crossed out, at which point the desired amount of money has been won, or until the player runs out of money to wager.
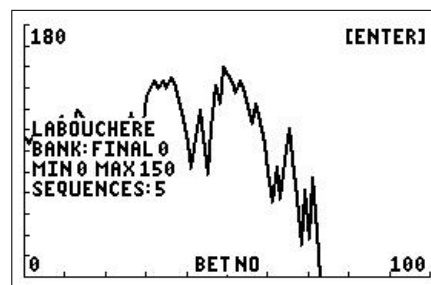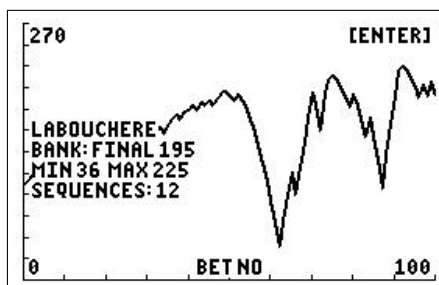
(e) Reverse Labouchère   (sequence 1,2,3,4)

In this version after a win, instead of deleting numbers from the list, the player adds the previous bet amount to the end of the list. He or she continues building up the list until they hit the table maximum. After a loss, the player deletes the outside numbers and continues working on the shorter list. The player starts their list again if they run out of numbers to bet.

The Reverse Labouchère system is often used because where the Labouchère list represents how much the player wants to win each bet, a reverse Labouchère line represents the most that the player is able to lose.
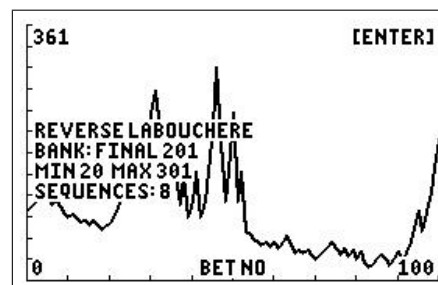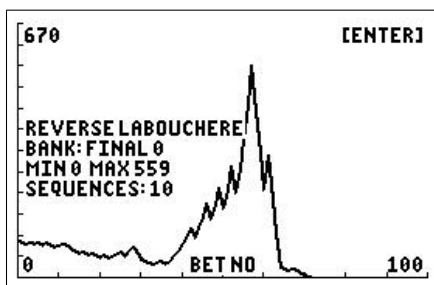
The Martingale and Labouchère systems are dangerous in that they can produce a long string of small wins which don't prepare you for the massive loss which is around the corner.

18.5.5   Tossing coins

Many people misunderstand the Law of Averages.

A sales book said something along the lines of "If 1 in 2 people are interested in buying your product then, if you get a knockback at one house, **the law of averages** says you **will** get a sale at the very next house." It makes you wonder whether this guy had ever tried it himself.

It's like saying that if red came up on the roulette wheel last time, black **must** come up next. If that were the case, you could correctly predict the next umpteen trillion spins of the wheel and retire tomorrow.

The Law of Averages (or the Law of Large Numbers) actually states that, if the probability of an event is p, then after n trials, the number of times this event occurs, x say, is such that

$$\frac{x}{n} \to p \quad \text{as} \quad n \to \infty.$$

The Law of Averages does **not** say that one event or a series of events causes another event to occur next in order to 'even things out'. In fact the opposite is true; the Law of Averages depends for its validity on each event being independent — that is, unaffected by earlier events.

The COINTOSS/CNTOSSCE program simulates tossing a coin (fair or biased). Enter the probability of *heads* and the number of trials (up to 999). The program plots the ratio H/N versus N, where H is the total number of *heads* in N tosses. The program also displays the value of N beyond which H/N remains within 10% of the probability of *heads*.

Values of H/N are stored in list L1; you should clear this when you have finished.
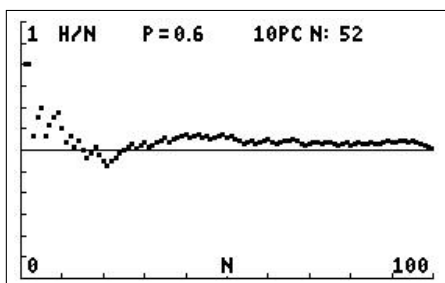
The program tells us that the ratio remained within 10% of 0.6 after 52 trials.

18.6   Solutions to exercises

Exercises   page 35

1. Evaluate each of the following by hand, then check with your calculator.

   (a) $6! = 720$ **(b)** $^6P_4 = 360$ **(c)** $^8C_5 = 56$.

2. **(a)** $13! = 6,227,020,800$ **(b)** $^{15}P_9 = 1,816,214,400$ **(c)** $^{35}C_{22} = 1,476,337,800$.

3. Generate 100 random integers between 1 and 6, simulating throwing a die. Store them in an appropriately named list.

```
randInt(1,6,100)→LRAND
{6 4 1 4 3 1 5 3 5 2 3 4 …
```

```
        1-Var Stats
List:RAND
FreqList:
Calculate
```

```
        1-Var Stats
x̄=3.38
Σx=338
Σx²=1422
Sx=1.680428036
σx=1.672004785
n=100
minX=1
↓Q₁=2
```

From 1-Var Stats, **(a)** the mean is 3.38; **(b)** the standard deviation is 1.67; and **(c)** the median is 4. Answers will vary here.

Exercises   page 226

1. If $X \sim \text{bin}(8, 0.4)$, to 4 decimal places,

   (a) $P(X=4) = 0.2322$
   (b) $P(X \leqslant 1) = 0.1064$
   (c) $P(X \geqslant 7) = 0.0085$

   (d) $P(1 \leqslant X \leqslant 3) = 0.5773$
   (e) $P(X < 6) = P(X \leqslant 5) = 0.9502$.

2. If $X \sim \text{bin}(25, 0.2)$, to 4 decimal places,

   (a) $P(X=4) = 0.1867$
   (b) $P(X \leqslant 10) = 0.9944$
   (c) $P(X \geqslant 7) = 0.2200$

   (d) $P(5 \leqslant X \leqslant 8) = 0.5326$
   (e) $P(X > 4) = 1 - P(X \leqslant 4) = 0.5793$.

Exercises   page 226

1. If $X \sim \text{bin}(10, 0.4)$, display the distribution of X graphically.

```
Plot1:L1,L2
                    ⊠
                 □
                     □
           □
                       □
     □
                    □
                  □ □  ▫
X=4              Y=0.2508227
```
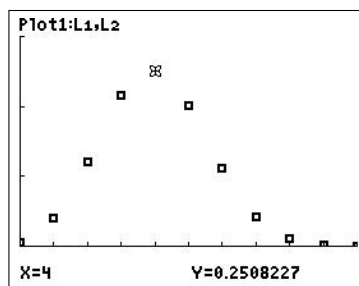
window $[0, 10, 1] \times [0, 0.3, 0.1]$

2. On the one graph, plot the distribution of:

$X_1 \sim \text{bin}(10, 0.1)$ use Plot1 with a square marker;

$X_2 \sim \text{bin}(10, 0.5)$ use Plot2 with a + marker;

$X_3 \sim \text{bin}(10, 0.9)$ use Plot3 with a dot marker.

Here, I've done line graphs rather than scatterplots to show the 3 distributions better.

window $[0, 10, 1] \times [0, 0.5, 0.1]$

Comment on how changing the value of p changes the shape of the distribution.

As p increases, the peak of the distribution moves to the right.

Exercises   **page 40**

If $X \sim \text{geom}(0.2)$, to 4 decimal places,

(a) $P(X=5)=0.0819$ **(c)** $P(X \geqslant 7)=0.2621$

(b) $P(X \leqslant 8)=0.8322$ **(d)** $P(3 \leqslant X \leqslant 9)=0.5058$.

Exercises   **page 41**

1. If $X \sim \text{geom}(0.4)$, display the distribution of X graphically for $1 \leqslant x \leqslant 8$.
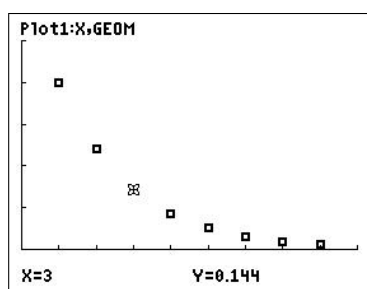
window $[0, 9, 1] \times [0, 0.5, 0.1]$

2. On the one graph, plot the distribution of:

$X_1 \sim \mathrm{geom}(0.1)$ use Plot1 with a square marker;

$X_2 \sim \mathrm{geom}(0.5)$ use Plot2 with a + marker;

Here, I've done line graphs rather than scatterplots to show the two distributions better.
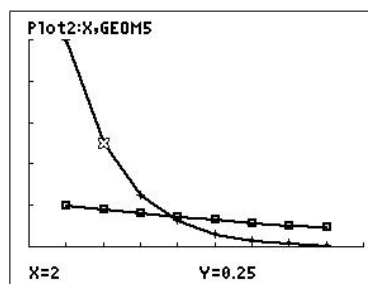
window $[0, 9, 1] \times [0, 0.5, 0.1]$

Comment on how changing the value of p changes the shape of the distribution.

As p increases, the distribution has a larger initial value but decays more rapidly.

Exercises   page 42

If $X \sim \mathrm{hypg}(100, 15, 5)$,

(a) $P(X=3)=0.0216$ **(c)** $P(X>3)=0.0016$

(b) $P(X \leqslant 3)=0.9984$ **(d)** $P(1 \leqslant X \leqslant 2)=0.5411$.

Exercises   page 42

1. If $X \sim \mathrm{hypg}(100, 15, 8)$, display the distribution of X graphically for $0 \leqslant x \leqslant 8$.

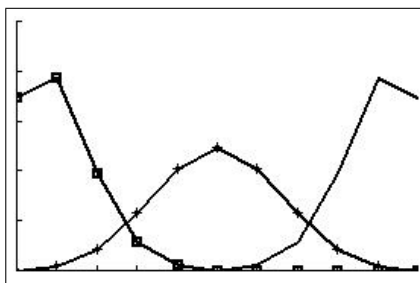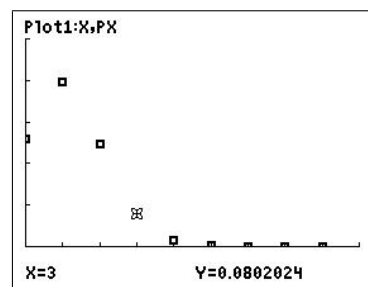window $[0, 9, 1] \times [0, 0.5, 0.1]$

2. On the one graph, plot the distribution of:

$X \sim \text{hypg}(100, 15, 8)$ use Plot1 with a square marker;

$X \sim \text{hypg}(100, 30, 8)$ use Plot2 with a + marker;

Here, I've done line graphs rather than scatterplots to show the two distributions better.
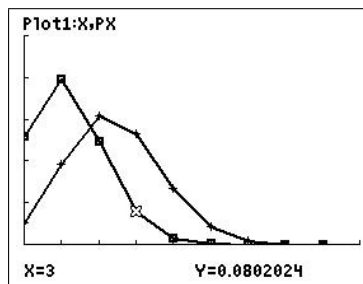
window $[0, 9, 1] \times [0, 0.5, 0.1]$

Comment on how changing the value of k changes the shape of the distribution.

As k increases, the distribution has a smaller initial value, and a smaller peak, shifted to the right.

Exercises   page 188

If $X \sim \text{pois}(0.5)$, to 4 decimal places,

(a) $P(X = 0) = 0.6065$ **(c)** $P(X \geqslant 1) = 0.3935$

(b) $P(X \leqslant 3) = 0.9982$ **(d)** $P(1 \leqslant X \leqslant 4) = 0.3933$.

Exercises   page 44

1. If $X \sim \text{pois}(1.5)$, display the distribution of X graphically for $0 \leqslant x \leqslant 6$.
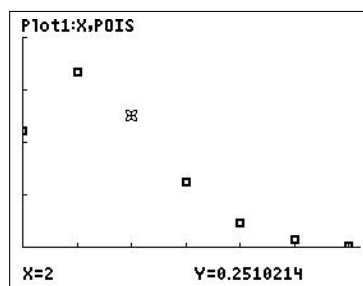
window $[0, 6.2, 1] \times [0, 0.4, 0.1]$

2. On the one graph, for $0 \leqslant x \leqslant 10$, plot the distribution of:

$X_1 \sim \text{pois}(2)$ use Plot1 with a square marker;

$X_2 \sim \text{pois}(5)$ use Plot2 with a + marker;

$X_2 \sim \text{pois}(9)$ use Plot2 with a dot marker.

Here, I've done line graphs rather than scatterplots to show the 3 distributions better.
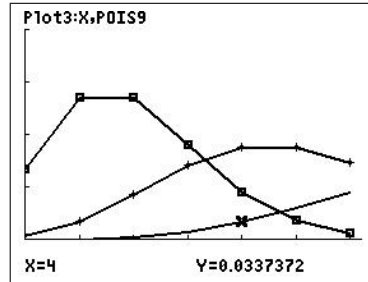
window $[0, 6.2, 1] \times [0, 0.4, 0.1]$

Comment on how changing the value of λ changes the shape of the distribution.

The the peak moves to the right and decreases in amplitude.

Exercises   page 47

1. If $Z \sim \text{N}(0, 1)$, to 4 decimal places,

(a) $P(-1.5 \leqslant Z \leqslant 2) = 0.9104$ **(c)** $P(Z \geqslant 1.6) = 0.0548$

(b) $P(Z \leqslant -0.8) = 0.2119$ **(d)** $P(-1 \leqslant Z \leqslant 1) = 0.6827$.

2. If $X \sim \text{N}(100, 20)$, to 4 decimal places,

(a) $P(88 \leqslant X \leqslant 112) = 0.9927$ **(c)** $P(X \leqslant 107) = 0.9412$

(b) $P(100 \leqslant X \leqslant 105) = 0.3682$ **(d)** $P(X \geqslant 97) = 0.7488$.

3. If $Z \sim \text{N}(0, 1)$, find the value of z to 2 decimal places if

(a) $P(Z \leqslant z) = 0.8413$: $x = 1.00$ **(c)** $P(Z \geqslant z) = 0.9772$: $x = -2.00$

(b) $P(Z \leqslant z) = 0.95$: $x = 1.64$ **(d)** $P(|Z| \leqslant 0.95)$: $x = 1.96$.

4. If $X \sim \text{N}(10, 4)$, find the value of x to 1 decimal place if

(a) $P(X \leqslant x) = 0.05$: $x = 6.7$

(b) $P(X \geqslant x) = 0.90$: $x = 7.4$ **(c)** $P(X \geqslant x) = 0.025$: $x = 13.9$

Exercises   page 48

If $X \sim \mathrm{N}(60, 25)$, display each of the following probabilities as an area under the normal pdf curve.
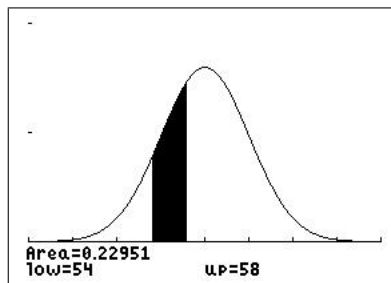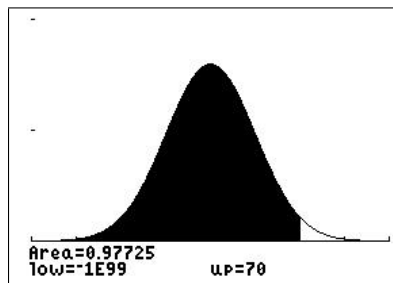
      (a) $P(54 \leqslant X \leqslant 58)$ **(b)** $P(X \leqslant 70)$ **(c)** $P(X \geqslant 62.5)$
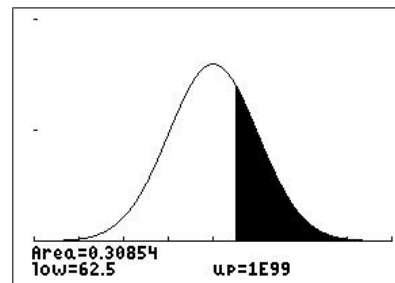
window $[40, 80, 5] \times [0, 0.1, 0.05]$

Done using the *ShadeNorm* command.

Graphically compare the normal approximation to the binomial for $p = 0.5$ when $n = 5$ and $n = 20$.
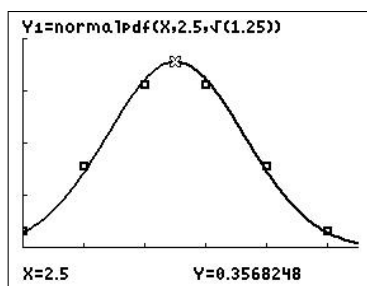
               $n = 5$ **$n = 20$**
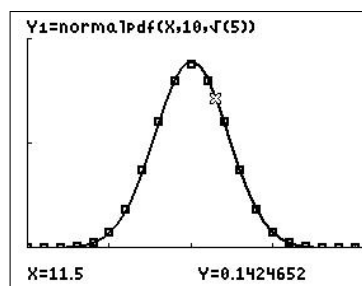
window $[0, 5.5, 1] \times [0, 0.4, 0.1]$ window $[0, 20.5, 5] \times [0, 0.2, 0.1]$

Exercises   page 50

If $X \sim \mathrm{Exp}(0.2)$, to 4 decimal places,

(a) $P(3 \leqslant X \leqslant 5) = 0.1809$ **(b)** $P(X \leqslant 5) = 0.6321$ **(c)** $P(X \geqslant 2) = 0.6703$

19   Matrix and Vector Operations

The material here is directly relevant to the topics *Vectors in the plane*, *Vectors in three dimensions* and *Matrices* in the Australian Curriculum for Specialist Mathematics.

Some of the material is taken from lecture notes on Linear Algebra written for first-year students at UNSW Canberra by Colin Pask and extended for graphics calculators by Peter McIntyre.

As with all technology, you should do the first few calculations by hand to make sure you understand the method. Use the technology firstly to check your hand calculations and then when the numbers make hand calculation difficult or prone to error.

19.1   Note on matrix and vector notation

Many books use bold italic uppercase letters for matrices, e.g. $\boldsymbol{M}$, and bold italic lowercase letters for vectors, e.g. $\boldsymbol{v}$. However, students can't write these forms and, in the interest of their distinguishing between scalars, vectors and matrices, we ask them to underline vectors and double-underline matrices. Accordingly, we do that here too to encourage them to do it.

Of course, not many take notice, resulting in expressions such as $\boldsymbol{M} = 2$. One can only try.

19.2   Basic matrix operations

The basic matrix operations on the calculator work just like the number operations with one or two minor exceptions.

Let's start out with matrices $\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $\underline{\underline{C}} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$.

19.2.1   Putting matrices in your calculator

Press $\boxed{\text{matrix}}$, then $\boxed{\blacktriangleright}$ $\boxed{\blacktriangleright}$ (or $\boxed{\blacktriangleleft}$) to get to EDIT (figure below left). You are now ready to edit one of the ten matrices [A]–[J]: press $\boxed{1}$ to select [A].

Now input the order (rows × columns) and the elements of [A] — the cursor indicates what is required. Press the desired number and $\boxed{\text{enter}}$. The cursor moves on and leads you through the elements row by row.

Make sure you press $\boxed{\text{enter}}$ after the last element (figure below right).
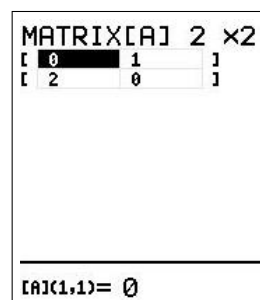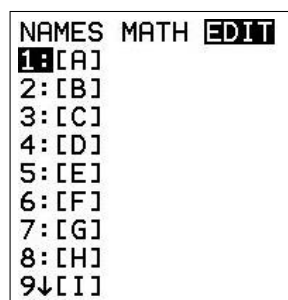
Press $\boxed{\text{matrix}}$ $\boxed{\blacktriangleleft}$ again, but press $\boxed{2}$ this time to input [B]. Finally input [C].

Press $\boxed{\text{2nd}}$ $\boxed{\text{quit}}$ to return to the Home screen.

19.2.2   Displaying matrices

You call up matrices by pressing $\boxed{\text{matrix}}$ and the appropriate number.

For example, pressing $\boxed{\text{matrix}}$ $\boxed{2}$ will display [B] on the screen.
Pressing $\boxed{\text{enter}}$ will show you the elements of (numbers in) [B].

```
[B]
                    [[1 2 3]
                     [4 5 6]]
```

19.2.3   Adding and subtracting matrices

Matrices can only be added or subtracted if they have the same dimensions, that is the same number of rows and the same number of columns.
We can add/subtract multiples of [A] and [C], but not with [B].

```
[A]+[C]
                    [[-1  3 ]
                     [5   -4]]
[A]-[C]
                    [[1   -1]
                     [-1  4 ]]
2[A]-3[C]
                    [[3   -4]
                     [-5  12]]
```

19.2.4   Multiplying matrices

Press $\boxed{\text{matrix}}$ $\boxed{1}$ $\boxed{\text{matrix}}$ $\boxed{2}$ to display [A][B] and press $\boxed{\text{enter}}$ to display the numerical result

$$\underline{\underline{AB}} \;=\; \begin{bmatrix} 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}.$$

The result is stored in Ans (as is the result of any calculation).

```
[A][B]
                    [[4 5 6]
                     [2 4 6]]
[A]Ans
                    [[2 4  6 ]
                     [8 10 12]]
```

If you now want to multiply by $\underline{A}$ again, i.e. to find $\underline{\underline{A}}\big(\underline{\underline{AB}}\big)$, press $\boxed{\text{matrix}}$ $\boxed{1}$ $\boxed{\text{2nd}}$ $\boxed{\text{ans}}$ (on the $\boxed{(-)}$ key) to display [A]Ans, then $\boxed{\text{enter}}$ to give the numerical result

$$\underline{\underline{A}}\big(\underline{\underline{AB}}\big) \;=\; \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Note: If you want to evaluate the expression $\underline{\underline{A}}\big(\underline{\underline{AB}}\big)$ using brackets, you need to put in the multiplication sign before the brackets, that is evaluate $[A]*\big([A]\,[B]\big)$. This is because a bracket following a matrix, $[A](1,2)$ for example, is used to denote the element of a matrix, here the $1,2$ element, i.e. the number in the first row and second column.

To evaluate $\underline{\underline{A}}\big(\underline{\underline{AB}}\big)$, it is much simpler to omit the brackets and just evaluate $[A]\,[A]\,[B]$, thereby using one of the laws of vector multiplication.

19.2.5   Squaring matrices

We could also work out $\underline{\underline{AAB}}$ as $\underline{\underline{A}}^2\underline{\underline{B}}$.

Press $\boxed{\text{matrix}}$ $\boxed{1}$ $\boxed{x^2}$ $\boxed{\text{matrix}}$ $\boxed{2}$ to display [A]^2[B] and $\boxed{\text{enter}}$ to give the same result:

$$\underline{\underline{A}}^2\underline{\underline{B}} \;=\; \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

```
[A]²[B]
                  [[2 4  6 ]
                   [8 10 12]]
[A]^4
                     [[4 0]
                      [0 4]]
[A]⁻¹
                    [[0 0.5]
                     [1 0  ]]
```

Integer powers of a matrix are produced the same way as with numbers (for positive integers). Only square matrices can be raised to a power.

$$\underline{\underline{A}}^4 \;=\; \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

The only negative integer 'power' that works is -1 $\left(\text{using the } \boxed{x^{-1}} \text{ key}\right)$: however, [A]$^{-1}$ produces the **inverse** of (square) matrix $\underline{\underline{A}}$.

The reciprocal of a matrix is not defined, nor is division of one matrix by another.

19.2.6   More-involved expressions

Extensions work just as you would expect. For example, to work out $\underline{\underline{AB}} + 3\underline{\underline{B}}$, press $\boxed{\text{matrix}}$ $\boxed{1}$ $\boxed{\text{matrix}}$ $\boxed{2}$ $+3$ $\boxed{\text{matrix}}$ $\boxed{2}$ to display '[A][B] + 3[B]', and $\boxed{\text{enter}}$ to give

$$\underline{\underline{AB}} + 3\underline{\underline{B}} \;=\; \begin{bmatrix} 7 & 11 & 15 \\ 14 & 19 & 24 \end{bmatrix}.$$

```
[A][B]+3[B]
                  [[7  11 15]
                   [14 19 24]]
Ans→[C]
                  [[7  11 15]
                   [14 19 24]]
```

Note: Again, if you want to evaluate an expression involving brackets, such as $\underline{\underline{A}}\left(\underline{\underline{B}} + \underline{\underline{C}}\right)$, you need to put in the multiplication sign before the brackets, that is evaluate $\underline{\underline{A}} * \left(\underline{\underline{B}} + \underline{\underline{C}}\right)$.

19.2.7   Storing matrices

If you wanted to keep the previous answer for later use, you might store it in matrix [C] by pressing $\boxed{\text{STO►}}$ $\boxed{\text{matrix}}$ $\boxed{3}$ $\boxed{\text{enter}}$ (figure above).

Notice that the calculator automatically gives [C] the correct order.

19.2.8   Illegal operations

The calculator will not let you do invalid operations.

For example, if you try to calculate $\underline{\underline{A}}+\underline{\underline{B}}$ by entering $\boxed{\text{matrix}}$ $\boxed{1}$ + $\boxed{\text{matrix}}$ $\boxed{2}$, you will see [A]+[B] on the screen, but pressing $\boxed{\text{enter}}$ produces the message ERR: DIM MISMATCH. *Why?*

19.2.9   Other matrix operations

A number of operations are contained in the $\boxed{\text{matrix}}$ math menu.

- *determinant*: $\det([A]) = -2$.

- *transpose*: $[B]^{\mathrm{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

- *dimension*: $\dim([B]) = \{2\ 3\}$: 2 rows $\times$ 3 columns.

```
det([A])
                        -2.
[B]ᵀ
                     [[1 4]
                      [2 5]
                      [3 6]]
dim([B])
                      {2 3}
```

```
Fill(1,[A])
                       Done
[A]
                     [[1 1]
                      [1 1]]
identity(2)
                     [[1 0]
                      [0 1]]
```

- *'filling' a matrix*: $\mathrm{Fill}\big(1, [A]\big)$ produces a matrix of 1s: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

- *identity matrix*: $\mathrm{identity}(n)$ produces the $n \times n$ identity matrix.

- *row-echelon form*: $\mathrm{ref}\big([A]\big)$ produces the row-echelon form of [A] (Gaussian elimination — see Section 19.3).

- *reduced-row-echelon form*: $\mathrm{rref}\big([A]\big)$ produces the reduced row-echelon form of [A] (Gauss-Jordan elimination — see Section 19.3).

See the TI-84 Guidebook for details on the other $\boxed{\text{matrix}}$ MATH menu items.

Some applications of matrices in population modelling can be found in *Population Modelling 3: Matrix Models*, Chapter 20.

19.3   Systematic Gauss Elimination

19.3.1   The method

The endpoint of Gauss elimination or Gauss reduction in solving a system of simultaneous equations is a matrix in **row-echelon** form, characterised by:

  (a) the first non-zero element in each row is a 1 (called a **pivot**);

  (b) all elements in the column below a pivot are 0.

To reduce a matrix to row-echelon form, we use **elementary row operations**.[26] The three elementary row operations are:

  A. exchange two rows: $R_i \rightleftharpoons R_j$

  B. multiply a row by a constant: $R_i \rightarrow cR_i, \ c \neq 0$

  C. add a constant multiple of one row to another: $R_i \rightarrow R_i + cR_j, \ c \neq 0$.

Procedure

  1. If the first column of the matrix is all zeros, 'cross out' this column to leave a smaller matrix.

  2. If the element in the first row and first column (top left element) of the matrix is 0, exchange Row 1 with another row (Operation A).

  (Sometimes it is convenient to do this even if the top left element is non-zero, so as to avoid fractions in Step 3.)

  3. Multiply the top row by a constant to make the first element 1 (Operation B).

  4. Use Operation C to obtain 0s **below** the 1 (pivot) by adding multiples of **Row 1** to each successive row $(R_i \rightarrow R_i + cR_1, \ i = 2, 3, \dots)$.

  5. 'Cross out' the first row and first column to leave a smaller matrix.

  6. Go back and start at Step 1 on this smaller matrix.

PTO

---

[26]Each row operation corresponds to the multiplication of the matrix by a corresponding elementary matrix.

Notes

- The only choice in this systematic version of Gauss elimination is which rows to interchange in Step 2. All the other operations are prescribed.

- The solutions to the simultaneous equations described by the row-echelon matrix are the same as those of the original matrix, hence the usefulness of the method. Write down the equations corresponding to the row-echelon matrix and use back substitution to find the solutions.

- The matrix changes wherever we perform an elementary row operation, so that we cannot use equal signs between the steps. Use a $\sim$ instead. The matrices are said to be (row-) equivalent.

- We can proceed in a similar manner to obtain 0s above the pivots too, the **reduced row-echelon** matrix. This is called *Gauss-Jordan elimination*. A unique solution can be read directly from the final matrix, without the need for back substitution. By hand, however, Gauss elimination and back substitution are usually quicker than Gauss-Jordan elimination.

19.3.2   Using the calculator

The TI-84/CE has two built-in commands *ref* and *rref* in the $\boxed{\text{matrix}}$ MATH menu.

ref$(matrix)$, where *matrix* is one of the ten matrices [A] – [J] used by the calculator, produces the row-echelon form of *matrix* (Gauss elimination).

rref$(matrix)$ produces the reduced row-echelon form of *matrix* (Gauss-Jordan elimination).

Using ▶Frac in the $\boxed{\text{math}}$ menu converts the elements to fractions where possible: for example ref$([A])$ ▶Frac displays the row-echelon form of [A] with elements as fractions.

The GAUSS/GAUSSCE program[27]

The programs just automate these procedures.

Store the matrix in [A] using $\boxed{\text{matrix}}$ EDIT. Run GAUSS/GAUSSCE. At the pauses, the matrix displayed is in

- row-echelon form — decimal version.

- row-echelon form — fraction version.

- reduced row-echelon form — decimal version.

- reduced row-echelon form — fraction version.

Press $\boxed{\text{enter}}$ to move through the different forms and $\boxed{\text{enter}}$ after the final form to complete the program. If you want to stop at any intermediate stage, press $\boxed{\text{on}}$ Quit.

After running the program, the original matrix remains in [A], with the row-echelon form stored in [D] and the reduced row-echelon form in [E]. Use $\boxed{\text{matrix}}$ to recall either [D] or [E] for further calculations.

---

[27]available at *canberramaths.org.au* under *Resources*

Examples: Gauss elimination

$$
\begin{bmatrix} 10 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.4 & 0.1 & 0.1 \\ 0 & 1 & -1 & -8.5 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2/5 & 1/10 & 1/10 \\ 0 & 1 & -1 & -17/2 \\ 0 & 0 & 1 & 7 \end{bmatrix}
$$

```
                              ref([A])
                                   [[1 0.4 0.1 0.1 ]
                                    [0 1    -1  -8.5]
[A]                                 [0 0    1    7   ]].
              [[10 4 1 1]     Ans▶Frac
               [6   2 1 4]         [[1 2/5 1/10 1/10 ]
               [1   0 0 0]]         [0 1    -1    -17/2]
                                    [0 0    1     7    ]]
```

Using the *ref* command

$$
\begin{bmatrix} 0 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.\dot{3} & 0.1\dot{6} & 0.\dot{6} \\ 0 & 1 & 0.25 & 0.25 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 1/6 & 2/3 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 7 \end{bmatrix}
$$

$$
\begin{bmatrix} 0 & 4 & 1 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0.25 & 0.25 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Example: Gauss-Jordan elimination

$$
\begin{bmatrix} 1 & 2 & 3 & 5 \\ 5 & 1 & 3 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

```
                              REDUCED ROW-ECHELON FORM
[A]
              [[1 2 3 5]                        [[1 0 0 0]
               [5 1 3 2]                         [0 1 0 0]
               [4 1 1 1]                         [0 0 1 0]
               [2 4 3 1]]                        [0 0 0 1]]
```

Using the GAUSSCE program

19.3.3   Exercises

These exercises assume you have covered the theory of solving simultaneous linear equations using Gauss elimination and the inverse-matrix method. Solutions are on page 77.

1. Enter the matrix below carefully into [A] using ⌐matrix⌐ EDIT.

$$
\begin{bmatrix}
0 & 1 & 1 & -2 & -3 \\
1 & 2 & -1 & 0 & 2 \\
2 & 4 & 1 & -3 & -2 \\
1 & -4 & -7 & -1 & -19
\end{bmatrix}
$$

This matrix is the augmented matrix $\underline{\underline{P}}\,|\,\underset{\sim}{q}$ for the following system of equations, in matrix form $\underline{\underline{P}}\,\underset{\sim}{v} = \underset{\sim}{q}$:

$$
\begin{aligned}
x_2 + \; x_3 - 2x_4 &= \;\; -3 \\
x_1 + 2x_2 - \; x_3 \quad\;\; &= \;\;\;\; 2 \\
2x_1 + 4x_2 + \; x_3 - 3x_4 &= \;\; -2 \\
x_1 - 4x_2 - 7x_3 - \; x_4 &= \;\; -19
\end{aligned}
$$

Solve this system with Method:

A using the calculator to find the row-echelon form (Gaussian elimination), then back substitution (you have to do this);

B using the calculator to find the reduced row-echelon form (Gauss-Jordan elimination), from which you can read off the answer;

C using $\underset{\sim}{v} = \underline{\underline{P}}^{-1}\underset{\sim}{q}$. Do you understand why this works?

A little manipulation on the calculator allows us to evaluate $\underline{\underline{P}}^{-1}\underset{\sim}{q}$.

Change [A] to the matrix $\underline{\underline{P}}$ by changing the column dimension to 4 using ⌐matrix⌐ EDIT: this chops off the last column of [A].

Store the column matrix $\underset{\sim}{q}$ in [B], then evaluate $[A]^{-1}[B]$.

PTO

2. Solve the following systems of equations using each of the three methods using the calculator to do the Gauss or Gauss-Jordan elimination.

Check your answers by substituting them back into the equations.

Can you use Method C to solve these? What happens in (b) and (c) below? Can you explain? What is $\det\left(\underline{\underline{P}}\right)$? What does this tell you about the inverse matrix?

(a)
$$\begin{aligned} 2x - 5y + 5z &= 17 \\ x - 2y + 3z &= 9 \\ -x + 3y &= -4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 \qquad - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 \qquad + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

Answers

(a) $x=1$, $y=-1$, $z=2$

(b) $x_1=1+2t$, $x_2=2+3t$, $x_3=t$; t any non-zero number

(c) no solution.

19.3.4 Solutions

1. Enter the matrix below carefully into [A] using $\boxed{\text{matrix}}$ EDIT.

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

This matrix is the augmented matrix $\underline{\underline{\boldsymbol{P}}} \mid \underset{\sim}{\boldsymbol{q}}$ for the following system of equations, in matrix form $\underline{\underline{\boldsymbol{P}}}\underset{\sim}{\boldsymbol{v}} = \underset{\sim}{\boldsymbol{q}}$:

$$
\begin{aligned}
x_2 + x_3 - 2x_4 &= -3 \\
x_1 + 2x_2 - x_3 &= 2 \\
2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\
x_1 - 4x_2 - 7x_3 - x_4 &= -19
\end{aligned}
$$

A. Solve this system using the calculator to find the row-echelon form (Gaussian elimination), then back substitution (you have to do this);

Either run the GAUSS/GAUSSCE program (left-hand screen below) or the *ref* command (right-hand screen) to give the row-echelon form of the matrix.

Using $\boxed{\text{math}}\ \boxed{1}$ (▶Frac) converts decimals to fractions where possible.

```
▶Frac FORM

    [[1 2 1/2 -3/2  -1]
     [0 1 5/4 -1/12 3 ]
     [0 0 1    -1   -2]
     [0 0 0    1    3 ]]
```

```
ref([A])▶Frac
    [[1 2 1/2 -3/2  -1]
     [0 1 5/4 -1/12 3 ]
     [0 0 1    -1   -2]
     [0 0 0    1    3 ]]
```

The corresponding (equivalent) equations are

$$
\begin{aligned}
x_1 + 2x_2 + \frac{1}{2}x_3 - \frac{3}{2}x_4 &= -1 \\
x_2 + \frac{5}{4}x_3 - \frac{1}{12}x_4 &= 3 \\
x_3 - x_4 &= -2 \\
x_4 &= 3
\end{aligned}
$$

The last equation gives $x_4 = 3$.

Substituting this into the third equation gives $x_3 = -2 + x_4 = -2 + 3 = 1$.

Substituting these into the second equation gives

$$x_2 \;=\; 3 - \frac{5}{4}x_3 + \frac{1}{12}x_4 \;=\; 3 - \frac{5}{4} + \frac{3}{12} \;=\; 2.$$

Substituting these into the first equation gives

$$x_1 \;=\; -1 - 2x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \;=\; -1 - 4 - \frac{1}{2} + \frac{9}{2} \;=\; -1.$$

Therefore, $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$.

B. using the calculator to find the reduced row-echelon form (Gauss-Jordan elimination), from which you can read off the answer;

Either run the GAUSS/GAUSSCE program again or use the *rref* command to give the reduced-row-echelon form of the matrix.

```
REDUCED ROW-ECHELON FORM

                [[1 0 0 0 -1]
                 [0 1 0 0 2 ]
                 [0 0 1 0 1 ]
                 [0 0 0 1 3 ]]
```

The corresponding (equivalent) equations are

$$x_1 \;=\; -1$$
$$x_2 \;=\; 2$$
$$x_3 \;=\; 1$$
$$x_4 \;=\; 3,$$

the answer directly.

C. using $\underset{\sim}{v} = \underset{=}{P}^{-1}\underset{\sim}{q}$. Do you understand why this works?

Change [A] from the 4×5 augmented matrix to the 4×4 matrix $\underset{=}{A}$ by changing the column dimension to 4 using $\boxed{\text{matrix}}$ EDIT: this chops off the last column of [A] (top matrix in the figure below).

Store the column matrix $\underset{\sim}{b}$ in [B], then evaluate $[A]^{-1}[B]$.

```
        [[0 1   1  -2]
         [1 2  -1  0 ]
         [2 4   1  -3]
         [1 -4 -7 -1]]
[B]                         [A]⁻¹[B]
         [[-3 ]                      [[-1]
          [2  ]                       [2 ]
          [-2 ]                       [1 ]
          [-19]]                      [3 ]]
```

This method works here because the matrix $\underset{=}{P}$ is invertible, corresponding to the case in which there is a unique solution.

Again the answer is $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$.

2. Solve the following systems of equations using each of the three methods using the calculator to do the Gauss or Gauss-Jordan elimination.

Check your answers by substituting them back into the equations.

(a)
$$
\begin{aligned}
2x - 5y + 5z &= 17 \\
x - 2y + 3z &= 9 \\
-x + 3y &= -4
\end{aligned}
$$

In matrix form,
$$
\begin{bmatrix} 2 & -5 & 5 \\ 1 & -2 & 3 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 9 \\ -4 \end{bmatrix},
$$

with augmented matrix
$$
\begin{bmatrix} 2 & -5 & 5 & 17 \\ 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \end{bmatrix}.
$$

The row-echelon form from GAUSS/GAUSSCE (Method A) is
$$
\begin{bmatrix} 1 & -2.5 & 2.5 & 8.5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.
$$

The corresponding equations are
$$
\begin{aligned}
x - 2.5y + 2.5z &= 8.5 \\
y + z &= 1 \\
z &= 2
\end{aligned}
$$

The last equation gives $z = 2$ directly. Substituting this into the second equation gives $y = -1$, and these into the first equation $x = 1$.

The unique solution from Method A is $x = 1$, $y = -1$ and $z = 2$.

PTO

Using GAUSS/GAUSSCE for Method B, Gauss-Jordan elimination, starting with the augmented matrix in [A]:

```
MATRIX[A] 3 ×4
[ 2      -5     5      17     ]
[ 1      -2     3      9      ]
[ -1     3      0      -4     ]
```

```
REDUCED ROW-ECHELON FORM
                        [[1 0 0 1 ]
                         [0 1 0 -1]
                         [0 0 1 2 ]]
```

The right-hand column of the reduced row-echelon form gives the solution; each equation now only contains one variable.

Following the hint in Question 1 for putting $\underset{=}{\boldsymbol{P}}$ back in matrix [A], the RHS of the original system in [B] and evaluating $[A]^{-1}[B]$ gives the Method C result

```
[A]
              [[2   -5 5]
               [1   -2 3]
               [-1 3  0]]
[B]
               [[17]
                [9 ]
                [-4]]
```

```
[A]⁻¹[B]
                  [[1 ]
                   [-1]
                   [2 ]]
```

which gives the solution directly.

Substituting the solution $x=1$, $y=-1$ and $z=2$ back into the original equations:

$$\text{LHS} \;=\; 2\times 1 - 5\times -1 + 5\times 2 \;=\; 17 \;=\; \text{RHS}.$$

$$\text{LHS} \;=\; 1 - 2\times -1 + 3\times 2 \;=\; 9 \;=\; \text{RHS}.$$

$$\text{LHS} \;=\; -1 + 3\times -1 \;=\; -4 \;=\; \text{RHS}.$$

The solution is verified.

PTO

(b)
$$x_1 + x_2 - 5x_3 = 3$$
$$x_1 \qquad - 2x_3 = 1$$
$$2x_1 - x_2 - x_3 = 0$$

In matrix form,

$$\begin{bmatrix} 1 & 1 & -5 \\ 1 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$

with augmented matrix

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix}.$$

GAUSS/GAUSSCE gives the row-echelon form (Method A)

$$\begin{bmatrix} 1 & -0.5 & -0.5 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The bottom row of all zeros tells us there is an infinite number of solutions.

The corresponding equations are

$$x_1 + x_2 - 5x_3 = 3$$
$$x_2 - 3x_3 = 2$$

Here, x_3 is a free variable: let $x_3 = t$, $t \in \mathbb{R}$.

The second equation gives $x_2 = 2 + 3t$, and the first $x_1 = 1 + 2t$.

The solution is $x_1 = 1 + 2t$, $x_2 = 2 + 3t$ and $x_3 = t$, $t \in \mathbb{R}$

Using GAUSS/GAUSSCE for Method B, Gauss-Jordan elimination,

```
MATRIX[A] 3 x4
[ 1       1       -5      3      ]
[ 1       0       -2      1      ]
[ 2      -1       -1      0      ]
```

```
REDUCED ROW-ECHELON FORM

        [[1 0 -2 1]
         [0 1 -3 2]
         [0 0 0  0]]
```

The right-hand column of the reduced row-echelon form does not gives the solution directly here. However, back substitution is easier with the reduced row-echelon form than with the row-echelon form.

Putting $\underset{=}{\boldsymbol{P}}$ back in matrix [A], the RHS of the original system in [B] and evaluating $[A]^{-1}[B]$ gives an error message: *No inverse exists.*

Method C cannot be used, consistent with the fact that there is not a unique solution.

Substituting the solution $x_1 = 1+2t$, $x_2 = 2+3t$ and $x_3 = t$ back into the original equations:

$$\text{LHS} = 1+2t + 2+3t - 5t = 3 = \text{RHS.}$$

$$\text{LHS} = 1+2t - 2t = 1 = \text{RHS.}$$

$$\text{LHS} = 2(1+2t) - (2+3t) - t = 0 = \text{RHS.}$$

The solution is verified.

(c)
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 \phantom{{}-x_2} + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

In matrix form,

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & -3 & 5 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \\ 1 \end{bmatrix},$$

with augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}$$

GAUSS/GAUSSCE gives the row-echelon form (Method A)

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{17}{3} & -\frac{10}{3} \\ 0 & 0 & 1 & \frac{67}{6} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The bottom row tells us there is no solution: the corresponding equation is $0x_1 + 0x_2 + 0x_2 = 1$.

Using GAUSSCE for Method B, Gauss-Jordan elimination,

```
MATRIX[A] 4 x4
[ 1      -1      2      4      ]
[ 1      0       1      6      ]
[ 2      -3      5      4      ]
[ 3      2       -1     1      ]
```

```
REDUCED ROW-ECHELON FORM
                        [[1 0 0 0]
                         [0 1 0 0]
                         [0 0 1 0]
                         [0 0 0 1]]
```

Again no solution.

Putting $\underline{\underline{P}}$ back in matrix [A], the RHS of the original system in [B] and evaluating $[A]^{-1}[B]$ gives an error message: *No inverse exists.*

Method C cannot be used, consistent with the fact that there is not a unique solution.

What is $\det\left(\underline{\underline{P}}\right)$? What does this tell you about the inverse matrix?

In (b) and (c), $\det\left(\underline{\underline{P}}\right)=0$, confirming that an inverse matrix does not exist.

19.4   Applications

19.4.1   Data analysis and curve fitting

A very large number of measurements, surveys and so on generate data points (x_i, y_i), $i = 1, 2, 3, \ldots, n$, and we would like to find ways to *represent* concisely all the results.

Mathematically we check to see if there is some formula or equation we can devise that will reproduce the data. We will then be able to use that formula to make predictions.

Two situations and approaches come to mind, and both almost always lead us into linear-algebra problems.

Situation 1

We have scattered data points, and we look for the curve that fits best through the data points. It may be a straight line or some other curve that we believe is useful or the right one to use for some other reason.
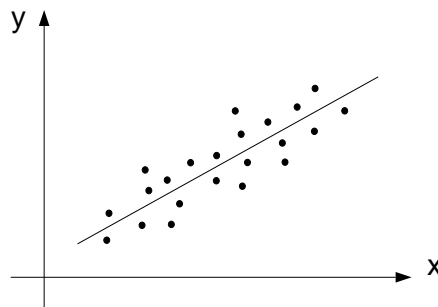
This is the sort of problem Gauss solved when devising a theory for finding the orbits of planets or other astronomical bodies when a few measurements were available, but with limited accuracy.

The *Method of Least Squares* is commonly used, and leads to a linear-algebra problem for the numbers defining the detailed curves. The least-squares algorithm for a number of different curves is built into the curve-fitting commands on your graphics calculator.

Situation 2

The data points seem to lie on a smooth curve, and we look for a curve that passes through all points (an exact fit).
If we find a formula for that curve, we can use it to give y values for x values between the data points — this is called *interpolation*.
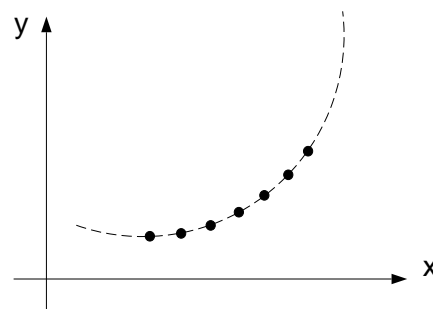
We can also use the formula to get y values when x is outside the set of x values for which data are available, e.g. using the dashed curve. This is called *extrapolation*, and can be dangerous, as the curve might in practice behave quite differently beyond the measured data points.

Obvious question: *Which kind of curve to fit to the data?*

This is a matter of experience. Sometimes an underlying theory will suggest the sort of curve to use. Generally we would try to use the simplest sorts of functions so the working is easy. One very commonly used simple form is a polynomial:

$$y \;=\; a_0 \;+\; a_1 x \;+\; a_2 x^2 \;+\; a_x^3 \;+\; a_4 x^4 \;+\; \cdots \;+\; a_m x^m.$$

The data points are then used to find the coefficients $a_0, a_1, a_2, \ldots, a_m$.

Example: Find a polynomial passing through the three points $(1, 2)$, $(2, 1)$ and $(3, 2)$.[28]

Define the variables

There are three data points, so we can fit a unique quadratic polynomial

$$y \;=\; a_0 \;+\; a_1 x \;+\; a_2 x^2,$$

which has three unknowns. We have to find values for a_0, a_1 and a_2.

Formulate the equations

Fitting data point 1 gives: $2 \;=\; a_0 \;+\; a_1 \;+\; a_2 \qquad (x = 1, \, y = 2)$
       data point 2 gives: $1 \;=\; a_0 \;+\; 2a_1 \;+\; 4a_2 \qquad (x = 2, \, y = 1)$
       data point 3 gives: $2 \;=\; a_0 \;+\; 3a_1 \;+\; 9a_2 \qquad (x = 3, \, y = 2).$

Solve the equations

Solving these equations (Gauss elimination) gives $a_0 = 5$, $a_1 = -4$ and $a_2 = 1$.

Answer the question

The required quadratic polynomial is $y = 5 - 4x + x^2$.

Some observations

1. Because of our choice of function to fit, we reduced the problem to one of solving linear equations.

2. Choosing a function where the unknown parameters enter non-linearly, for example $y = Ax^B \sin(Cx)$, would lead to nasty simultaneous non-linear equations for A, B and C when the data points are fitted.

3. If the powers of x in the polynomial were replaced by some other functions of x, e.g. instead of 1, x, x^2, x^3, ..., we use $\sin(1)$, $\sin(x)$, $\sin(2x)$, $\sin(3x)$, ..., we would still get linear equations for the coefficients a_0, a_1, a_2, a_3, ... when fitting the data points. Such things are also commonly used when applying mathematics in scientific and engineering work.

4. *What would happen if we had tried other polynomials?*
     A line?   Our three equations for a_0 and a_1 are now inconsistent. We cannot fit a line through these three points.
     A cubic?   Our three equations for a_0, a_1, a_2, a_3 are consistent, but the solution involves a parameter. There is an infinite number of different cubics that can be fitted through these three points.

---

[28]Fitting polynomials up to quartics using the built-in calculator routines is covered in Chapter 5 of Volume 1 of this book. Fitting polynomials of any degree using a calculator program is covered in Chapter 21 of this volume.

Solutions to the Exercises here are in Section 19.4.5.

Exercise

Find a polynomial of degree 2 that passes through the points $(2, 5)$, $(3, 2)$ and $(4, 5)$.

Check that each point does lie on the polynomial. You may do this either algebraically or graphically.

Answer: $y = 29 - 18x + 3x^2$.

Exercise

In the example on the previous page, do the calculations suggested in Point 4, that is

  (a) fitting a straight line to these three points;

  (b) fitting a cubic polynomial to these three points.

In the second case, write down the equation of the cubic that fits the points (it should contain a parameter) and plot the cubic for a few values of the parameter, say -1, 0 and 1, to verify that it does indeed pass through the three points.

Think about extrapolation in these cases. Is it likely to be accurate?

Answers: (a) no solution; (b) $y = 5 - 6t + (-4 + 11t)x + (1 - 6t)x^2 + tx^3, \; t \in \mathbb{R}$.

19.4.2   Balancing chemical reactions

Suppose we are investigating the effects of phosphoric acid H_3PO_4 on calcium Ca.

The molecules produced are calcium phosphate $Ca_3P_2O_8$ and hydrogen gas H_2.
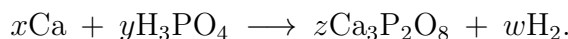
In order to see how this can work (if it can work!), we proceed as follows.

Define the variables

Let the numbers of molecules involved be x, y, z and w respectively.

Formulate the equations

The chemical reaction is

$$x\text{Ca} + y\text{H}_3\text{PO}_4 \longrightarrow z\text{Ca}_3\text{P}_2\text{O}_8 + w\text{H}_2.$$

We now balance the numbers of atoms involved:

$$
\begin{array}{llrclcrccc}
\text{Ca} & : & x & = & 3z & \quad\text{or}\quad & x & -3z & & = & 0 \\
\text{H} & : & 3y & = & 2w & \quad\text{or}\quad & 3y & & -2w & = & 0 \\
\text{P} & : & y & = & 2z & \quad\text{or}\quad & y & -2z & & = & 0 \\
\text{O} & : & 4y & = & 8z & \quad\text{or}\quad & 4y & -8z & & = & 0
\end{array}
$$

So we have a set of *homogeneous linear equations* for x, y, z and w.

Solve the equations

$$
\begin{array}{c}
\text{augmented} \\
\text{matrix}
\end{array}
\quad
\begin{bmatrix}
1 & 0 & -3 & 0 & 0 \\
0 & 3 & 0 & -2 & 0 \\
0 & 1 & -2 & 0 & 0 \\
0 & 4 & -8 & 0 & 0
\end{bmatrix}
\quad
\begin{array}{c}
\text{reduces to} \\
\text{row-echelon} \\
\text{form}
\end{array}
\quad
\begin{bmatrix}
1 & 0 & -3 & 0 & 0 \\
0 & 1 & -2 & 0 & 0 \\
0 & 0 & 1 & -\frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Converting back to equations and solving gives

$$w = t \qquad z = \frac{1}{3}t \qquad y = \frac{2}{3}t \qquad x = t \qquad t \in \mathbb{R}.$$

Answer the question

This is a case where the physical application tells us how to choose the value of t.

    First, we are talking about numbers of molecules, so t must give positive integer values for x, y, z and w.

    Second, it is the convention to give the smallest numbers of molecules that can be involved — in any reaction, the actual numbers of molecules involved will just be multiples of those numbers, depending on how much of the chemicals is used.

So we take $t = 3$, and the reaction is written as

$$3\text{Ca} + 2\text{H}_3\text{PO}_4 \longrightarrow \text{Ca}_3\text{P}_2\text{O}_8 + 3\text{H}_2.$$
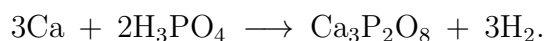
Note that if this had not been a valid reaction (say if we had left out the H_2), the homogeneous equations would have given us *only the trivial solution* to tell us that fact.

19.4.3   Networks

A very large number of modelling problems involve networks.

Think of: supply of water, gas, electricity, broadband connection; streets; airline
routes; railways; electric circuits; neural networks — biological and arti-
ficial; flows of money between banks, businesses, customers; . . . .

The design and optimisation of these networks are mathematical problems of great impor-
tance; whole branches of Mathematics have grown up to tackle different aspects of networks.

Here are two small examples showing how simple flows in networks can be calculated. Both
involve the idea of a

Linear Conservation Principle at a junction
What goes into a junction must also come out.
Otherwise there would be a pile-up at the junction!

If x measures the thing flowing in the network in this
example, $25 + x_1 = x_2 + x_3$ (input = output).
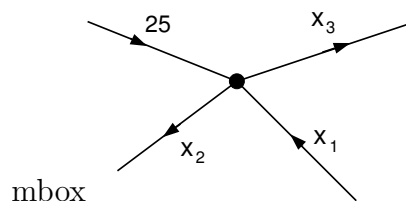
mbox

Note that x_i is defined to be positive in the direction of the arrow. If x_i turns out negative
in the solution, this means the arrow is reversed (the x_i are 1D vectors).

19.4.4   Traffic flow

This is a problem in the design and management of city
streets. The networks may be very large, and a model
allows planners to see the effect of design variations, ac-
cident problems, street closures, etc.
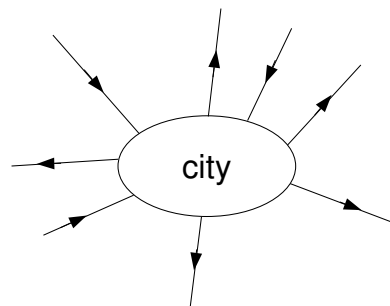
Here is a small piece of a network so we can see the
type of equations generated. The units in the flow are
vehicles per hour.
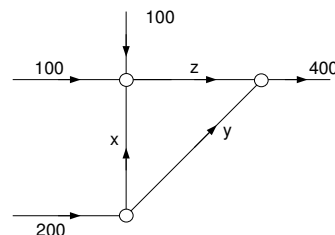
Define the variables

Call the unknown traffic flows x, y and z vehicles per hour, as shown on the diagram (directions
must be specified here too by the arrows).

PTO

Formulate the equations

The conservation of vehicles at each of the three junctions gives the equations — **input on the left, output on the right**; starting at the bottom junction:

$$200 \ = \ x + y$$

$$100 + 100 + x \ = \ z$$

$$y + z \ = \ 400.$$

Re-arranging these equations into standard form gives

$$
\begin{array}{rcrcrcl}
x & + & y & & & = & 200 \\
-x & & & + & z & = & 200 \\
& & y & + & z & = & 400
\end{array}
$$
and these are equations we can solve using Gauss elimination.

Solve the equations

Augmented matrix
$$
\left[\begin{array}{ccc|c}
1 & 1 & 0 & 200 \\
-1 & 0 & 1 & 200 \\
0 & 1 & 1 & 400
\end{array}\right]
$$
converts to row-echelon form
$$
\left[\begin{array}{ccc|c}
1 & 1 & 0 & 200 \\
0 & 1 & 1 & 400 \\
0 & 0 & 0 & 0
\end{array}\right].
$$
Check!

The solution is

$$z = t \qquad y = 400 - t \qquad x = t - 200 \qquad t \in \mathbb{R}.$$

Answer the question

We could now use this model to see the effect, for example, of making the x street one way. Then we would require

$$x > 0, \quad \text{implying} \quad t > 200,$$

and that tells us the traffic flows will be $z > 200$ and $y < 200$.

Maybe there are roadworks such that $y < 50$. *What happens?*

Of course, that is easy to see in our simple example, but in a big, complex city model, the effects of street changes may be quite hard to assess without a mathematical model.

Exercise

The flow of traffic (in vehicles per hour) around a Rome roundabout is shown below.
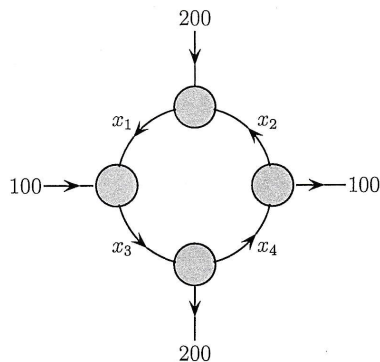
(a) Solve this system for x_1, x_2, x_3 and x_4.

Write your solution in terms of a parameter t.

(b) For what values of the parameter t is the solution realistic, i.e. the traffic travels only in the direction of the arrows?

(c) Find the traffic flow if roadworks block the road corresponding to x_2.

(d) What happens if a group of demonstrators blocks the road corresponding to x_4?

How could the traffic authorities fix this problem temporarily?

Exercise

The flow of traffic in vehicles per hour through a network of streets is shown in the figure below.
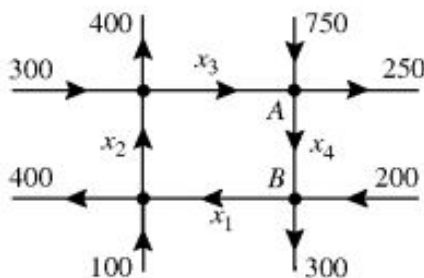
(a) Write down the system of equations that describes the traffic flow and convert to matrix form.

(b)   **(i)** Convert the matrix to row-echelon form.

(ii) Hence find x_1, x_2, x_3 and x_4.

(c) The local traffic authority wants the loop $x_1 \to x_2 \to x_3 \to x_4$ to function as a one-way rectangular 'roundabout' (rectabout?) in the direction shown by the arrows in the figure.

For what values of the parameter in your solution in (b)(ii) is this possible?

Where on the rectabout is the traffic flow greatest?

(d) The traffic authority wants to close down completely for resealing the road between the intersections A and B, i.e. that corresponding to x_4.
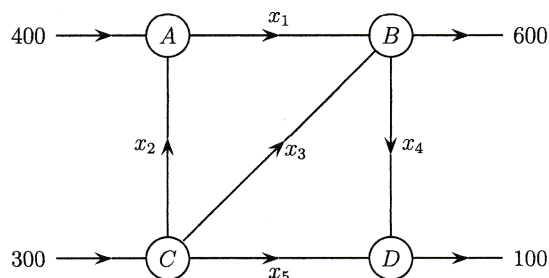
Can traffic still flow in the network?

Sketch the diagram of the resulting traffic flow (with arrows) in this case.

How is this different from normal operation?

Exercise

The flow of traffic in vehicles per hour through a network of streets is shown in the figure below. A, B, C and D are intersections.

(a) Write down the system of equations that describes the traffic flow and convert to matrix form.

(b) **(i)** Convert the matrix to row-echelon form.

   (ii) Explain why x_3 and x_5 are free variables.

   (iii) Hence find x_1, x_2, x_3, x_4 and x_5.

(c) Roadworks have closed down completely the road corresponding to x_3, and restricted the flow on the road corresponding to x_5 to 100 vehicles per hour.

   (i) Find the traffic flow on the other roads.

   (ii) In terms of the traffic-flow diagram, explain why your answer in (i) makes sense.

(d) The road engineers (who hadn't done the theoretical modelling beforehand) later realise that they could have resurfaced one other road while carrying out these roadworks, thereby saving a lot of money.
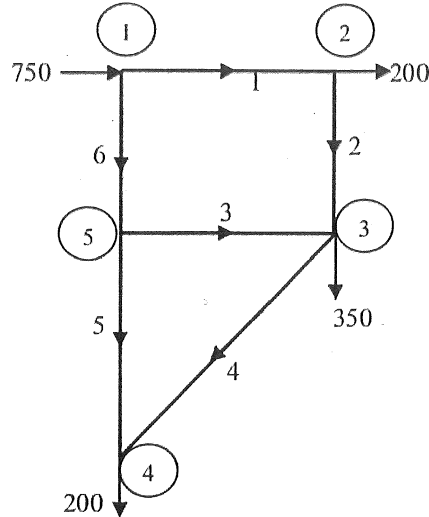
Exercise

The flow of traffic in vehicles per hour through a network of streets is shown in the figure below. The numbers in circles are intersection labels; the numbers on the roads are the road labels.

(a) Write down the system of equations that describes the traffic flow in terms of the variables x_1 to x_6.

(b) **(i)** Write the equations in matrix form.

   (ii) Convert the augmented matrix to row-echelon form using Gauss elimination.

   (iii) Write out the equations corresponding to the row-echelon form.

   (iv) Which are the free variables?

   Hint: A free variable is one in which the corresponding column of the augmented matrix does not contain a pivot (leading 1). This means that a free variable is not the first variable in any of the equations corresponding to the augmented matrix. There are then two parameters: call them t and p.

   (v) Hence find x_1 to x_6.

(c) The Road Traffic Authority is investigating the possiblity of making Road 3 a pedestrian plaza.

   (i) Is this possible? If so, find the traffic flow on the other roads.

   (ii) What other road might it make sense to close to simplify the traffic flow.
   Describe the traffic flow in this case.

PTO

(d) The Road Traffic Authority wants to close Roads 1 and 3 (this is before it becomes a pedestrian plaza) for a week-long festival.

    (i) Is this possible? If so, find the traffic flow on the other roads.

    (ii) In terms of the traffic-flow diagram, explain why your answer in (i) makes sense.

(e) The Road Traffic Authority needs to close roads 5 and 6 for an extended period for major building works.

    (i) Is this possible? If so, find the traffic flow on the other roads.

    (ii) In terms of the traffic-flow diagram, explain why your answer in (i) makes sense.

Exercise

The flow of traffic in vehicles per hour through a network of streets in a CBD is shown in the figure below.
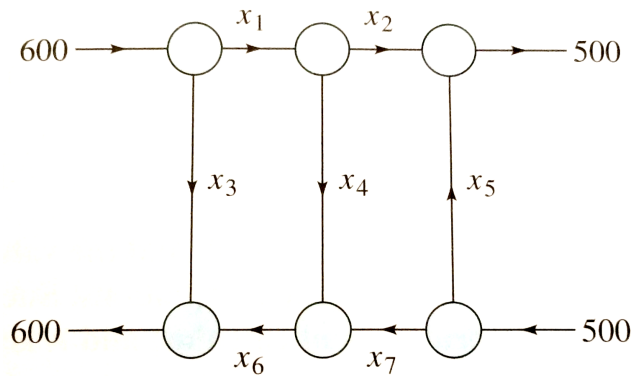

(a) Write down the system of equations that describes the traffic flow in terms of the variables x_1 to x_7.

(b)   **(i)** Write the equations in matrix form.

    (ii) Convert the augmented matrix to row-echelon form using Gauss-Jordan elimination.

    (iii) Write out the equations corresponding to the row-echelon form.

    (iv) Which are the free variables?

        There are then two parameters: call them t and p.

    (v) Hence find x_1 to x_7.

(c) Can the roads be closed one at a time for maintenance while still maintaining traffic flow?

Can drivers still pass through the CBD (enter at one side, leave at the other) in all cases?

Exercise

A section of gas pipeline has the flows (in appropriate units) shown below. Here, gas can only flow in the indicated directions.
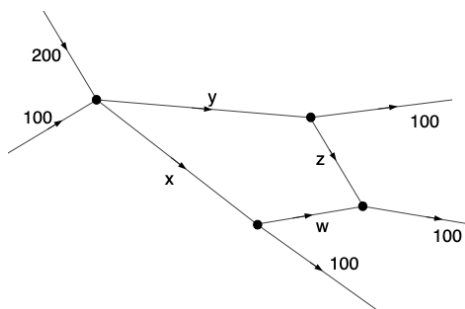
(a) Using the Linear Conservation Principle, find the flows x, y, z and w.

(b) Could the system function if a pipe was closed for maintenance so that

   (i) $z = 0$?

   (ii) $w = 0$?

(c) (i) Could the system function if the flow in the y pipe was restricted so that $y \leqslant 50$?

   (ii) Is there a minimum allowed for the flow y? Explain why your answer makes sense by referring to the diagram.

19.4.5   Solutions

Exercise   (page 90)

Find a polynomial of degree 2 that passes through the points $(2, 5)$, $(3, 2)$ and $(4, 5)$.

Define variables

Let the polynomial of degree 2 (quadratic) be $y = a_0 + a_1 x + a_2 x^2$.

Formulate the equations

Substituting the values for x and y for each point into the equation of the quadratic gives

$$
\begin{aligned}
5 &= a_0 + 2a_1 + 4a_2 & (x = 2, \, y = 5) \\
2 &= a_0 + 3a_1 + 9a_2 & (x = 3, \, y = 2) \\
5 &= a_0 + 4a_1 + 16a_2 & (x = 4, \, y = 5)
\end{aligned}
$$

Solve the equations

The augmented matrix is therefore

$$
\begin{bmatrix}
1 & 2 & 4 & 5 \\
1 & 3 & 9 & 2 \\
1 & 4 & 16 & 5
\end{bmatrix}.
$$

Gauss elimination produces the row-echelon matrix

$$
\begin{bmatrix}
1 & 2 & 4 & 5 \\
0 & 1 & 5 & -3 \\
0 & 0 & 1 & 3
\end{bmatrix}.
$$

The solution, from back substitution, is $a_2 = 3$, $a_1 = -18$ and $a_0 = 29$.

Answer the question

The required polynomial is therefore $y = 29 - 18x + 3x^2$.

Check that each point does lie on the polynomial.

Algebraically, we substitute an x value into the RHS of the polynomial and show that we get the corresponding y value.

$y(2) = 29 - 36 + 12 = 5$, as required.

$y(3) = 29 - 54 + 27 = 2$, as required.

$y(4) = 29 - 72 + 48 = 5$, as required.

Graphically, plot $y = 29 - 18x + 3x^2$ and use Trace to verify that the points lie on the curve.

Exercise   (page 86)

(a) Fit a straight line to the three points $(1, 2)$, $(2, 1)$ and $(3, 2)$.

Fitting a straight line $y = a_0 + a_1 x$ gives the equations

$$
\begin{aligned}
2 &= a_0 + a_1 \\
1 &= a_0 + 2a_1 \\
2 &= a_0 + 3a_1,
\end{aligned}
$$

in matrix form

$$
\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}
=
\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.
$$

Carrying out Gauss elimination on the matrix

$$
\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}
\quad \text{gives} \quad
\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix},
$$

showing that the equations are inconsistent: the last row gives $0a_0 + 0a_1 = 2$.

A straight line cannot be fitted exactly to these three points; they are not collinear.

(b) Fit a cubic polynomial to these three points.

In this case, write down the equation of the cubic that fits the points (it should contain a parameter) and plot the cubic for a few values of the parameter to verify that it does indeed pass through the three points.

Think about extrapolation in these cases. Is it likely to be accurate?

Fitting the data points $(1, 2)$, $(2, 1)$ and $(3, 2)$ to a general cubic

$$
y = a_0 + a_1 x + a_2 x^2 + a_3 x^3
$$

gives the equations

$$
\begin{aligned}
2 &= a_0 + a_1 + a_2 + a_3 \\
1 &= a_0 + 2a_1 + 4a_2 + 8a_3 \\
2 &= a_0 + 3a_1 + 9a_2 + 27a_3,
\end{aligned}
$$

in matrix form

$$
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}
=
\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.
$$

Gauss elimination on the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 1 \\ 1 & 3 & 9 & 27 & 2 \end{array}\right]$$

gives, using the calculator,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & 5 \\ 0 & 1 & 0 & -11 & -4 \\ 0 & 0 & 1 & 6 & 1 \end{array}\right],$$

with corresponding equations

$$a_0 + 6a_3 = 5$$

$$a_1 - 11a_3 = -4$$

$$a_2 + 6a_3 = 1.$$

a_3 is a free variable: set $a_3 = t$, $t \in \mathbb{R}$.

Then, $a_2 = 1 - 6t$, $a_1 = -4 + 11t$ and $a_0 = 5 - 6t$.

Hence the cubic is

$$y = 5 - 6t + (-4 + 11t)x + (1 - 6t)x^2 + tx^3 \qquad t \in \mathbb{R}.$$

The figure below shows the three points (generated by Plot1) and the graphs of the cubic with $t = -1$, $t = 0$ and $t = 1$, respectively. Store the value of t in the T memory before graphing.
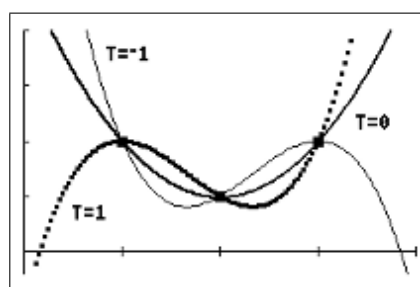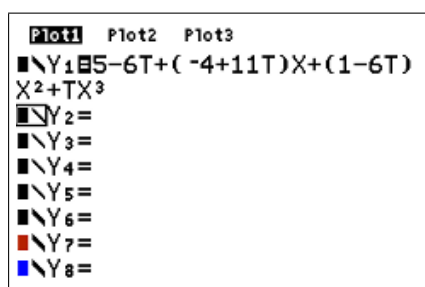
window $[0, 4, 1] \times [-0.4, 4, 1]$

All three curves fit the three given points, but clearly extrapolation is going to give quite different values, depending on which curve you choose.

For example,

$$\begin{aligned} y(0.5) &= 1.375 & t &= 1 \\ &= 3.25 & t &= 0 \\ &= 5.125 & t &= -1. \end{aligned}$$

Exercise   (page 91)

The flow of traffic (in vehicles per hour) around a Rome roundabout is shown below.

(a) Solve this system for x_1, x_2, x_3 and x_4.

Write your solution in terms of a parameter t.
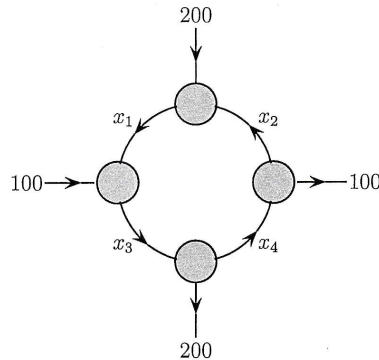
Define variables

The variables are defined in the diagram.

Formulate the equations

From conservation of vehicles at each junction (number of vehicles entering an inter-section = number of vehicles leaving the intersection), starting at the top junction and going clockwise:

$$200 + x_2 = x_1$$

$$x_4 = x_2 + 100$$

$$x_3 = x_4 + 200$$

$$x_1 + 100 = x_3.$$

Solve the equations

Re-arranging into standard form gives

$$
\begin{array}{rcrcrcrcr}
x_1 & - & x_2 & & & & & = & 200 \\
 & & x_2 & & & - & x_4 & = & -100 \\
 & & & & x_3 & - & x_4 & = & 200 \\
x_1 & & & - & x_3 & & & = & -100
\end{array}
$$

The augmented matrix is therefore
$$
\left[
\begin{array}{cccc|c}
1 & -1 & 0 & 0 & 200 \\
0 & 1 & 0 & -1 & -100 \\
0 & 0 & 1 & -1 & 200 \\
1 & 0 & -1 & 0 & -100
\end{array}
\right].
$$

Gauss elimination produces the row-echelon matrix

$$
\begin{bmatrix}
1 & -1 & 0 & 0 & 200 \\
0 & 1 & 0 & -1 & -100 \\
0 & 0 & 1 & -1 & 200 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

There is no leading 1 corresponding to x_4, so we let $x_4 = t$, where $t \in \mathbb{R}$.

Then, $x_3 = 200 + t$, $x_2 = -100 + t$, $x_1 = 100 + t$.

(b) For what values of t is the solution realistic, i.e. the traffic is one way in the direction of the arrows.

For one-way traffic in the direction of the arrows, all variables must be positive or zero. For x_4, we require $t \geqslant 0$; for x_3, $t \geqslant -200$; for x_2, $t \geqslant 100$; for x_1, $t \geqslant -100$.

The t values that satisfy all these are $t \geqslant 100$.

(c) Find the traffic flow if roadworks block the road corresponding to x_2.

If the road corresponding to x_2 is blocked, $x_2 = 0$. Therefore, $t = 100$, giving $x_4 = 100$, $x_3 = 300$ and $x_1 = 200$. These are all possible values, so the roundabout can cope when the road corresponding to x_2 is blocked.

(d) What happens if a group of demonstrators blocks the road corresponding to x_4? How could the traffic authorities fix this problem temporarily?

If the road corresponding to x_4 is blocked, $x_4 = 0$. Therefore, $t = 0$, giving $x_3 = 200$, $x_2 = -100$ and $x_1 = 100$.

The value for x_2 is negative, meaning the traffic must go along this part of the road in the opposite direction to normal. The roundabout cannot function as normal.

The problem could be fixed temporarily by allowing traffic to flow in the opposite direction to the arrow for x_2.

Exercise   (page 90)

The flow of traffic in vehicles per hour through a network of streets is shown in the figure below.
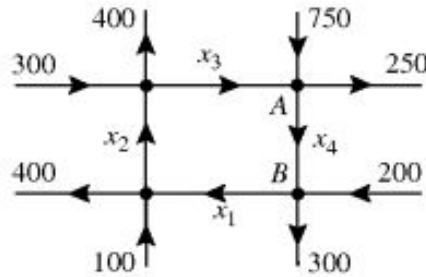
(a) Write down the system of equations that describes the traffic flow and convert to matrix form.

Equating inputs (LHS below) and outputs (RHS below) at each intersection, starting at the bottom left intersection and going clockwise gives the following equations.

$$100 + x_1 = 400 + x_2 \quad \text{or} \quad x_1 - x_2 = 300.$$

$$300 + x_2 = 400 + x_3 \quad \text{or} \quad x_2 - x_3 = 100.$$

$$750 + x_3 = 250 + x_4 \quad \text{or} \quad x_3 - x_4 = -500.$$

$$200 + x_4 = 300 + x_1 \quad \text{or} \quad x_1 - x_4 = -100.$$

In matrix form,

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 300 \\ 100 \\ -500 \\ -100 \end{bmatrix}.$$

(b)   **(i)** Convert the matrix to row-echelon form.
The augmented matrix is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 1 & 0 & 0 & -1 & -100 \end{bmatrix}.$$

Gauss elimination gives the row-echelon form

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(ii) Hence find x_1, x_2, x_3 and x_4.

We know there is an infinite number of solutions because of the bottom row of zeroes. The free variable (no corresponding pivot) is x_4.

Let $x_4 = p \in \mathbb{R}$.

Then, from Row 3, $x_3 - x_4 = -500$, so that $x_3 = p - 500$.

From Row 2, $x_2 - x_3 = 100$, so that $x_3 = p - 400$.

From Row 1, $x_1 - x_2 = 300$, so that $x_1 = p - 100$.

Therefore, the solution is

$$\begin{aligned} x_1 &= p - 100 \\ x_2 &= p - 400 \\ x_3 &= p - 500 \\ x_4 &= p \qquad p \in \mathbb{R}. \end{aligned}$$

(c) The local traffic authority wants the loop $x_1 \to x_2 \to x_3 \to x_4$ to function as a one-way rectangular 'roundabout' (rectabout?) in the direction shown by the arrows in the figure.

For what values of the parameter in your solution in (b)(ii) is this possible?

For the rectabout to function as shown by the arrows, we require all flows to be non-negative. From x_3, this requires $p \geqslant 500$.

Where on the rectabout is the traffic flow greatest?

With $p = 500$, $x_1 = 400$, $x_2 = 100$, $x_3 = 0$ and $x_4 = 500$, so the x_4 flow, at 500 vehicles per hour, is the greatest.

(d) The traffic authority wants to close down completely for resealing the road between the intersections A and B, i.e. that corresponding to x_4.

Can traffic still flow in the network? How is this different from normal operation?

We have $x_4 = p = 0$, so that $x_1 = -100$, $x_2 = -400$, $x_3 = -500$ and $x_4 = 0$. The negative signs mean that the traffic flows x_1, x_2 and x_3 are in the opposite direction to the arrows.

It is therefore possible to close down completely the road between the intersections A and B but the rectabout would have traffic going in the opposite direction to normal.

Exercise   (page 91)

The flow of traffic in vehicles per hour through a network of streets is shown in the figure below. A, B, C and D are intersections.
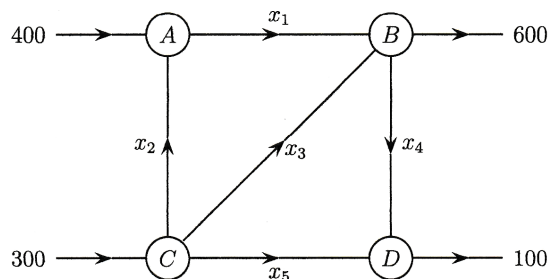
(a) Write down the system of equations that describes the traffic flow and and convert to matrix form.

Equating inputs and outputs at each intersection gives:

Intersection A: $400 + x_2 = x_1$, or $x_1 - x_2 = 400$.

Intersection B: $x_1 + x_3 = x_4 + 600$, or $x_1 + x_3 - x_4 = 600$.

Intersection C: $300 = x_2 + x_3 + x_5$, or $x_2 + x_3 + x_5 = 300$.

Intersection D: $x_4 + x_5 = 100$.

Putting the equations together gives

$$
\begin{aligned}
x_1 + x_2 &= 400 \\
x_1 + x_3 - x_4 &= 600 \\
x_2 + x_3 + x_5 &= 300 \\
x_4 + x_5 &= 100.
\end{aligned}
$$

In matrix form,

$$
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
1 & 0 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5
\end{bmatrix}
=
\begin{bmatrix}
400 \\ 600 \\ 300 \\ 100
\end{bmatrix}.
$$

(b)   **(i)** Convert the matrix to row-echelon form.

The augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array}\right].$$

Gauss elimination gives the row-echelon form

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

   (ii) Explain why x_3 and x_5 are free variables.

There are no pivots (leading 1s) in Columns 3 and 5, corresponding to x_3 and x_5.

   (iii) Hence find x_1, x_2, x_3, x_4 and x_5.

Let $x_3 = p \in \mathbb{R}$, $x_5 = t \in \mathbb{R}$.

Then, from Row 3, $x_4 + x_5 = 100$, so that $x_4 = 100 - t$.

From Row 2, $x_2 + x_3 - x_4 = 200$.
Therefore, $x_2 = 200 - x_3 + x_4 = 200 - p + (100 - t) = 300 - p - t$.

From Row 1, $x_1 - x_2 = 400$, so that $x_1 = 400 + x_2 = 400 + 300 - p - t = 700 - p - t$.

Therefore, the solution is

$$
\begin{aligned}
x_1 &= 700 - p - t \\
x_2 &= 300 - p - t \\
x_3 &= p \\
x_4 &= 100 - t \\
x_5 &= t, \qquad p, t \in \mathbb{R}.
\end{aligned}
$$

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(c) Roadworks have closed down completely the road corresponding to x_3, and restricted the flow on the road corresponding to x_5 to 100 vehicles per hour.

(i) Find the traffic flow on the other roads.

We have $x_3 = p = 0$ and $x_5 = t = 100$.
Therefore,

$$
\begin{aligned}
x_1 &= 600 \\
x_2 &= 200 \\
x_3 &= 0 \\
x_4 &= 0 \\
x_5 &= 100.
\end{aligned}
$$

(ii) In terms of the traffic-flow diagram, explain why your answer in (i) makes sense.

Of the 300 vehicles entering Intersection C, only 100 can go to Intersection D $(x_5 = 100)$.

The other 200 must go to Intersection A $(x_2 = 200)$, adding to the 400 entering that intersection.

These 600 vehicles then proceed to Intersection B $(x_1 = 600)$ and exit there.

(d) The road engineers (who hadn't done the theoretical modelling beforehand) later realise that they could have resurfaced one other road while carrying out these roadworks, thereby saving a lot of money.

Which road was that and why?

We have $x_4 = 0$, so the road between Intersections B and D also carries no traffic.

Exercise   (page 92

The flow of traffic in vehicles per hour through a network of streets is shown in the figure below. The numbers in circles are intersections, the numbers on the roads, the road labels.
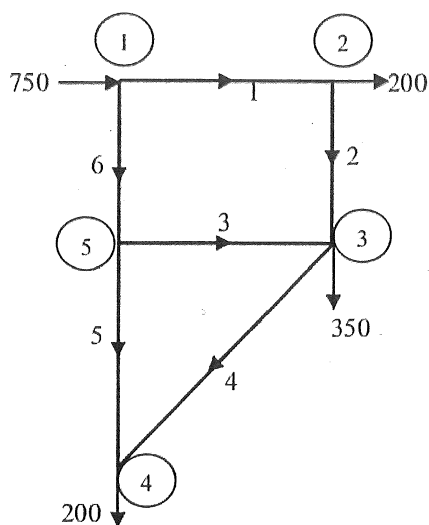
(a) Write down the system of equations that describes the traffic flow in terms of the variables x_1 to x_6.

Equating inputs (LHS below) and outputs (RHS below) at each intersection in numerical order gives:

Intersection 1: $750 = x_1 + x_6$ or $x_1 + x_6 = 750$.

Intersection 2: $x_1 = x_2 + 200$ or $x_1 - x_2 = 200$.

Intersection 3: $x_2 + x_3 = x_4 + 350$ or $x_2 + x_3 - x_4 = 350$.

Intersection 4: $x_4 + x_5 = 200$.

Intersection 5: $x_6 = x_3 + x_5$ or $x_3 + x_5 - x_6 = 0$.

Putting the equations together gives

$$x_1 + x_6 = 750$$

$$x_1 - x_2 = 200$$

$$x_2 + x_3 - x_4 = 350$$

$$x_4 + x_5 = 200$$

$$x_3 + x_5 - x_6 = 0.$$

(b) **(i)** Write the equations in matrix form.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6
\end{bmatrix}
=
\begin{bmatrix}
750 \\ 200 \\ 350 \\ 200 \\ 0
\end{bmatrix}.
$$

(ii) Convert the augmented matrix to row-echelon form using Gauss elimination.

The augmented matrix is

$$
\left[
\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 0 & 1 & 750 \\
1 & -1 & 0 & 0 & 0 & 0 & 200 \\
0 & 1 & 1 & -1 & 0 & 0 & 350 \\
0 & 0 & 0 & 1 & 1 & 0 & 200 \\
0 & 0 & 1 & 0 & 1 & -1 & 0
\end{array}
\right],
$$

with Gauss elimination giving the row-echelon form

$$
\left[
\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 0 & 1 & 750 \\
0 & 1 & 0 & 0 & 0 & 1 & 550 \\
0 & 0 & 1 & -1 & 0 & -1 & -200 \\
0 & 0 & 0 & 1 & 1 & 0 & 200 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\right].
$$

(iii) Write out the equations corresponding to the row-echelon form.

$$
\begin{aligned}
x_1 + x_6 &= 750 \\
x_2 + x_6 &= 550 \\
x_3 - x_4 - x_6 &= -200 \\
x_4 + x_5 &= 200.
\end{aligned}
$$

(iv) Which are the free variables?

No equation starts with x_5 or x_6 (no leading 1s in the corresponding 5th and 6th columns in the row-echelon form): these are the free variables.

PTO

(v) Hence find x_1 to x_6.

Let $x_5 = t$, $x_6 = p$; $p, t \in \mathbb{R}$.

From the last equation, $x_4 = 200 - x_5 = 200 - t$.

From the third equation, $x_3 = -200 + x_4 + x_6 = -200 + (200 - t) + p = p - t$.

From the second equation, $x_2 = 550 - x_6 = 550 - p$.

From the first equation, $x_1 = 750 - x_6 = 750 - p$.

Therefore, the solution is

$$\begin{bmatrix} 750 - p \\ 550 - p \\ p - t \\ 200 - t \\ t \\ p \end{bmatrix}.$$

(c) The Road Traffic Authority is investigating the possiblitiy of making Road 3 a pedestrian plaza.

(i) Is this possible? If so, find the traffic flow on the other roads.

To close Road 3 means $p = t$.

Therefore, we have for the x_i,

$$\begin{bmatrix} 750 - p \\ 550 - p \\ 0 \\ 200 - p \\ p \\ p \end{bmatrix}.$$

Therefore, this arrangement is possible; there is still a free parameter to adjust the flow of traffic on the other roads.

(ii) What other road might it make sense to close to simplify the traffic flow. Describe the traffic flow in this case.

It seems sensible to have $x_4 = 0$ too, so that vehicles either turn right at Intersection 1 if they want to exit at Intersection 4 or they go $1 \to 2 \to 3$, exiting at the appropriate intersection.

$x_4 = 0$ means $p = 200$, giving this behaviour.

PTO

(d) The Road Traffic Authority wants to close Roads 1 and 3 (this is before it becomes a pedestrian plaza) for a week-long festival.

  (i) Is this possible? If so, find the traffic flow on the other roads.

  To close Road 3 means $p=t$, and to close Road 1 means $p=750$.
  Therefore, we have for the x_i,

  $$\begin{bmatrix} 0 \\ -200 \\ 0 \\ -550 \\ 750 \\ 750 \end{bmatrix}.$$

  Therefore, this arrangement is possible.

  (ii) In terms of the traffic-flow diagram, explain why your answer in (i) makes sense.

  The 750 vehicles entering the system at Intersection 1 all turn right and continue on to Intersection 4.

  200 exit here, and the remaining 550 go to Intersection 3 in a direction opposite to the arrow for x_4.

  350 exit at Intersection 3 and the remaining 200 go on to Intersection 2, where they exit.

  Hence we have a series of one-way streets: $1 \to 5 \to 4 \to 3 \to 2$, all of which is possible.

(d) The Road Traffic Authority needs to close Roads 5 and 6 for an extended period for major building works.

  (i) Is this possible? If so, find the traffic flow on the other roads.

  To close roads 5 and 6 means $p = t = 0$.
  Therefore, we have for the x_i,

 $$\begin{bmatrix} 750 \\ 550 \\ 0 \\ 200 \\ 0 \\ 750 \end{bmatrix}.$$

  Therefore, this arrangement is possible.

  (ii) In terms of the traffic-flow diagram, explain why your answer in (i) makes sense.

  The traffic again follows a one-way route, this time $1 \to 2 \to 3 \to 4$, the opposite direction to the previous case, with the appropriate numbers exiting at Intersections 2, 3 and 4.

Exercise   (page 93)

The flow of traffic in vehicles per hour through a network of streets in a CBD is shown in the figure below.
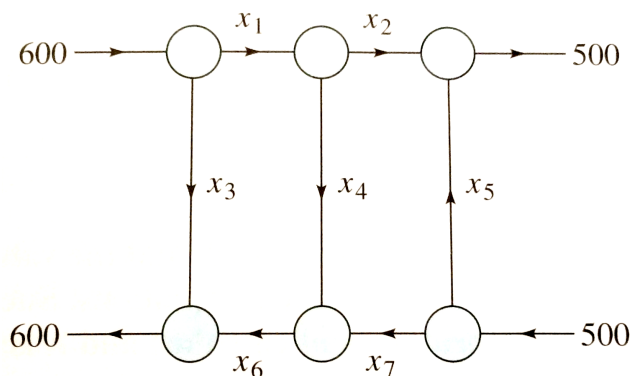
(a) Write down the system of equations that describes the traffic flow in terms of the variables x_1 to x_7.

Equating inputs (LHS below) and outputs (RHS below) at each intersection, strating top left and going clockwise gives:

Intersection 1: $600 = x_1 + x_3$ or $x_1 + x_3 = 600$.

Intersection 2: $x_1 = x_2 + x_4$ or $x_1 - x_2 - x_4 = 0$.

Intersection 3: $x_2 + x_5 = 500$.

Intersection 4: $500 = x_5 + x_7$ or $x_5 + x_7 = 500$.

Intersection 5: $x_4 + x_7 = x_6$ or $x_4 - x_6 + x_7 = 0$.

Intersection 6: $x_3 + x_6 = 600$.

Putting the equations together gives

$$
\begin{aligned}
x_1 \; + \; x_3 \; &= \; 650 \\
x_1 \; - \; x_2 \; - \; x_4 \; &= \; 0 \\
x_2 \; + \; x_5 \; &= \; 500 \\
x_5 \; + \; x_7 \; &= \; 500 \\
x_4 \; - \; x_6 \; + \; x_7 \; &= \; 0 \\
x_3 \; + \; x_6 \; &= \; 600.
\end{aligned}
$$

(b)   **(i)** Write the equations in matrix form.

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7
\end{bmatrix}
=
\begin{bmatrix}
600 \\ 0 \\ 500 \\ 0 \\ 600 \\ 500
\end{bmatrix}.
$$

(ii) Convert the augmented matrix to row-echelon form using Gauss-Jordan elimination.

The augmented matrix is

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 600 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 500
\end{bmatrix}
$$

with Gauss-Jordan elimination giving the row-echelon form

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

(iii) Write out the equations corresponding to the row-echelon form.

$$
\begin{aligned}
x_1 - x_6 &= 0 \\
x_2 - x_7 &= 0 \\
x_3 + x_6 &= 600 \\
x_4 - x_6 + x_7 &= 0 \\
x_5 + x_7 &= 500.
\end{aligned}
$$

(iv) Which are the free variables?

No equation starts with x_6 or x_7 (no leading 1s in the corresponding 6th and 7th columns in the row-echelon form): these are the free variables.

(v) Hence find x_1 to x_7.

Let $x_6 = t$, $x_7 = p$; $p, t \in \mathbb{R}$.
From the first equation, $x_1 = x_6 = t$.
From the second equation, $x_2 = x_7 = p$.
From the third equation, $x_3 = 600 - x_6 = 600 - t$.
From the fourth equation, $x_4 = x_6 - x_7 = t - p$.
From the fifth equation, $x_5 = 500 - x_7 = 500 - p$.

Therefore, the solution is

$$\begin{bmatrix} t \\ p \\ 600-t \\ t-p \\ 500-p \\ t \\ p \end{bmatrix}.$$

(c) Can the roads be closed one at a time for maintenance while still maintaining traffic flow?

Can drivers still pass through the CBD (enter at one side, leave at the other) in all cases?

Closing the first road, setting $x_1 = 0$, means $t = 0$. This means $x_6 = 0$ also, so we can do these two roads at once.

The solution in this case is

$$\begin{bmatrix} 0 \\ p \\ 600 \\ -p \\ 500-p \\ 0 \\ p \end{bmatrix}.$$

Though traffic can still flow, the two halves of the CBD are effectively isolated. Traffic entering top left can only turn right immediately twice and head back out in the direction they came from on a parallel road. Traffic entering from the right can turn right at either of the firsdt two intersections but must then turn right again and head back out in the direction they came from.

Closing the second road, setting $x_2 = 0$, means $p = 0$. This means $x_7 = 0$ also, so we can again do two roads at once. This is just the mirror image of the first case, with drivers entering from the right leaving back that way, having no choice of roads, while drivers entering from the right have a choice of two intersections at which to turn right but then must return in the direction from which they came.

Setting $x_4 = 0$ means $t = p$, with solution

$$\begin{bmatrix} t \\ t \\ 600-t \\ 0 \\ 500-t \\ t \\ t \end{bmatrix}.$$

Traffic can flow freely in most directions and drivers can transit through the CBD.

The other two cases, $x_3 = 0$ and $x_5 = 0$ are similar, as one would expect from the diagram.

Exercise   (page 94)

A section of gas pipeline has the flows (in appropriate units) shown below. Gas can only flow in the indicated directions. vspace-3mm
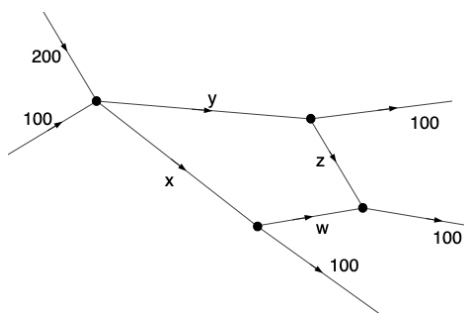
(a) Using the Linear Conservation Principle, find the flows x, y, z and w.

   Define variables

   The variables are defined in the diagram.

   Formulate the equations

   Equating inputs (LHS) and outputs (RHS) at each junction, clockwise from top left, gives

$$\begin{aligned} 200 + 100 &= x + y \\ y &= z + 100 \\ z + w &= 100 \\ x &= w + 100. \end{aligned}$$

Solve the equations

Rewriting the equations systematically gives

$$
\begin{aligned}
x + y &= 300 \\
y - z &= 100 \\
z + w &= 100 \\
x - w &= 100,
\end{aligned}
$$

with augmented matrix (be careful to put in 0 for missing terms)

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 300 \\
0 & 1 & -1 & 0 & 100 \\
0 & 0 & 1 & 1 & 100 \\
1 & 0 & 0 & -1 & 100
\end{bmatrix}.
$$

Gauss elimination converts this to the row-echelon form

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 300 \\
0 & 1 & -1 & 0 & 100 \\
0 & 0 & 1 & 1 & 100 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Back substitution gives $w = t$, $z = 100 - t$, $y = 200 - t$ and $x = 100 + t$, $\;t \in \mathbb{R}$.

Answer the question

The flows are given by $x = 100 + t$, $y = 200 - t$, $z = 100 - t$ and $w = t$, where t is an arbitrary number.

As we want flows here to be non-negative, we must have $0 \leqslant t \leqslant 100$.

(b) Could the system function if a pipe was closed for maintenance so that

    (i) $z = 0$?

      If $z = 0$, we have $t = 100$, and all flows are still non-negative.
      The system still functions.

    (ii) $w = 0$?

      If $w = 0$, we have $t = 0$, and again all flows are still non-negative.
      The system still functions.

(c)   **(i)** Could the system function if the flow in the y pipe was restricted so that $y \leqslant 50$?

      If $y \leqslant 50$, we have $t \geqslant 150$. In this case, z would be negative, which is not allowed.
      The system will not function if $y \leqslant 50$.

    (ii) Is there a minimum allowed for the flow y? Explain why your answer makes sense by referring to the diagram.

      The maximum allowable value for t here is 100, otherwise $z < 0$. This means the minimum possible value for y is 100.

      This makes sense when you look at the flow diagram. The only way for the top output to be 100 is if there is a flow of at least 100 through y.

19.5   Linear Transformations

Our original aim was to extend the single equation in one variable $ax = b$ to systems of linear equations in multiple variables; we did that by defining $\underline{\underline{A}}\underline{x} = \underline{b}$, where $\underline{\underline{A}}$ and $\underline{b}$ are known and $\underline{x}$ is to be found.

Instead of thinking of $\underline{b}$ as a given vector and $\underline{x}$ the unknown, we could consider the case where $\underline{x}$ is given. $\underline{\underline{A}}\underline{x}$ then generates a new vector $\underline{b}$; vector $\underline{x}$ is transformed into vector $\underline{b}$ by the matrix $\underline{\underline{A}}$.

These are *linear transformations*. Writing out the equations in full (e.g. see Section 19.5.4) shows that the components of the new vector are just linear combinations or sums of the components of the first vector; no squares, cubes, exponentials, etc are involved.

Linear transformations are common when vector and matrix theory is applied. We shall look at the case of operations on position vectors in a plane.

Let $\underline{\underline{T}}$ be a matrix that operates on a 2D position vector $\underline{r}_1$ to transform it into $\underline{r}_2$:

$$\underline{r}_2 = \underline{\underline{T}}\underline{r}_1 \quad \text{or} \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Our task is to find the matrices $\underline{\underline{T}}$ that describe important geometric operations.
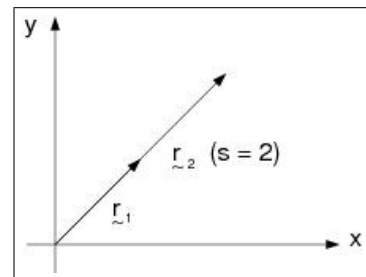
19.5.1   Simplest case: Scaling

The matrix that scales (magnifies) a vector by s is

$$\underline{\underline{S}}(s) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} = s\,\underline{\underline{I}}.$$

$$\underline{r}_2 = \underline{\underline{S}}(s)\,\underline{r}_1 = s\,\underline{\underline{I}}\,\underline{r}_1 = s\underline{r}_1.$$

Example
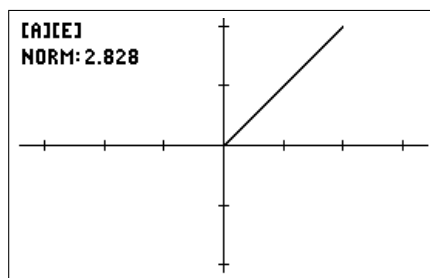
$$\underline{\underline{S}}(2)\,\underline{r}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Note: The inverse of $\underline{\underline{S}}(s)$ is $\underline{\underline{S}}(1/s)$: $\underline{\underline{S}}(s)\underline{\underline{S}}(1/s) = \underline{\underline{I}}$, $s \neq 0$.

Check using your calculator with $\underline{\underline{S}}(2)$ in [A]. In the TRANSFRM program, vector $\underset{\sim}{r}_1$ is stored in matrix [E] and displayed, with label and norm, in blue at the top left of the screen. The transformed vector [A] [E], label and norm are in black.
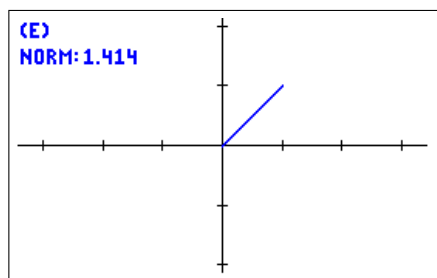
Exercises

Solutions to the exercises start on page 125.

Use your calculator to generate the following vectors, given $\underset{\sim}{r}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $\underline{\underline{S}}(2)\,\underset{\sim}{r}_1$ **(b)** $\underline{\underline{S}}(-1)\,\underset{\sim}{r}_1$ **(c)** $\underline{\underline{S}}(-2)\,\underset{\sim}{r}_1$.

19.5.2   Reflections

Let $\underline{\underline{M}}_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. What is the effect of $\underline{\underline{M}}_x$ on a general position vector $\underset{\sim}{r}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$?

$\underset{\sim}{r}_2 = \underline{\underline{M}}_x \underset{\sim}{r}_1$ in component form is $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ -y_1 \end{bmatrix}$.

So $\underline{\underline{M}}_x$ changes y into $-y$. It acts to reflect or **mirror** the vector in the x axis.
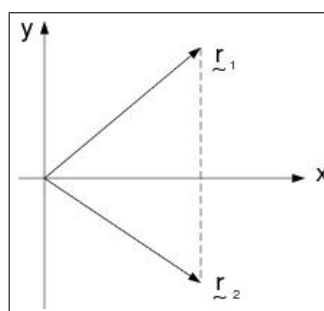
Example

Given $\underset{\sim}{r}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (stored in [E]), plot on your calculator $\underline{\underline{M}}_x \underset{\sim}{r}_1$.
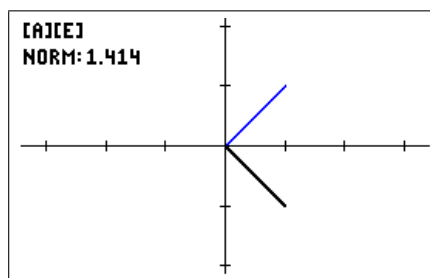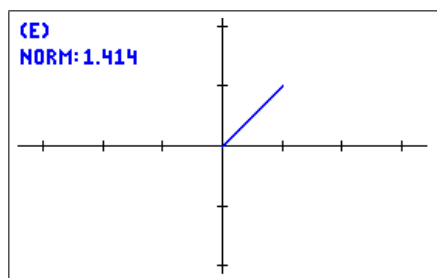
Similarly we find:

Words	**Picture**	**Algebra**
Reflect the vector in the y axis.		$\mathbf{r}_2 = \mathbf{M}_y\, \mathbf{r}_1,$ where $\mathbf{M}_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$
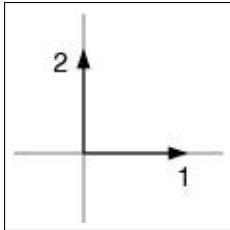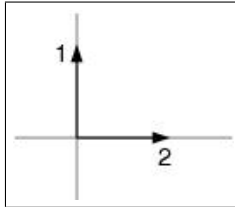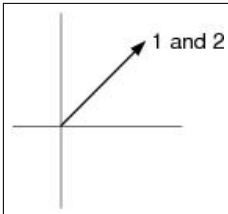
Check: $\mathbf{M}_y\, \mathbf{r}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ y_1 \end{bmatrix}.$
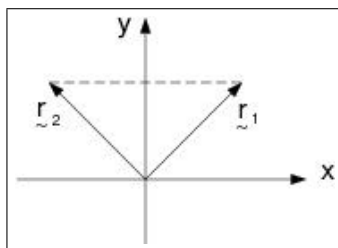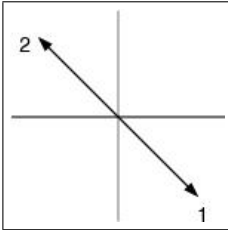
How about $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

Try a few cases $\mathbf{M}\, \mathbf{r}_1$, with:

$$\mathbf{r}_1 = a\, \mathbf{i} \qquad \begin{bmatrix} a \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ a \end{bmatrix}$$

$$\mathbf{r}_1 = a\, \mathbf{j} \qquad \begin{bmatrix} 0 \\ a \end{bmatrix} \rightarrow \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\mathbf{r}_1 = b\, \mathbf{i} + b\, \mathbf{j} \qquad \begin{bmatrix} b \\ b \end{bmatrix} \rightarrow \begin{bmatrix} b \\ b \end{bmatrix}$$

$$\mathbf{r}_1 = c\, \mathbf{i} - c\, \mathbf{j} \qquad \begin{bmatrix} c \\ -c \end{bmatrix} \rightarrow \begin{bmatrix} -c \\ c \end{bmatrix}$$

We conclude that $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M_{x\leftrightarrow y}$ describes or *represents* the operation of reflection in the line $y = x$.

Aside: *Could you prove that in general*, i.e. go beyond checking a few special cases as we did, to show it is a general result?

More generally, the matrix describing reflections in the line through the origin making an angle θ with the x axis is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}.$$

Exercises

Use your calculator to generate the following vectors, given $r_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $M_x\, r_1$ **(b)** $M_y\, r_1$ **(c)** $M_{x\leftrightarrow y}\, r_1$ **(d)** reflect r_1 in the line $y = -x$.

Observations about reflection matrices

1. Reflecting a vector twice through a mirror gets back the original vector.

   Mathematically, $M\,M r_1 = r_1$.

   So $M\,M = I$.
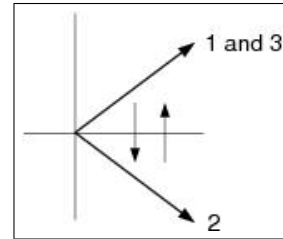
A reflection matrix M is its own inverse: $M^{-1} = M$.

Check: $M_{x\leftrightarrow y}\, M_{x\leftrightarrow y} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Note that we used a geometric argument to deduce a property of matrices *representing* a *geometric operation*.

2. For any reflection or mirror operation, there are two special vectors:

   one is not changed at all: $M v_1 = v_1$;

   the other just reverses direction: $M v_2 = -v_2$.

Example: For M_x

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
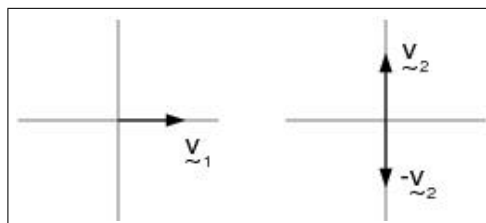
Example: For $\underline{\underline{M}}_{x \leftrightarrow y}$

$$\underset{\sim}{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \underset{\sim}{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$
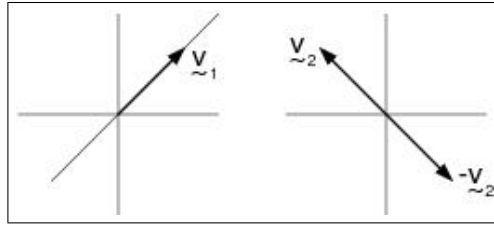
Of course we can replace the 1s by any number a.

19.5.3   Rotations

Instead of trying out special cases, I will now give you the general formula. We will then prove that it is correct, see a few examples and explore the algebra a little.

Theorem

The matrix that represents the operation of rotating a vector by an angle α in the anti-clockwise direction is the rotation matrix

$$\underline{\underline{R}}(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

Proof

Let $\underset{\sim}{r_1}$ be a vector with length or norm
$\|\underset{\sim}{r_1}\| = r$ and at an angle θ to the x axis.

Then $\underset{\sim}{r_1} = r\cos(\theta)\,\underset{\sim}{i} + r\sin(\theta)\,\underset{\sim}{j}$,

or $\underset{\sim}{r_1} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}.$
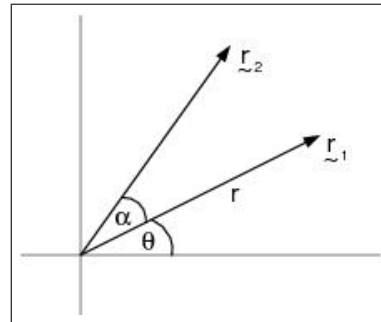
We now operate on $\underset{\sim}{r_1}$ with the matrix $\underline{\underline{R}}(\alpha)$.

$$\underline{\underline{R}}(\alpha)\,\underset{\sim}{r_1} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} r\big(\cos\alpha\cos\theta - \sin\alpha\sin\theta\big) \\ r\big(\sin\alpha\cos\theta + \cos\alpha\sin\theta\big) \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\big(\alpha+\theta\big) \\ r\sin\big(\alpha+\theta\big) \end{bmatrix} \qquad \text{using trigonometric identities}$$

$$= \underset{\sim}{r_2}.$$

We see that $\underset{\sim}{r_2}$ has exactly the same form as $\underset{\sim}{r_1}$, but the angle θ has been increased to $\theta+\alpha$. This says that $\underset{\sim}{r_1}$ has been rotated through an angle α to get $\underset{\sim}{r_2}$, so $\underline{\underline{R}}(\alpha)$ is the rotation matrix for this operation.

Remember: This describes an anti-clockwise rotation. If the angle α is *negative*, the rotation will be *clockwise* through an angle $|\alpha|$.

Examples

1. **Specifying a rotation matrix**

   The matrix that rotates vectors by 90° or $\pi/2$ radians is

$$\underline{\underline{R}}\,(\pi/2) \;=\; \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

   The matrix that rotates vectors by $\pi/3$ radians or 60° is

$$\underline{\underline{R}}\,(\pi/3) \;=\; \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}.$$

2. **Identifying a rotation matrix**

   Suppose $\underline{\underline{A}} \;=\; \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}.$

   Because $-1/2 = \sin(-\pi/6)$ and $\sqrt{3}/2 = \cos(-\pi/6)$, we can identify $\underline{\underline{A}}$ as the matrix representing a rotation by $-\pi/6$ or $-30°$, so that $\underline{\underline{A}} = \underline{\underline{R}}\,(-\pi/6)$.

   This is actually a rotation by $+30°$, but clockwise rather than anti-clockwise.

3. **Using a rotation matrix**

   Find the vector $\underset{\sim}{r_2}$ produced when $\underset{\sim}{r_1} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$ is rotated by 45° or $\pi/4$ radians.

$$\underset{\sim}{r_2} \;=\; \underline{\underline{R}}\,(\pi/4)\,\underset{\sim}{r_1} \;=\; \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 6 \\ 10 \end{bmatrix} \;=\; \begin{bmatrix} -2\sqrt{2} \\ 8\sqrt{2} \end{bmatrix}.$$

   You could check this by making sure the length of the vector has not been changed:

$$\|\underset{\sim}{r_1}\| \;=\; \sqrt{6^2+10^2} \;=\; \sqrt{136}\,;$$

$$\|\underset{\sim}{r_2}\| \;=\; \sqrt{(-2\sqrt{2})^2+(8\sqrt{2})^2} \;=\; \sqrt{136} \;=\; \|\underset{\sim}{r_1}\|.$$

Exercises

Use your calculator to generate the following vectors, given $\underset{\sim}{r_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $\underline{\underline{R}}\,(\pi/4)\,\underset{\sim}{r_1}$ (b) $\underline{\underline{R}}\,(-\pi/3)\,\underset{\sim}{r_1}$ (c) $\underline{\underline{R}}\,(3\pi/2)\,\underset{\sim}{r_1}$ (d) $\underline{\underline{R}}\,(5\pi/3)\,\underset{\sim}{r_1}.$

19.5.4   Properties of rotation matrices

Because $\underline{\underline{R}}(\alpha)$ represents a geometric operation, we should be able to use the geometric results to predict properties of rotation matrices.

1. **Inverses:** What is $\underline{\underline{R}}^{-1}(\alpha)$?

   Geometrically, if we rotate a vector by α and then by $-\alpha$, we get back to where we started, i.e.

$$\text{first we get}\quad \underset{\sim}{r}_2 \;=\; \underline{\underline{R}}(\alpha)\,\underset{\sim}{r}_1,$$

$$\text{then}\quad \underset{\sim}{r}_3 \;=\; \underline{\underline{R}}(-\alpha)\,\underset{\sim}{r}_2 \;=\; \underline{\underline{R}}(-\alpha)\,\underline{\underline{R}}(\alpha)\,\underset{\sim}{r}_1,$$

$$\text{so that}\quad \underset{\sim}{r}_3 \;=\; \underset{\sim}{r}_1.$$

   This *suggests* that:

   > The inverse of $\underline{\underline{R}}(\alpha)$ is $\underline{\underline{R}}(-\alpha)$: $\underline{\underline{R}}^{-1}(\alpha) = \underline{\underline{R}}(-\alpha).$

   Proof: Remembering that $\cos(-\alpha) = \cos(\alpha)$ and $\sin(-\alpha) = -\sin(\alpha)$:

$$\underline{\underline{R}}(-\alpha)\,\underline{\underline{R}}(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \underline{\underline{I}}.$$

   Exercise: For the second part of the proof, show that $\underline{\underline{R}}(\alpha)\underline{\underline{R}}(-\alpha) = \underline{\underline{I}}$.

2. **Doing two rotations**

   If we rotate a vector by α_1 then by α_2, *geometrically* we get the same result as if we did one single rotation by $\alpha_1+\alpha_2$. This *suggests* that

   > $$\underline{\underline{R}}(\alpha_2)\,\underline{\underline{R}}(\alpha_1) = \underline{\underline{R}}(\alpha_1+\alpha_2).$$

Proof: A simple application of trigonometric identities.

$$\underset{=}{\boldsymbol{R}}(\alpha_2)\,\underset{=}{\boldsymbol{R}}(\alpha_1) \;=\; \begin{bmatrix} \cos\alpha_2 & -\sin\alpha_2 \\ \sin\alpha_2 & \cos\alpha_2 \end{bmatrix} \begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha_2\cos\alpha_1 - \sin\alpha_2\sin\alpha_1 & -\cos\alpha_2\sin\alpha_1 - \sin\alpha_2\cos\alpha_1 \\ \sin\alpha_2\cos\alpha_1 + \cos\alpha_2\sin\alpha_1 & -\sin\alpha_2\sin\alpha_1 + \cos\alpha_2\cos\alpha_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) \end{bmatrix}$$

$$= \underset{=}{\boldsymbol{R}}(\alpha_1+\alpha_2).$$

Examples

(a) $\underset{=}{\boldsymbol{R}}(\pi/3)\,\underset{=}{\boldsymbol{R}}(\pi/6) = \underset{=}{\boldsymbol{R}}(\pi/2)$.

Check:

$$\underset{=}{\boldsymbol{R}}(\pi/3)\,\underset{=}{\boldsymbol{R}}(\pi/6) \;=\; \begin{bmatrix} \dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \end{bmatrix} \begin{bmatrix} \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \underset{=}{\boldsymbol{R}}(\pi/2),$$

and similarly $\underset{=}{\boldsymbol{R}}(\pi/6)\,\underset{=}{\boldsymbol{R}}(\pi/3) = \underset{=}{\boldsymbol{R}}(\pi/2)$.

(b) $\underset{=}{\boldsymbol{R}}(-\pi/3)\,\underset{=}{\boldsymbol{R}}(\pi/3) = \underset{=}{\boldsymbol{R}}(0)$.

Check:

$$\underset{=}{\boldsymbol{R}}(-\pi/3)\,\underset{=}{\boldsymbol{R}}(\pi/3) \;=\; \begin{bmatrix} \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ -\dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \end{bmatrix} \begin{bmatrix} \dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underset{=}{\boldsymbol{R}}(0) = \underset{=}{\boldsymbol{I}},$$

verifying the fact that $\underset{=}{\boldsymbol{R}}^{-1}(\pi/3) = \underset{=}{\boldsymbol{R}}(-\pi/3)$.

Exercises

Use your calculator to generate the following vectors, given $\underset{\sim}{\boldsymbol{r}}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $\underset{=}{\boldsymbol{R}}(\pi/4)\,\underset{=}{\boldsymbol{R}}(\pi/2)\,\underset{\sim}{\boldsymbol{r}}_1$ (b) $\underset{=}{\boldsymbol{R}}(3\pi/4)\,\underset{\sim}{\boldsymbol{r}}_1$.

3. Rotation matrices commute

Geometrically, if we rotate a vector by α_1 and then by α_2, we get just the same result as rotating first by α_2 and then by α_1.

The first case is represented by $\quad \underline{\underline{R}}(\alpha_2)\,\underline{\underline{R}}(\alpha_1)\,\underset{\sim}{r},$

and the second by $\quad\quad\quad\quad \underline{\underline{R}}(\alpha_1)\,\underline{\underline{R}}(\alpha_2)\,\underset{\sim}{r}.$

This *suggests* that

Rotation matrices commute: $\underline{\underline{R}}(\alpha_1)\,\underline{\underline{R}}(\alpha_2) = \underline{\underline{R}}(\alpha_2)\,\underline{\underline{R}}(\alpha_1).$

Proof

As $\alpha_1+\alpha_2 = \alpha_2+\alpha_1$, then $\underline{\underline{R}}(\alpha_1+\alpha_2) = \underline{\underline{R}}(\alpha_2+\alpha_1).$

Therefore, from the result in 2 above, $\underline{\underline{R}}(\alpha_2)\,\underline{\underline{R}}(\alpha_1) = \underline{\underline{R}}(\alpha_1)\,\underline{\underline{R}}(\alpha_2).$

You can also prove it by direct calculation.

Exercises

Use your calculator to generate the following vectors, given $\underset{\sim}{r}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $\underline{\underline{R}}\,(\pi/4)\,\underline{\underline{R}}\,(\pi/2)\,\underset{\sim}{r}_1$ (b) $\underline{\underline{R}}\,(\pi/2)\,\underline{\underline{R}}\,(\pi/4)\,\underset{\sim}{r}_1.$

19.5.5   Mixing operations

We can now multiply together these various matrices to represent a string of transformations of a vector. If we do four operations, for example, to get the final vector $\underset{\sim}{r}_f$, then[29]

$$\underset{\sim}{r}_f \;=\; \underline{\underline{T}}_4\,\underline{\underline{T}}_3\,\underline{\underline{T}}_2\,\underline{\underline{T}}_1\,\underset{\sim}{r}_1.$$

We can use geometric thinking to decide properties of the matrix products.

Example

What is the algebra for

Take a vector $\underset{\sim}{v}_1$, rotate it by $45°$ and double its length to get a new vector $\underset{\sim}{v}_2$?

We now know that $\underset{\sim}{v}_2 = \underline{\underline{T}}\underset{\sim}{v}_1$, where $\underline{\underline{T}}$ is the appropriate transformation matrix built up using a rotation and a scaling:

$$\underline{\underline{T}} \;=\; \underline{\underline{S}}(2)\,\underline{\underline{R}}\,(\pi/4) \quad\quad \text{order!}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{-1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}.$$

---

[29]Work from right to left when doing multiple matrix operations on vectors.

What could you say about the transformation $\underline{\underline{R}}(\pi/4)\,\underline{\underline{S}}(2)$?

The theory of linear transformations is developed in greater detail in many Linear Algebra textbooks.

Exercises

Use your calculator to generate the following vectors, given $\underset{\sim}{r}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $\underline{\underline{R}}(\pi/2)\,\underline{\underline{M}}_x\,\underset{\sim}{r}_1$ **(b)** $\underline{\underline{M}}_x\,\underline{\underline{R}}(\pi/2)\,\underset{\sim}{r}_1$ **(c)** $\underline{\underline{R}}(\pi/4)\,\underline{\underline{S}}(-2)\,\underset{\sim}{r}_1$.

19.5.6   Concluding remarks on matrices

1. We have now defined matrices and taken the first steps in developing their algebra.

2. Simultaneous linear equations are written as $\underline{\underline{A}}\underset{\sim}{x} = \underset{\sim}{b}$.

3. The solution for square systems can be given by $\underset{\sim}{x} = \underline{\underline{A}}^{-1}\underset{\sim}{b}$; it is the existence of $\underline{\underline{A}}^{-1}$ that guarantees that $\underline{\underline{A}}\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution.

4. Matrices also tell us how to linearly transform one vector $\underset{\sim}{v}_1$ into another: $\underset{\sim}{v}_2 = \underline{\underline{T}}\underset{\sim}{v}_1$. This idea has applications to things like workforce and population evolution, and geometric transformations.

19.5.7 Problems

Solutions to the problems start on page 129.

1. (a) Calculate the effect of multiplying the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ by

   $M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Draw a diagram to show what happens in each case.
   Check with your calculator.

   (b) What geometric operation does multiplication by M appear to represent? You can check your deduction using the general formula given in Section 19.5.2.

   (c) What is the effect of operating twice on a vector with M, i.e. multiplying by M^2? What is M^{-1}?

2. (a) What geometric operation does the matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ represent?

   (b) Calculate A^4 and A^8.

   (c) How did you know **geometrically** those would be the answers?

   (d) Use a geometric argument to find A^{-1} and check by calculating AA^{-1}.

   (e) What geometric operation is represented by $T = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$?

3. If r_1 is transformed into r_2 by an operation represented by T_1 followed by an operation represented by T_2, then $r_2 = T_2 T_1 r_1$. What sequence of operations do AM and MA represent, if M and A are the matrices in Questions 1 and 2?

   Draw a diagram to show what happens to the vector $r = i + j$ in each case.

   Check with your calculator.

   So do we know $AM = MA$ or $AM \neq MA$? Why?

4. Find the determinant of the general rotation matrix

$$R(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

19.5.8   Solutions to exercises

In the TRANSFRM program, vector r_1 is stored in matrix [E] and displayed, with label and norm, in blue at the top left of the screen. The transformed vectors, label and norm are in black.

```
(E)
NORM: 2.236
```

Exercises   page 115

Use your calculator to generate the following vectors, given $r_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $\underline{\underline{S}}(2)\,r_1$

```
[A][E]
NORM: 4.472
```

(b) $\underline{\underline{S}}(-1)\,r_1$

```
[A][E]
NORM: 2.236
```

(c) $\underline{\underline{S}}(-2)\,r_1$

```
[A][E]
NORM: 4.472
```

Exercises   page 117

Use your calculator to generate the following vectors, given $r_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $\underline{\underline{M}}_x \, r_1$

```
[A][E]
NORM: 2.236
```

(b) $\underline{\underline{M}}_y \, r_1$

```
[A][E]
NORM: 2.236
```

(c) $\underline{\underline{M}}_{x \leftrightarrow y} \, r_1$

```
[A][E]
NORM: 2.236
```

(d) reflect r_1 in the line $y = -x$.

```
[A][E]
NORM: 2.236
```

Exercises   page 119

Use your calculator to generate the following vectors, given $\underset{\sim}{r_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $\underset{=}{R}(\pi/4)\underset{\sim}{r_1}$

```
[A][E]
NORM: 2.236
```

(b) $\underset{=}{R}(-\pi/3)\underset{\sim}{r_1}$

```
[A][E]
NORM: 2.236
```

(c) $\underset{=}{R}(3\pi/2)\underset{\sim}{r_1}$

```
[A][E]
NORM: 2.236
```

(d) $\underset{=}{R}(5\pi/3)\underset{\sim}{r_1}$.

```
[A][E]
NORM: 2.236
```

Exercises   page 121

Use your calculator to generate the following vectors, given $\underset{\sim}{r}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $\underset{=}{R}(\pi/4)\,\underset{=}{R}(\pi/2)\,\underset{\sim}{r}_1$


```
[A][B][E]
NORM: 2.236
```

(b) $\underset{=}{R}(\pi/2)\,\underset{=}{R}(\pi/4)\,\underset{\sim}{r}_1$


```
[B][A][E]
NORM: 2.236
```

Exercises   page 122

Use your calculator to generate the following vectors, given $\underset{\sim}{r}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

(a) $\underset{=}{R}(\pi/2)\,\underset{=}{M}_x\,\underset{\sim}{r}_1$


```
[A][B][E]
NORM: 2.236
```

(b) $\underset{=}{M}_x\,\underset{=}{R}(\pi/2)\,\underset{\sim}{r}_1$.


```
[B][A][E]
NORM: 2.236
```
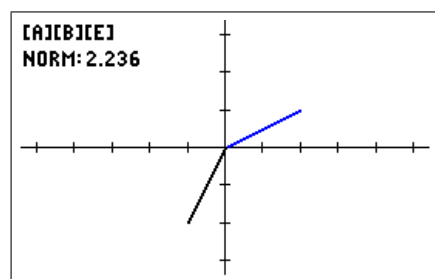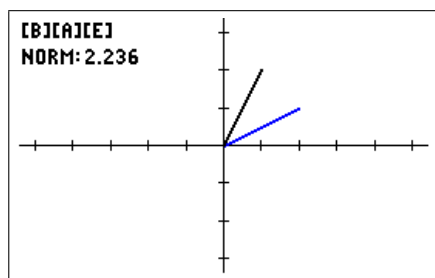
(c) $\underline{\underline{R}}\left(\pi/4\right)\underline{\underline{S}}\left(-2\right)\underset{\sim}{r}_1$
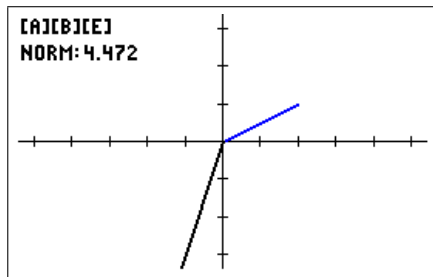
19.5.9   Solutions to problems

1. **(a)** Calculate the effect of multiplying the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ by

$\underline{\underline{M}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Draw a diagram to show what happens in each case.

$\underline{\underline{M}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
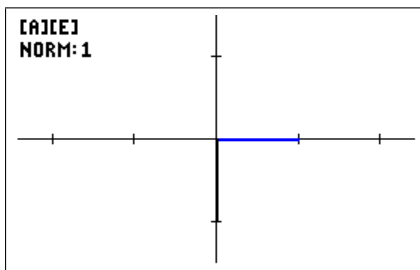
$\underline{\underline{M}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$
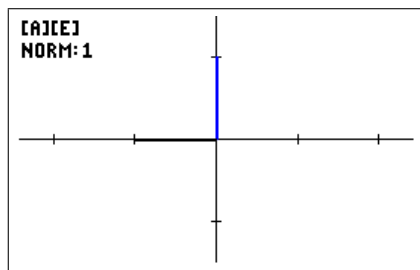
$\underline{\underline{M}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
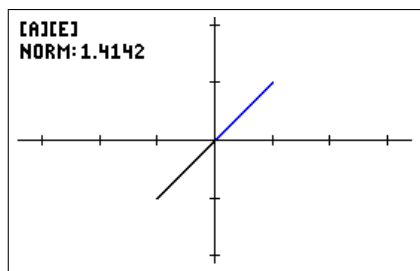
$\underline{\underline{M}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
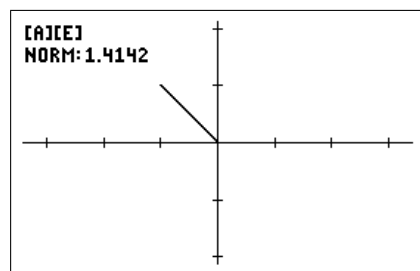
(b) What geometric operation does multiplication by $\underline{\underline{M}}$ appear to represent?

Reflection in the line $y = -x$.

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(c) What is the effect of operating twice on a vector with $\underline{\underline{M}}$, i.e. multiplying by $\underline{\underline{M}}^2$? What is $\underline{\underline{M}}^{-1}$?

Geometrically, operating twice with $\underline{\underline{M}}$ gets back to the original vector.

Algebraically therefore, $\underline{\underline{M}}^2 = \underline{\underline{I}}$ and so $\underline{\underline{M}}^{-1} = \underline{\underline{M}}$. Check by calculating $\underline{\underline{M}}^2$.

2. (a) What geometric operation does the matrix $\underline{\underline{A}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ represent?

Comparing $\underline{\underline{A}}$ with the general form of the rotation matrix in the Notes, we see that $\cos(\alpha) = \sin(\alpha) = 1/\sqrt{2}$, so that $\underline{\underline{A}}$ represents a rotation of $45°$ or $\pi/4$ radians in the anti-clockwise direction, i.e. $\underline{\underline{A}} = \underline{\underline{R}}(\pi/4)$.

(b) Calculate $\underline{\underline{A}}^4$ and $\underline{\underline{A}}^8$.

$$\underline{\underline{A}}^4 = \left(\underline{\underline{A}}^2\right)^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$\underline{\underline{A}}^8 = \left(\underline{\underline{A}}^4\right)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{I}}.$$

(c) How did you know **geometrically** those would be the answers?

$\underline{\underline{A}}^4$ represents 4 rotations by $\pi/4$, i.e. a rotation by π:

$$\underline{\underline{A}}^4 = \underline{\underline{R}}(\pi) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$\underline{\underline{A}}^8$ represents 8 rotations by $\pi/4$, i.e. a rotation by 2π:

$$\underline{\underline{A}}^8 = \underline{\underline{R}}(2\pi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{I}}.$$

(d) Use a geometric argument to find $\underline{\underline{A}}^{-1}$ and check by calculating $\underline{\underline{A}}\underline{\underline{A}}^{-1}$.

$\underline{\underline{A}}^{-1}$ will represent a rotation by $-\pi/4$, so that $\underline{\underline{A}}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$

$$\underline{\underline{A}}\underline{\underline{A}}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{I}}.$$

(e) What geometric operation is represented by $\underline{\underline{T}} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$?

$\underline{\underline{T}} = 2\underline{\underline{A}} = 2\underline{\underline{R}}(\pi/4)$, so it represents a rotation by $\pi/4$ and doubling the length of the vector.

3. If r_1 is transformed into r_2 by an operation represented by T_1 followed by an operation represented by T_2, then $r_2 = T_2 T_1 r_1$. What sequence of operations do AM and MA represent, if M and A are the matrices in Questions 1 and 2?

   Draw a diagram to show what happens to the vector $r = i + j$ in each case.

AMr

MAr

So do we know $AM = MA$ or $AM \neq MA$? Why?

AM represents a reflection in the line $y = -x$ followed by a rotation by $45°$.

MA represents a rotation by $45°$ followed by a reflection in the line $y = -x$.

Geometrically, AM and MA operating on $r = i + j$ produce different vectors, so that $AM \neq MA$.

If you check, you will find

$$AM = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad MA = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

4. Find the determinant of the general rotation matrix $R(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$.

$$\det R(\alpha) = \det \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} = \cos^2(\alpha) + \sin^2(\alpha) = 1.$$

All rotation matrices have a determinant of 1. This corresponds to the fact that they do not change the length of a vector (they have a scaling factor of 1).

19.6   Eigenvalues and eigenvectors

19.6.1   Theory

Given an $n \times n$ matrix $\underline{\underline{A}}$, the eigenvalues of $\underline{\underline{A}}$ are the constants λ and the eigenvectors of $\underline{\underline{A}}$ the corresponding (non-zero) n-dimensional vectors $\underset{\sim}{v}$ ($n \times 1$ matrices) satisfying the equation

$$\underline{\underline{A}} \underset{\sim}{v} \;=\; \lambda \underset{\sim}{v}.$$

The eigenvalues λ are the zeros or roots of the nth-degree characteristic polynomial

$$p(x) \;=\; \det\!\left(\underline{\underline{A}} - x\,\underline{\underline{I}}\right),$$

where $\underline{\underline{I}}$ is the $n \times n$ identity matrix.

A nice way to find the eigenvalues on a TI-84/CE is graphically: set

$$Y_1 \;=\; \det\!\big([A] - X\,\mathrm{identity}(n)\big),$$

the characteristic polynomial, where you have to put in a value for n, the dimension of $[A]$, and graph in the usual way with an appropriate window.

Use *zero* (in the $\boxed{\text{calc}}$ menu) to find the zeros of the polynomial: these are the eigenvalues λ_i, $i = 1, 2, \ldots$.

To find the eigenvectors, we have to solve the homogeneous matrix equation

$$\left(\underline{\underline{A}} - \lambda_i\,\underline{\underline{I}}\right)\underset{\sim}{v} \;=\; \underset{\sim}{\mathbf{0}}$$

for each eigenvalue λ_i found above.

On the calculator, we do this by Gauss-Jordan elimination, either using the command *rref* on the matrix $\underline{\underline{A}} - \lambda_i\,\underline{\underline{I}}$ or using the GAUSS/GAUSSCE program with $\underline{\underline{A}} - \lambda_i\,\underline{\underline{I}}$ stored in matrix $[A]$.

19.6.2   Example

Find the eigenvalues and corresponding eigenvectors of

$$\underline{\underline{A}} \;=\; \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}.$$

This is a simple problem that can all be done easily by hand. We use it to illustrate the calculator method, which can be used for more-complicated problems.

Eigenvalues

Put $\underline{\underline{A}}$ into matrix $[A]$ in your calculator and set $Y_1 = \det\!\big([A] - X\,\mathrm{identity}(2)\big)$.

$$\boxed{\blacksquare \backslash Y_1 \boxminus \mathtt{det([A]-Xidentity(2))}}$$

Start with $\mathrm{Xmin} = -5$ and $\mathrm{Xmax} = 5$. Press $\boxed{\text{graph}}$ to plot the characteristic polynomial, here a quadratic (because $[A]$ is 2×2). It is clear that the zeros lie in the range $0 < x < 3$. Change the window to magnify this region.
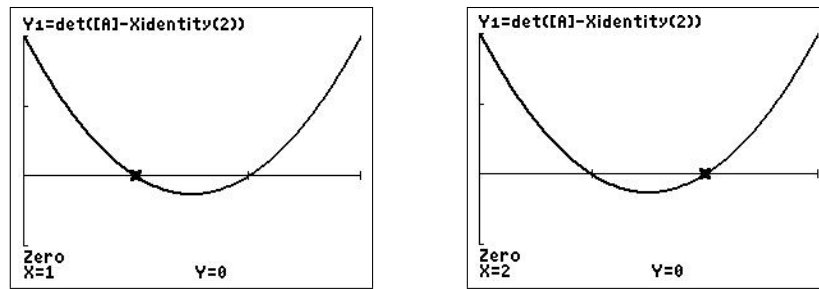
window $[0, 3, 1] \times [-1, 2, 1]$

Using *zero* in the $\boxed{\text{calc}}$ menu gives the eigenvalues as $\lambda_1 = 1$ and $\lambda_2 = 2$ (figures above).

In this case, it is easy to show by hand that the characteristic polynomial is $p(\lambda) = \lambda^2 - 3\lambda + 2$, with zeros 1 and 2.

Eigenvectors

$\boldsymbol{\lambda_1 = 1}$: We have to find $\underset{\sim}{\boldsymbol{v}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, such that $\underset{=}{\boldsymbol{A}}\underset{\sim}{\boldsymbol{v}} = \underset{\sim}{\boldsymbol{v}}$ or $\left(\underset{=}{\boldsymbol{A}} - \underset{=}{\boldsymbol{I}}\right)\underset{\sim}{\boldsymbol{v}} = \underset{\sim}{\boldsymbol{0}}$, i.e.

$$\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1}$$

To use the GAUSS/GAUSSCE program, put $\underset{=}{\boldsymbol{A}} - \underset{=}{\boldsymbol{I}}$ in matrix [A].

However, we want to use $\underset{=}{\boldsymbol{A}}$, now in [A], for the second eigenvector too, so first store [A] in [B].

Then store [B]−identity(2) in [A] and run GAUSS/GAUSSCE.

Alternatively execute the command ref$\big([\text{B}]-\text{identity}(2)\big)$.

We should really use the augmented matrix here, with a third column of 0s from the right-hand side of Eq. (1), but these remain 0 in the Gauss elimination, so we just remember to put them back in at the end.

```
[A]→[B]
            [[3   2]
            [-1  0]]
[B]-identity(2)→[A]
            [[2   2 ]
            [-1  -1]]
Pr9mGAUSS
```

```
REDUCED ROW-ECHELON FORM
                    [[1 1]
                    [0 0]]
```

The reduced row-echelon form of $\underset{=}{\boldsymbol{A}} - \underset{=}{\boldsymbol{I}}$, with the third column of 0s now back in, is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the bottom row is all zeros.

There **must** be at least one row of zeros (the bottom row(s) after Gauss-Jordan or Gauss elimination) in all eigenvector problems because there is always an infinite number of solutions to the eigenvector equations, Eq. (1) here.

Eigenvalues are only defined to an arbitrary multiplicative constant.

The corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The bottom row tells us that x_2 is arbitrary, so we set $x_2 = t$, where t is any non-zero number. The top row tells us that $x_1 + x_2 = 0$, so that $x_1 = -x_2 = -t$.

The eigenvector v_1 corresponding to $\lambda_1 = 1$ is therefore

$$v_1 = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero[30]) multiples of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We usually say that this vector is the eigenvector, with the understanding that all non-zero multiples of it are also eigenvectors.

Check:
$$Av_1 = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = v_1.$$

$\lambda_2 = 2$: We have to solve $Av = 2v$ or $(A - 2I)v = 0$ for v.

Store $[B] - 2\,\text{identity}(2)$ in $[A]$ and run GAUSS/GAUSSCE.

Alternatively execute $\text{ref}([B] - 2\,\text{identity}(2))$.

Both give the reduced row-echelon form of $A - 2I$, with the third column of 0s back in, as

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $x_2 = t$, where t is any number, and $x_1 + 2x_2 = 0$, so that $x_1 = -2x_2 = -2t$.

The eigenvector v_2 corresponding to $\lambda_2 = 2$ is therefore

$$v_2 = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The eigenvector is an arbitrary (non-zero) multiple of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Check that $Av_2 = 2v_2$.

---

[30]If $t = 0$, we obtain a zero vector, which, by definition, cannot be an eigenvector.

19.6.3   EIGENV5/EIGENV5CE — finds real eigenvalues and eigenvectors

The EIGENV5/EIGENV5CE program[31] combines plotting the characteristic polynomial, a root finder to find the eigenvalues and the GAUSS/GAUSSCE program to find the eigenvectors of the $n \times n$ matrix [A], all semi-automatically.

The description below is rather long but the process is relatively quick to carry out once you've had some practice.

Example

Find the eigenvectors and eigenvalues of

$$\underline{\underline{A}} \; = \; \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

1. Store $\underline{\underline{A}}$ (here 3×3) in matrix [A] using $\boxed{\text{matrix}}$ EDIT.

2. Run the EIGENV5/EIGENV5CE program. Choose FIND THE EIGENVALUES.

    This graphs the characteristic polynomial, whose zeros give the real eigenvalues, using a window $[-10, 10, 1] \times [-5, 5, 1]$.
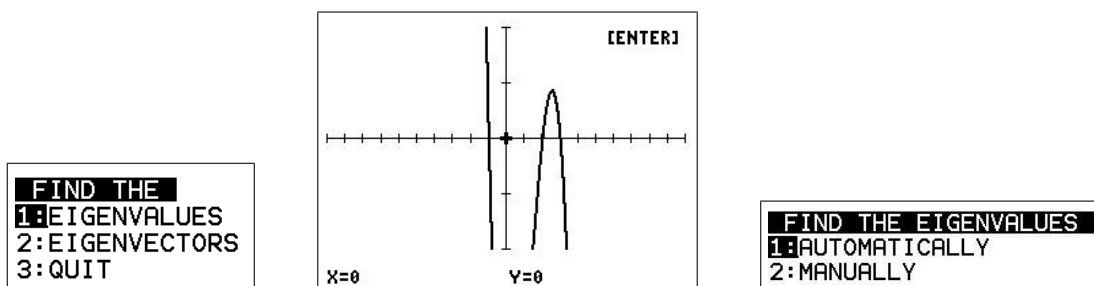
3. After the polynomial has been graphed, you have the option to have its zeros (the real eigenvalues) found automatically or you can do it manually using *zero* in the $\boxed{\text{calc}}$ menu.

4. If you choose Automatic mode, the calculator first asks if you want to change the X scale (change Xmin/Xmax). The program only finds zeros that are on the screen.
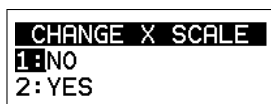
    If the zeros are not all on screen (*how many should there be?*) or if they are concentrated in a small part of the X range,[32] select YES. The calculator will ask you to change Xmin/Xmax, then it will replot the graph.

    Keep doing this step until all the zeros are on screen and spread out. Then select NO.

---

[31]available at *canberramaths.org.au* under *Resources*

[32]The zero finder works along the graph pixel by pixel until it finds a sign change in the function between two adjacent pixels. If the zeros are very concentrated, there may be more than one zero in a pixel. In this case, at least one of these zeros may not be found or the same one may be found twice. It's best to choose Xmin/Xmax so that the zeros are spread out as far as possible ON THE SCREEN.

5. The calculator will find (approximations to) the zeros, place a small square at the relevant point on the graph and list on-screen the values found, rounded to 3 decimal places. The full (but still approximate) values for the zeros/eigenvalues are stored in list L5 (smallest to largest) for later use in finding the eigenvectors.

For the example here, you should obtain the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.
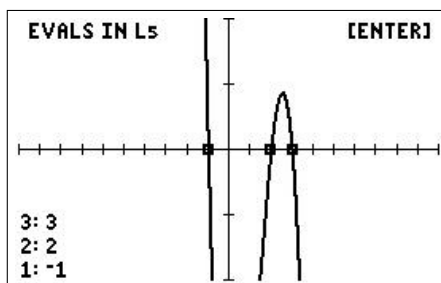
6. The program now asks if you want to find the eigenvectors. Answer YES.[33]

7. Choose which eigenvalue you want to use by selecting its <u>index</u> in L5: $1, 2, 3, \cdots$.

8. The program will return the eigenvalue and eigenvector (use the right arrow to scroll if necessary). Alternatively it may return the message

   Reduced row-echelon matrix needs interpreting.

This happens if the z component of the eigenvector is zero, if there is more than one row of all zeros or if $\underset{\sim}{\underset{=}{A}}\underset{\sim}{v} \neq \lambda\underset{\sim}{v}$ (the program checks).

Rather than the program choose between one of several possibilities, it asks you to look at the reduced row-echelon form of the augmented matrix and work out the eigenvector.

9. After it has calculated an eigenvector, the program asks whether you want to find further eigenvectors. The last eigenvector is stored in matrix [B].

In our example, the eigenvectors corresponding to $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 3$ are

$$\underset{\sim}{v_1} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \qquad \underset{\sim}{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \underset{\sim}{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

---

[33]but see Point 11.

10. You can run EIGENV5/EIGENV5CE at any time to find the eigenvectors (choose FIND THE EIGENVECTORS? at the start), provided the eigenvalues are still in list L5 and the matrix in [A].

    Alternatively you can store the eigenvalues in L5 manually ($\boxed{\text{stat}}$ Edit), run the program and choose FIND THE EIGENVECTORS from the first menu.

    The appropriate matrix has to be stored in matrix [A] of course.

11. *Repeated eigenvalues* show up as the graph of the characteristic equation touching, but not crossing, the x axis. The root finder will therefore not pick them up.

    If the x value (λ) where the graph touches the x axis is obvious from the graph, you can insert it into L5 via the program. At Point 6 above, there is an additional prompt allowing you to do this. You can then continue to run the program as normal.

    The program may find an eigenvector, if there is only one, or else it will display the reduced row-echelon form of the augmented matrix to allow you to work out the eigenvector(s).

    If the x value is not obvious from the graph, you can use *maximum* or *minimum* in the $\boxed{\text{calc}}$ menu to find it. You will need to quit out of the program to do this. Then press $\boxed{\text{graph}}$ to replot the graph. Store the value in L5, after the eigenvalues the calculator has already stored.

    Then re-run the EIGENV5/EIGENV5CE program, choose FIND THE EIGENVECTORS and put in the index of the eigenvalue you have just entered. If an eigenvalue is repeated, it may have more than one associated eigenvector.

19.6.4   Exercises

1. Find, using the calculator (but not EIGENV5/EIGENV5CE), the eigenvectors and eigenvalues of the matrix
$$\underline{\underline{A}} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}.$$
   Check with EIGENV5/EIGENV5CE and verify that $\underline{\underline{A}}\underline{v} = \lambda\underline{v}$ for each eigenvalue.

2. Show, using the calculator (but not EIGENV5/EIGENV5CE), that the eigenvalues of
$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$
   are -1, 1 and 2, with corresponding eigenvectors
$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

3. Find, using the calculator (but not EIGENV5/EIGENV5CE), the eigenvectors and eigenvalues of the matrix
$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$
   Check with EIGENV5/EIGENV5CE and verify that $\underline{\underline{A}}\underline{v} = \lambda\underline{v}$ for each eigenvalue.

19.6.5   Solutions

1. Find, using the calculator (but not EIGENV5/EIGENV5CE), the eigenvectors and eigenvalues of the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}.$$

Check with EIGENV5/EIGENV5CE and verify that $\underline{\underline{A}}\underset{\sim}{v} = \lambda\underset{\sim}{v}$ for each eigenvalue.

Eigenvalues

Put the 2×2 matrix $\underline{\underline{A}}$ into matrix [A] in your calculator.

Set $Y_1 = \det\big([A] - X\,\text{identity}(2)\big)$.

$$\boxed{\blacksquare\text{\textbackslash}Y_1 \blacksquare \text{det([A]-Xidentity(2))}}$$

Start with $\boxed{\text{zoom}}$ $\boxed{6}$. Press $\boxed{\text{graph}}$ to plot the characteristic polynomial, a quadratic here. It is clear that the zeros lie in the range $0 < x < 10$.

window $[0, 10, 1] \times [-10, 10, 5]$

Using *zero* in the $\boxed{\text{calc}}$ menu gives the eigenvalues as $\lambda_1 = 3$ and $\lambda_2 = 8$.

PTO

Eigenvectors

$\lambda_1 = 3$

Let $\underset{\sim}{v_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. Then $\left(\underset{=}{A} - \lambda \underset{=}{I}\right) \underset{\sim}{v_1} = \left(\underset{=}{A} - 3\underset{=}{I}\right) \underset{\sim}{v_1} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

To use the GAUSS/GAUSSCE program, we need to put $\underset{=}{A} - 3\underset{=}{I}$ in matrix [A]. As we want to use $\underset{=}{A}$, now in [A], for the second eigenvector, first store [A] in [B].

Then store [B]−identity(2) in [A] and run GAUSS/GAUSSCE.

We should really use the augmented matrix here with a third column of 0s; these remain 0 in the Gauss elimination, so we just remember to put them back in afterwards.

```
[A]

              [[4 2]
               [2 1]]
```

```
REDUCED ROW-ECHELON FORM

                [[1 0.5]
                 [0 0  ]]
```

This gives the reduced row-echelon form of $\underset{=}{A} - 3\underset{=}{I}$, with the third column of 0s back in, as

$$\begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the bottom row is all zeros. There must be at least one row of zeros (the bottom row) in all eigenvector problems.

Therefore, y_1 is a free variable. Set $y_1 = t \in \mathbb{R}$, $t \neq 0$.

The top row gives the equation $x_1 + 0.5 y_1 = 0$, so that $x_1 = -0.5 y_1 = -0.5t$.

The solution is therefore $y_1 = t$, $x_1 = -0.5t$, where t is any **non-zero** real number.

Therefore, $\underset{=}{A}$ has an eigenvalue $\lambda_1 = 3$, with corresponding eigenvector

$$\underset{\sim}{v_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -0.5t \\ t \end{bmatrix} = t \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad t \neq 0 \in \mathbb{R}.$$

Given that t is arbitrary, set $s = -\frac{1}{2}t$, $s \neq 0 \in \mathbb{R}$, so that

$$\underset{\sim}{v_1} = s \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Check your answer by verifying that $\underset{=}{A}\underset{\sim}{v_1} = 3\underset{\sim}{v_1}$.

$$\begin{aligned} \underset{=}{A}\underset{\sim}{v_1} &= \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -6 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= 3\underset{\sim}{v_1}. \end{aligned}$$

$\underline{\lambda_2 = 8}$

Let $\underset{\sim}{v_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then $\left(\underset{=}{A} - \lambda \underset{=}{I}\right) \underset{\sim}{v_1} = \left(\underset{=}{A} - 8\underset{=}{I}\right) \underset{\sim}{v_1} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Gauss elimination on the augmented matrix:

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, y_2 is a free variable. Set $y_2 = t \neq 0 \in \mathbb{R}$.

The top row gives the equation $x_2 - 2y_2 = 0$, so that $x_2 = 2y_2 = 2t$.

The solution is therefore $y_2 = t$, $x_2 = 2t$, where t is any **non-zero** real number.

Therefore, $\underset{=}{A}$ has an eigenvalue $\lambda_2 = 8$, with corresponding eigenvector

$$\underset{\sim}{v_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad t \neq 0 \in \mathbb{R}.$$

Check your answer by verifying that $\underset{=}{A}\underset{\sim}{v_2} = 8\underset{\sim}{v_2}$.

$$\underset{=}{A}\underset{\sim}{v_2} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ 8 \end{bmatrix}$$

$$= 8 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= 8\underset{\sim}{v_2}.$$

Note that we usually take $t = 1$ when writing down the eigenvectors, knowing that any non-zero multiple is also an eigenvector. However, sometimes we choose a value that removes fractions in the eigenvector, as we did with s above.

Using EIGENV5/EIGENV5CE
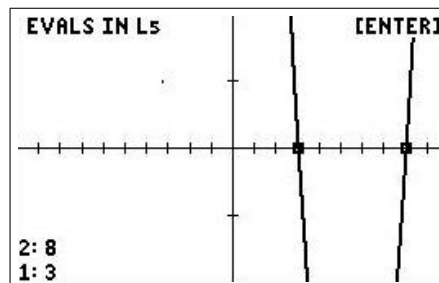
EIGENVALUE	EIGENVALUE
L5(1)=3	L5(2)=8
EIGENVECTOR [B]	EIGENVECTOR [B]
{-1/2 1}	{2 1}

2. Show, using the calculator (but not EIGENV5/EIGENV5CE), that the eigenvalues of

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

are -1, 1 and 2, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Eigenvalues

Put the 3×3 matrix A into matrix [A] in your calculator.

Set $Y_1 = \det\big([A] - X \, \text{identity}(3)\big)$.

$$\boxed{\blacksquare \backslash Y_1 \boxminus \text{det}([A]-X\text{identity}(3))}$$

Start with Xmin $= -5$ and Xmax $= 5$. Press $\boxed{\text{graph}}$ to plot the characteristic polynomial, here a cubic. It is clear that the zeros lie in the range $-2 < x < 3$.
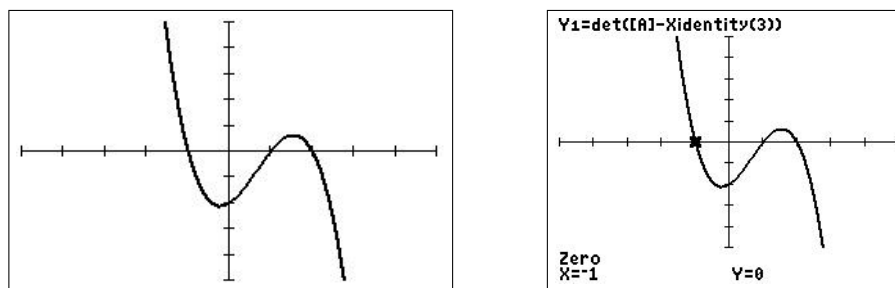
window $[-5, 5, 1] \times [-5, 5, 1]$

Using *zero* in the $\boxed{\text{calc}}$ menu gives the eigenvalues as $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 3$.

Eigenvectors

$\boldsymbol{\lambda_1 = -1}$: We have to find $v_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, such that $A v_1 = -v_1$ or $\big(A + I\big)v_1 = 0$,

i.e.

$$\left(\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To use the GAUSS/GAUSSCE program, we need to put $A + I$ in matrix [A]. As we want to use A, now in [A], for the second eigenvector, first store [A] in [B]. Then store [B]+identity(3) in [A] and run GAUSS/GAUSSCE.

This gives the reduced row-echelon form of $\underline{\underline{A}} + \underline{\underline{I}}$, with the fourth column of 0s put back in, as

$$
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Note again that the bottom row is all zeroes.

The bottom row tells us that x_3 is arbitrary, so we set $x_3 = t$, where t is any number. The second row gives $x_2 = 0$. The top row tells us that $x_1 - x_3 = 0$, so that $x_1 = x_3 = t$.

The eigenvector $\underset{\sim}{v_1}$ corresponding to $\lambda_1 = -1$ is therefore

$$
\underset{\sim}{v_1} =
\begin{bmatrix}
t \\
0 \\
t
\end{bmatrix}
= t
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}.
$$

The eigenvectors are arbitrary (non-zero) multiples of the vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$\boldsymbol{\lambda_2 = 1}$:   We have to solve $\underline{\underline{A}}\underset{\sim}{v_2} = \underset{\sim}{v_2}$ or $\left(\underline{\underline{A}} - \underline{\underline{I}} \right)\underset{\sim}{v_2} = \underset{\sim}{0}$ for $\underset{\sim}{v_2}$.

Store [B] − identity(3) in [A] and run GAUSS/GAUSSCE. The reduced row-echelon form of $\underline{\underline{A}} - \underline{\underline{I}}$, with the fourth column of 0s put back in, is

$$
\begin{bmatrix}
1 & 0 & -3 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Therefore, $x_3 = t$, where t is any number. The second row gives the equation $x_2 - 2x_3 = 0$, so that $x_2 = 2x_2 = 2t$. The top row gives the equation $x_1 - 3x_3 = 0$, so that $x_1 = 3x_3 = 3t$.

The eigenvector $\underset{\sim}{v_2}$ corresponding to $\lambda_2 = 2$ is therefore

$$
\underset{\sim}{v_2} =
\begin{bmatrix}
3t \\
2t \\
t
\end{bmatrix}
= t
\begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix}.
$$

The eigenvectors are arbitrary (non-zero) multiples of the vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

$\lambda_3 = 2$:   We have to solve $\underset{\displaystyle\sim}{\underline{\underline{A}}}v_3 = 2v_3$ or $\left(\underline{\underline{A}} - 2\,\underline{\underline{I}}\right)v_3 = \underset{\displaystyle\sim}{0}$ for v_3.

Store $[B] - 2\,\text{identity}(3)$ in $[A]$ and run GAUSS/GAUSSCE. The row-equivalent reduced row-echelon form of $\underline{\underline{A}} - 2\,\underline{\underline{I}}$, with the fourth column of 0s put back in, is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $x_3 = t$, where t is any number. The second row gives the equation $x_2 - 3x_3 = 0$, so that $x_2 = 3x_2 = 3t$. The top row gives the equation $x_1 - x_3 = 0$, so that $x_1 = x_3 = t$.

The eigenvector v_3 corresponding to $\lambda_3 = 2$ is therefore

$$v_3 \;=\; \begin{bmatrix} t \\ 3t \\ t \end{bmatrix} \;=\; t \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero) multiples of the vector $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

3. Find, using the calculator (but not EIGENV5/EIGENV5CE), the eigenvectors and eigenvalues of the matrix

$$\underline{\underline{A}} \;=\; \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Check with EIGENV5/EIGENV5CE and verify that $\underset{\displaystyle\sim}{\underline{\underline{A}}}v = \lambda v$ for all the eigenvalues.
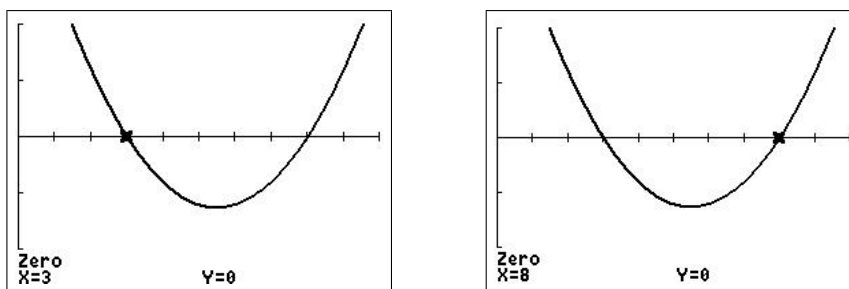
Eigenvalues

Put the 3×3 matrix $\underline{\underline{A}}$ into matrix $[A]$ in your calculator.

Set $Y_1 = \det\big([A] - X\,\text{identity}(3)\big)$.

Start with Xmin $= -5$ and Xmax $= 5$. Press $\boxed{\text{graph}}$ to plot the characteristic polynomial, here a cubic.
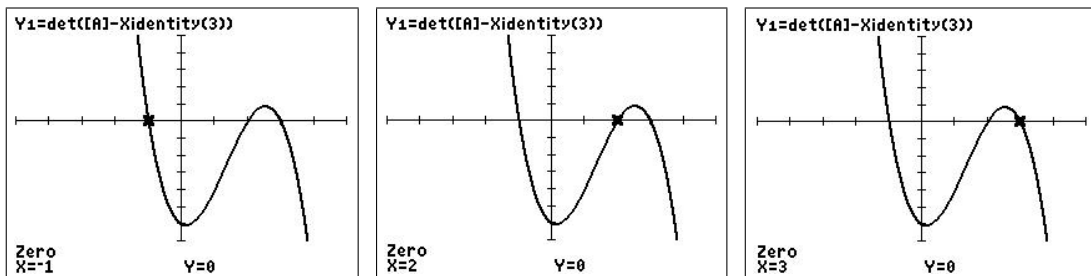
window $[-5, 5, 1] \times [-7, 5, -1]$

Using *zero* in the $\boxed{\text{calc}}$ menu gives the eigenvalues as $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

Eigenvectors

$\lambda_1 = -1$: We have to solve $\left(\underline{\underline{A}} + \underline{\underline{I}}\right)\underset{\sim}{v_1} = \underset{\sim}{0}$ for $\underset{\sim}{v_1}$.

Let $\underset{\sim}{v_1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. We need to find the components x, y and z. Then,

$$\left(\underline{\underline{A}} + \underline{\underline{I}}\right)\underset{\sim}{v_1} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The augmented matrix for this problem is 3×4, with the 4th column all zeros.

To use the GAUSS/GAUSSCE program, we need to put $\underline{\underline{A}} + \underline{\underline{I}}$ in matrix [A]. Store [A] in [B], then store [B]+identity(3) in [A] and run GAUSS/GAUSSCE.

This gives the reduced row-echelon form of $\underline{\underline{A}} + \underline{\underline{I}}$, with the 4th column of 0s back in, as

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

(note the bottom row of zeros), with corresponding equations

$$\begin{aligned} x + y &= 0 \\ y + 3z &= 0. \end{aligned}$$

Back substitution gives $z = t$, $y = -3t$, $x = 3t$.

Therefore the required eigenvector is $\underset{\sim}{v_1} = t\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$, where $t \neq 0 \in \mathbb{R}$.

Check that $\underline{\underline{A}}\underset{\sim}{v_1} = -\underset{\sim}{v_1}$.

$\underline{\lambda_2 = 2}$

Let $\underset{\sim}{v_2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.   Solve $\left(\underline{\underline{A}} - 2\underline{\underline{I}}\right)\underset{\sim}{v_2} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Gauss elimination

$$\begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

corresponding to equations

$$\begin{aligned} x - 2y &= 0 \\ y &= 0, \end{aligned}$$

with free variable z and solution (back substitution) $z = t$, $y = 0$, $x = 0$, with $t \neq 0 \in \mathbb{R}$.

Therefore, the required eigenvector is $\underset{\sim}{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Check that $\underline{\underline{A}}\underset{\sim}{v_2} = 2\underset{\sim}{v_2}$.

$\underline{\lambda_3 = 3}$

Let $\underset{\sim}{\boldsymbol{v}_3} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Solve $\left(\underline{\underline{\boldsymbol{A}}} - 3\,\underline{\underline{\boldsymbol{I}}}\right)\underset{\sim}{\boldsymbol{v}_3} = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Gauss elimination

$$\begin{bmatrix} -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

corresponding to equations

$$\begin{aligned} x - y &= 0 \\ y - z &= 0, \end{aligned}$$

with free variable z and solution (back substitution) $z = t$, $y = t$, $x = t$, with $t \neq 0 \in \mathbb{R}$.

Therefore, the required eigenvector is $\underset{\sim}{\boldsymbol{v}_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Check that $\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}_3} = 3\underset{\sim}{\boldsymbol{v}_3}$.

Using EIGENV5/EIGENV5CE

```
EVALS IN L5              [ENTER]




3: 3
2: 2
1: -1
```
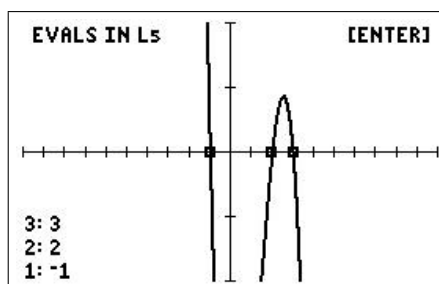
```
EIGENVALUE
L5(1)=-1

EIGENVECTOR [B]
              {3 -3 1}
```

```
EIGENVALUE
L5(2)=2

EIGENVECTOR [B]
              {0 0 1}
```

```
EIGENVALUE
L5(3)=3

EIGENVECTOR [B]
              {1 1 1}
```

19.7 Vector operations

19.7.1 Introduction

All the operations here, except one, are based on the dot or scalar product. This can be evaluated with the vectors stored as (column) matrices or as lists. If $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ are two vectors of the same dimension n, they can be represented on the calculator by the $n \times 1$ column matrices [A] and [B], say, or the components stored in lists, say, L_1 and L_2. Then the dot or scalar product is given by the matrix multiplication

$$\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} \; = \; [A]^T[B],$$

where the superscript T denotes the transpose, or by the list operation

$$\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} \; = \; \mathrm{sum}(L_1 L_2).$$

The list product $L_1 L_2$ produces a list whose components are the products of the corresponding components of the two lists; summing these then gives the scalar product.

Note that the scalar product is defined for vectors of any length, i.e. $\underset{\sim}{\boldsymbol{a}}, \underset{\sim}{\boldsymbol{b}} \in \mathbb{R}^n$; this is handled by both methods on the calculator.

The one exception mentioned above is the cross or vector product, which is only defined for three-dimensional vectors.

The list method is a little easier to use, so we continue with it. The operations described below are contained in the VECTOR/VECTORCE program,[34] which prompts for the vectors, then does the calculations.

19.7.2 Operations

Use $\boxed{\text{stat}}$ Edit to input the vectors or enclose them in { }, with commas, and store them to the relevant list.

Assume vector $\underset{\sim}{\boldsymbol{a}}$ is in list L_1, vector $\underset{\sim}{\boldsymbol{b}}$ in list L_2. *sum* is in the $\boxed{\text{list}}$ MATH menu.

Scalar multiplication $(\mathbb{R}^n)$

If c is a constant, the scalar multiple $c\underset{\sim}{\boldsymbol{a}} = c\, L_1$.

Scalar or dot product $(\mathbb{R}^n)$

$\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} = \mathrm{sum}(L_1 L_2).$

Vector or cross product $(\mathbb{R}^3$ only$)$

$\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}}$ is a vector, given in terms of the entries in the two lists by

$$\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}} \; = \; \big(L_1(2)\, L_2(3) - L_1(3)\, L_2(2), \;\; L_1(3)\, L_2(1) - L_1(1)\, L_2(3), \;\; L_1(1)\, L_2(2) - L_1(2)\, L_2(1)\big).$$

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[34]available at *canberramaths.org.au* under *Resources*

Norm $(\mathbb{R}^n)$

$$\text{norm } \underset{\sim}{\boldsymbol{a}} \;=\; \sqrt{(\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{a}})} \;=\; \sqrt{(\text{sum}(L_1\,L_1))}$$

Projection $(\mathbb{R}^n)$

The (vector) projection of $\underset{\sim}{\boldsymbol{b}}$ onto $\underset{\sim}{\boldsymbol{a}}$ is given by

$$\text{proj}\,\underset{\sim}{\boldsymbol{a}}\,\underset{\sim}{\boldsymbol{b}} \;=\; \frac{\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}}}{\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{a}}}\, \underset{\sim}{\boldsymbol{a}} \;=\; \text{sum}(L_1\,L_2)/\text{sum}(L_1\,L_2)\,L_2.$$

Angle between vectors $(\mathbb{R}^n)$

The angle θ between vectors $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ is given by

$$\cos(\theta) \;=\; \frac{\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}}}{\sqrt{(\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{a}})}\,\sqrt{(\underset{\sim}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{b}})}} \;=\; \text{sum}(L_1\,L_2)/\sqrt{(\text{sum}(L_1\,L_1))}/\sqrt{(\text{sum}(L_2\,L_2))}.$$

19.7.3   Exercises

1. Given vectors $\underset{\sim}{\boldsymbol{a}} = (1, 2, 3)$ and $\underset{\sim}{\boldsymbol{b}} = (-1, 0, 3)$, find

   (a) $\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}}$ *Answer*: 8.

   (b) $\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}}$ *Answer*: $(6, 6, –2)$.

   (c) norm $\underset{\sim}{\boldsymbol{b}}$ *Answer*: $\sqrt{10}$.

   (d) $\text{proj}\,\underset{\sim}{\boldsymbol{a}}\,\underset{\sim}{\boldsymbol{b}}$ *Answer*: $\left(\dfrac{4}{7}, \dfrac{8}{7}, \dfrac{12}{7}\right)$.

   (e) the angle between $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ in radians and degrees.   *Answer*: 0.828 rad; $47.5°$.

2. Given vectors $\underset{\sim}{\boldsymbol{a}} = \left(\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{7}\right)$ and $\underset{\sim}{\boldsymbol{b}} = \left(\dfrac{2}{3}, \dfrac{3}{5}, \dfrac{2}{7}, -1\right)$, find

   (a) $\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}}$ *Answer*: $\dfrac{47}{105}$.

   (b) norm $\underset{\sim}{\boldsymbol{b}}$ *Answer*: 1.373 (to 3DP).

   (c) $\text{proj}\,\underset{\sim}{\boldsymbol{a}}\,\underset{\sim}{\boldsymbol{b}}$ *Answer*: $\big(0.531,\ 0.354,\ 0.212,\ 0.152\big)$ (to 3DP).

   (d) the angle between $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ in radians and degrees.   *Answer*: 1.05 rad; $59.9°$.

20   Population Modelling 3: Matrix Models

20.1   Population-projection matrices

In 2005, the Australian Government's Productivity Commission released its research report, *Economic Implications of an Aging Australia*[35]. Planners were concerned that the proportion of older people was increasing, which would eventually put pressure on social services like health care. There would also be fewer people in the working population to pay for the required services and pensions. The distribution of the population with respect to age is changing.

To describe the population, we can divide it into age groups, for example into five-year groups, 0–5 years, 6–10 years, etc. It is also useful to split the population into males and females.

Here are some data taken from the Productivity Report, with the Australian population in one-year age groups and split into males and females.
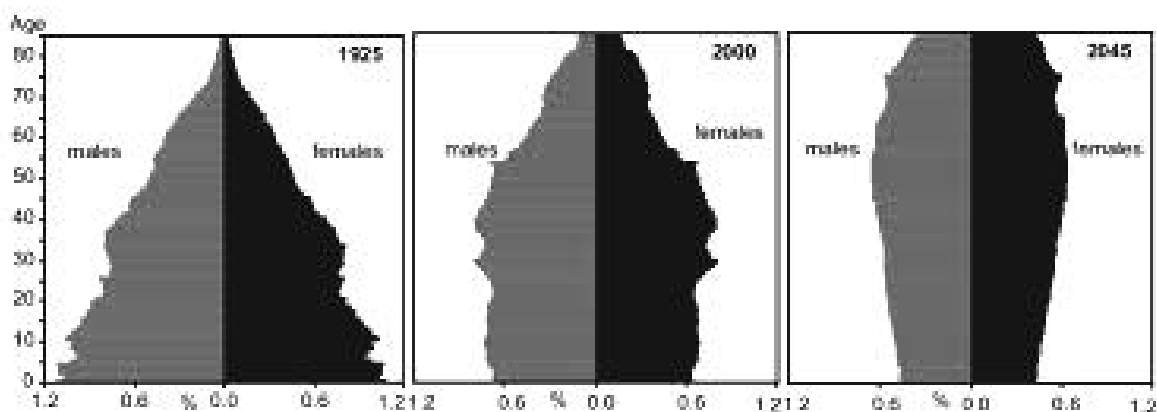
From pyramid to coffin. Changing age structure of the Australian population, 1925–2045.

Some of the results are measured data (1925 and 2000), while those for 2045 are clearly predictions. The trend is very clear, with a predominance of young people in 1925 (after World War 1) changing to a more even distribution in 2000 up to the age of the 'baby boomers' (40–55 years).

We follow this approach in the examples in this section: splitting a population into age classes and seeing how these classes evolve over time.

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[35]www.pc.gov.au/inquiries/completed/ageing/report

20.1.1   Simple model

Divide a population into numbers of young y_n and numbers of adults a_n.

We define the **population vector** $\underset{\sim}{p_n} = \begin{bmatrix} y_n \\ a_n \end{bmatrix}$.

We now need the rules that tell us how the population evolves each cycle, that is each time step. From observation, we find that each cycle **on average**:

- 48% of adults produce young;

- all young mature into adults;

- 20% of adults die.

This translates into the equations (*do it*)

$$
\begin{aligned}
y_{n+1} &= 0y_n + 0.48a_n &= 0y_n + 0.48a_n \\
a_{n+1} &= 1y_n + (1-0.2)a_n &= 1y_n + 0.8a_n.
\end{aligned}
$$

Written in matrix form, we have

$$
\begin{bmatrix} y_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0.48 \\ 1 & 0.8 \end{bmatrix} \begin{bmatrix} y_n \\ a_n \end{bmatrix},
$$

$$
\text{or} \qquad \underset{\sim}{p_{n+1}} = \underline{\underline{T}}\,p_n.
$$

The transition matrix $\underline{\underline{T}} = \begin{bmatrix} 0 & 0.48 \\ 1 & 0.8 \end{bmatrix}$ tells how the population evolves.

We explore this problem using a numerical experiment (*calculators ready*).

Suppose that at the start of Year 1 we have 100 young and 100 adults.

Then, in Year 2, $\quad \underset{\sim}{p_2} = \underline{\underline{T}}\underset{\sim}{p_1} = \begin{bmatrix} 0 & 0.48 \\ 1 & 0.8 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 48 \\ 180 \end{bmatrix}.$

Doing this calculation repeatedly gives the population evolution from year to year:

$$
\begin{bmatrix} 100 \\ 100 \end{bmatrix} \to \begin{bmatrix} 48 \\ 180 \end{bmatrix} \to \begin{bmatrix} 86.4 \\ 192 \end{bmatrix} \to \begin{bmatrix} 92.2 \\ 240 \end{bmatrix} \to \begin{bmatrix} 115.2 \\ 284.2 \end{bmatrix} \to \cdots
$$
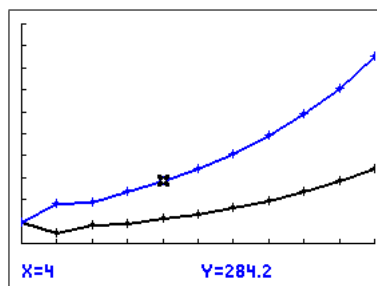
Populations of adults (top curve) and young over 10 cycles.

window $[0, 10, 1] \times [0, 1000, 100]$

The pattern here is not obvious, although the curves eventually look exponential. In fact you will find that, after enough cycles and no matter how we start off,

- the number of young is 0.4 times the number of adults;
- the overall population, young + adults, increases by 20% each year.

That is, *eventually*, the population vector behaves like

$$\cdots \rightarrow \begin{bmatrix} 0.4x \\ x \end{bmatrix} \rightarrow 1.2 \begin{bmatrix} 0.4x \\ x \end{bmatrix} \rightarrow (1.2)^2 \begin{bmatrix} 0.4x \\ x \end{bmatrix} \rightarrow (1.2)^3 \begin{bmatrix} 0.4x \\ x \end{bmatrix} \rightarrow \cdots ,$$

and only the value of x depends on how we start off.

We have a *steady-state distribution* in the population (relative numbers of young and adults), but the overall population is increasing by 20% each cycle, i.e.

$$\underline{\underline{T}}\, \underset{\sim}{p}_{ss} = 1.2\, \underset{\sim}{p}_{ss}, \tag{1}$$

where $\underset{\sim}{p}_{ss} = \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}$ or any multiple of it.

We come to a set of questions.

1. Why is the increase factor equal to 1.2?

2. Where does the 40 : 100 ratio for *young* : *old* come from?

3. How could we manage this population? For example, how could we stabilise it so that the 1.2 became 1? What would that do to the distribution of young and adults?
   Can we sort out all of this without going through the repeated-multiplications method?

The answers to these three questions come from considering the eigenvalues and eigenvectors of $\underline{\underline{T}}$: Eq. (1) is an eigenvalue equation for $\underline{\underline{T}}$.

The eigenvalues of $\underline{\underline{T}}$ are 1.2 and -0.4. Clearly, -0.4 is not physically relevant. The dominant eigenvalue 1.2 gives the growth rate of the population, with corresponding eigenvector $\underset{\sim}{p}_{ss}$.

To stabilise the population, we would have to change the non-zero elements of $\underline{\underline{T}}$ so that the dominant eigenvalue 1.2 became 1.

The corresponding eigenvector would then give the relative numbers of young and adults.

The eigenvalue method gives us the means to do these calculations without going through the repeated-multiplications method although the graphics generated by, for example, the POPMX4/POPMX4CE program are a quick way to see the overall picture.

Summary (general result)

1. The dominant eigenvalue of the transition matrix $\underline{\underline{T}}$ gives the rate of overall population growth (or decline).

2. The corresponding eigenvector gives the steady-state distribution of the population between the sub-classes.

20.1.2   Leslie matrices

The Leslie matrix is a discrete, age-structured model of population growth that is very popular in population ecology. It was invented by and named after Patrick H. Leslie.

We divide a population into a number of classes — here we shall assume three classes, referring to three age groups, young, adults and seniors, with respective numbers y, a and s. The population is then described by the vector or 3×1 column matrix

$$\underset{\sim}{\boldsymbol{v}} = \begin{bmatrix} y \\ a \\ s \end{bmatrix}.$$

A 3×3 transition matrix $\underline{\underline{\boldsymbol{T}}}$ tells us how the population evolves. For example, if the population to start with is

$$\underset{\sim}{\boldsymbol{v}_0} = \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix},$$

after one cycle it is

$$\underset{\sim}{\boldsymbol{v}_1} = \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \underline{\underline{\boldsymbol{T}}} \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix} = \underline{\underline{\boldsymbol{T}}}\underset{\sim}{\boldsymbol{v}_0}.$$

In problems leading to a Leslie transition matrix:

- in each cycle, members of the other classes produce a certain number of new young in Class 1;
- a certain fraction of each class survives to move into the next class; the rest die;
- all members of the top class die.

This leads to a Leslie matrix $\underline{\underline{\boldsymbol{T}}}$ that is zero everywhere except possibly:

- along the top row after the first element — the birth rates for each class;

- in the elements along the diagonal parallel to and just below the main diagonal — the survival rates for each class.

$$\begin{bmatrix} 0 & * & * & * & * & \cdots & * & * \\ * & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & * & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & * & 0 \end{bmatrix}$$

Leslie discovered these matrices in the 1940s when he pioneered this way of exploring how populations can develop. He taught himself matrix algebra while he was in hospital with TB.

Exercise: Leslie matrices and beetles 1   *Solutions are in Section 20.3.*

(a) During each cycle, each adult beetle produces on average 2.75 young and each senior beetle produces on average 2.5 young; one quarter of the young beetles survive to become adults; and one half of the adult beetles survive to become seniors. In a Leslie-matrix problem, all the seniors die.

Find the Leslie transition matrix $\underline{\underline{T}}$.

Good strategy: Write out the linear equations for y_1, a_1 and s_1 in terms of y_0, a_0 and s_0, and convert to matrix form.

$$y_1 = ?y_0 + ?a_0 + ?s_0$$

$$a_1 = ?y_0 + ?a_0 + ?s_0$$

$$s_1 = ?y_0 + ?a_0 + ?s_0$$

(b) If we start with 40 young and no adults or seniors, show that after one cycle

$$\underset{\sim}{v_1} = \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}.$$

You should have entered $\underline{\underline{T}}$ into [A], the initial $\underset{\sim}{v}$ into [B] and evaluated [A][B].

(c) Multiply repeatedly by $\underline{\underline{T}}$, and record $\underset{\sim}{v}$ and the total population $P = y + a + s$ after 11, 12 and 13 cycles.[36]

You might like to plot the populations too to help with the questions.

What happens to the total population? to the ratios of the numbers in the different classes?

Exercise: Leslie matrices and beetles 2

In the previous exercise we found that a certain population of beetles was described by the equations $\underset{\sim}{v_{n+1}} = \underline{\underline{T}}\underset{\sim}{v_n}$, where

$$\underline{\underline{T}} = \begin{bmatrix} 0 & 2.75 & 2.5 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

An orchardist has beetles in her trees and knows the beetle population evolves as described by this transition matrix,

At the start of last year she employed a pest-control company to reduce the beetle numbers; they did that (they claim) by using a spray that killed off the senior beetles.

This year the orchardist did a survey over 50 trees and found 800 young beetles, 200 adults and 100 seniors.

She now says the pest-control company failed and she wants her money back.

Does she have a case?

---

[36]the POP/POPCE program (Section 20.2) helps here

20.1.3   Populations and oscillations

Workers other than Leslie had independently used matrix algebra in population models. The first was Harro Bernardelli, who published a paper in 1941 in the *Journal of the Burma Research Society* with the title *Population Waves*.

Bernardelli's paper was unusual in focussing not on the eventual stability of the population structure, but on intrinsic oscillations in the population structure. He had observed oscillations in the age structure of the Burmese population between 1901 and 1931.

As an abstract model for such oscillations, he proposed a matrix model for the evolution of the population with

$$\underline{\underline{T}} = \begin{bmatrix} 0 & 0 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix},$$

and showed by numerical calculations that this gave rise to apparently permanent oscillations in the age structure.

Exercise

(a) Record the total population P at each cycle for 12 cycles using Bernardelli's matrix $\underline{\underline{T}}$

with $\underset{\sim}{v_0} = \begin{bmatrix} 1 \\ 0.01 \\ 0.01 \end{bmatrix}$, the initial populations in three age groupings in millions.

Do this by hand or using POP/POPCE (Section 20.2).

Plot P versus cycle number, joining up the points with straight lines (see Section 20.2).

Discuss your findings.

(b) Repeat (a) using $\underline{\underline{T}} = \begin{bmatrix} 0 & 0 & 5 \\ 0.7 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$.

Describe your results in words. Explain in terms of the entries in $\underline{\underline{T}}$.

Solutions are in Section 20.3.

20.1.4   Exercise: A bank workforce

The manager of a bank has in his workforce t_n trainee tellers and s_n senior tellers at the start of Year n. At the end of each year:

— the number of new trainee tellers recruited is set equal to the number of senior tellers in the bank that year;

— 36% of trainee tellers graduate to become seniors; 24% are allowed to break their contract and leave the bank, while the rest remain as trainees;

— 40% of senior tellers are promoted to higher positions; 20% of senior tellers leave; the remainder stay as senior tellers.

(a) Write down the equations for the teller workforce at the start of Year $n+1$ in terms of the workforce at the start of Year n.

(b) Convert the equations in (a) to matrix form.

(c) If there are 125 trainees and 125 senior tellers at the start of Year 1, how many of each are there at the start of Year 3?

(d) Is there a steady state here? If so, what is it?

(e) Find the eigenvalues and eigenvectors of the transition matrix and reconcile them with your answer in (d).

Solutions are in Section 20.3.

20.1.5   Exercise: Disease modelling

A population of mice in a medical experiment at the beginning of Week n is specified by the number h_n of healthy mice and the number i_n of mice infected by the virus under investigation.

During each week:

> 40% of the healthy mice contract the virus;
>
> 30% of the infected mice recover to become healthy again;
>
> 50% of the infected mice die.

(a) Find the linear equations that give the number of healthy mice h_{n+1} and the number of infected mice i_{n+1} at the beginning of Week $n+1$ in terms of the numbers h_n, i_n at the beginning of Week n.

(b) Define the mouse population vector $\underset{\sim}{\boldsymbol{v}}_n = \begin{bmatrix} h_n \\ i_n \end{bmatrix}$ and write your equations in matrix form.

(c) If the experiment starts at the beginning of Week 1 with 1000 healthy mice, find the populations of healthy and infected mice at the beginning of Week 2.

(d) Find the respective populations at the beginning of Week 8.

(e) Given 1000 mice to start with, how many would have to be healthy and how many infected to start with the 'steady state'?

What is the form of 'steady state' here?

What do you predict for the long-term future of this population?

Solutions are in Section 20.3.

20.1.6   Exercise: Killer whales

Based on a Year-12 project set by Margaret McLaughlan of St Francis Xavier College.

Teachers (and students) may find the programs WHALES/WHALESCE (Section 20.2) useful (but not essential) for experimenting with the population-projection matrix.

Leslie matrices are just a special case of a more general population-projection matrix. In the more general case, animals in a class may remain in that class for more than one cycle. The probability that an animal remains in a particular class for any given cycle is an element on the diagonal of the matrix, immediately above the entry in the Leslie matrix, which gives the probability of moving to the next class in any given cycle.

Therefore we have a matrix whose elements along the top row give the fecundity or birth rate per animal per cycle for each class, whose diagonal elements give the probability of an animal remaining in a particular class in any cycle and whose elements below the diagonal give the probability of an animal moving to the next class in any cycle. The fact that the latter two numbers in any column do not add up to 1 means that some animals in each class die each cycle.

For female killer whales, we have four classes — yearlings (individuals in the first year of life), juveniles (past the first year, but not mature), mature females and post-reproductive females. The mean period in the juvenile stage is 13.4 years and in the mature stage 22.1 years, with an overall lifetime of 80–90 years. Details in Brault and Caswell, *Pod-specific demography of killer whales*, Ecology 74, 1444–1454 (1993).

The population-projection matrix for female killer whales is given below. The time for one cycle is one year.

$$\underline{\underline{T}} = \begin{bmatrix} 0 & 0.0043 & 0.1138 & 0 \\ 0.9775 & 0.9111 & 0 & 0 \\ 0 & 0.0736 & 0.9534 & 0 \\ 0 & 0 & 0.0452 & 0.9804 \end{bmatrix}$$

1. A project you are involved in wants to re-introduce killer whales into an area of ocean from which they have disappeared. The project leader wants to know what is the best combination of juveniles and mature females to re-introduce, assuming an overall total of 50 (plus an appropriate number of adult males). You decide to model three options over 40 cycles (years).

   (a) 50 female juveniles.

   (b) 40 female juveniles and 10 mature females.

   (c) 50 mature females.

   What happens to the different population classes of female killer whales over time in each of these options (according to this model)?
   What is the best strategy for re-establishing the killer-whale population?

   Hint: (*manual method*) If $\underline{\underline{T}}$ is in the 4×4 matrix [A] and the initial population $\underset{\sim}{v_0}$ in the 4×1 matrix [B], executing the command [A] [B] $\rightarrow$ [B], then repeatedly pressing ⎡enter⎤ will generate successive population vectors $\underset{\sim}{v_1}$, $\underset{\sim}{v_2}$, $\underset{\sim}{v_3}$, etc.[37]
   The symbol $\rightarrow$ stands for the ⎡sto⎤ key.

---

[37]You could also run POP/POPCE (Section 20.2), repeatedly pressing ⎡enter⎤, to give the total population directly.

2. Does the trend in the total population depend on which option you choose?

3. The value of 0.1138 in the top row of $\underline{\underline{T}}$ gives the number of live births per mature female per cycle (year).

To what value could this birth rate fall before the total population starts to decrease?

This birth rate is clearly important for the overall survival of killer whales.

Hint: See the discussion in Section 20.2 on the eigenvalue λ_p of the matrix $\underline{\underline{T}}$.

4. How sensitive is the population to the survival rates of yearlings (0.9775), juveniles (0.9111), mature females (0.9534) and post-reproductive females (0.9804)?

You might like to quantify your answers here by determining what percentage decrease in each rate is needed to stop the population growing.

Do your answers make sense?

The WHALES/WHALESCE program (Section 20.2) does these calculations and also plots the populations.

Solution are in Section 20.3.

PTO

20.1.7   Exercise: Age distribution of trees in a forest

The difference-equation version of this problem can be found in *Population Modelling 2* in Volume 2 of this book.

The population of trees in a forest is split into four age groups: b_n is the number of baby trees ($0-15$ years old) at time-point n; y_n the number of young trees ($16-30$ years); m_n middle-aged trees ($31-45$ years old) ; and o_n old trees (more than 45 years old).

The time step for our difference equations is 15 years.

In order to simplify the model we make the following assumptions:

A. a certain percentage of trees in each age group dies in each time interval;

B. surviving trees age into the next age group each time step; old trees remain old trees (or die);

C. dead trees are replaced by an equal number of baby trees.

Define α, β, γ, δ as the fraction of dead trees in the respective age groups in each time interval. Then, the difference-equation model is

$$b_{n+1} \;=\; \alpha b_n + \beta y_n + \gamma m_n + \delta o_n \qquad \text{Assumption C} \qquad\qquad (2)$$

$$y_{n+1} \;=\; (1-\alpha)b_n \qquad\qquad\qquad \text{Assumptions A, B} \qquad\qquad (3)$$

$$m_{n+1} \;=\; (1-\beta)y_n \qquad\qquad\qquad \text{Assumptions A, B} \qquad\qquad (4)$$

$$o_{n+1} \;=\; (1-\gamma)m_n + (1-\delta)o_n \qquad \text{Assumptions A, B.} \qquad\qquad (5)$$

1. If the population of trees in time interval n is $N = b_n + y_n + m_n + o_n$, show that the population stays the same size after one more time step, and so by induction the population of trees is a constant N.

2. Equations $(1)-(4)$ are linear in the dependent variables so that matrices are a way of solving the problem. Define $\underset{\sim}{\boldsymbol{v}}_n = (b_n, y_n, m_n, o_n)^{\text{T}}$ (a column vector) and write down the matrix $\underline{\underline{\boldsymbol{T}}}$ that you would use in the matrix model $\underset{\sim}{\boldsymbol{v}}_{n+1} = \underline{\underline{\boldsymbol{T}}}\underset{\sim}{\boldsymbol{v}}_n$ of these equations.

3. Take $\alpha = 0.2$, $\beta = 0.5$, $\gamma = 0.3$, $\delta = 0.2$ and $N = 1000$. Write down the matrix $\underline{\underline{\boldsymbol{T}}}$ with these values.

4. Four initial conditions are needed in order to fully solve this matrix equation. Assume all baby trees initially.

   Run the matrix model in Problems 2 and 3 through 10 cycles (150 years). Alternatively, you could use the program POPMX4/POPMX4CE (Section 20.2) to calculate and plot your results. Do the individual populations appear to be stabilising?

5. Use the EIGENV5/EIGNV5CE program to find the eigenvalues of $\underline{\underline{\boldsymbol{T}}}$.

6. Find the eigenvector of $\underline{\underline{\boldsymbol{T}}}$ corresponding to $\lambda = 1$ (*explain why we use this value*) and relate this to the behaviour of the model that you found in Problem 4.

Solutions are in Section 20.3.

20.1.8   The population problem with harvesting

We return to a population problem similar to that in Section 20.1.1 but now we investigate how harvesting can be included. Theory tells us the 'exact' solution but graphics allow us to explore around that solution.

A population comprises y_n young and a_n adults at the start of Year n. Each year, 90% of the young survive to become adults, 90% of the adults have young and 20% of adults die. Thus,

$$\left[\begin{array}{c} y_{n+1} \\ a_{n+1} \end{array}\right] = \underline{\underline{T}} \left[\begin{array}{c} y_n \\ a_n \end{array}\right] = \left[\begin{array}{cc} 0 & 0.9 \\ 0.9 & 0.8 \end{array}\right] \left[\begin{array}{c} y_n \\ a_n \end{array}\right]. \qquad \left(\begin{array}{c} Check \\ that! \end{array}\right)$$

We now wish to harvest a fraction h of the adults each year but maintain a stable population.

Note that the same mathematics works if we are 'harvesting' some adults from a captive population of rare animals to release back into the wild.

The adult 'death' rate becomes $0.2+h$, so that $\underline{\underline{T}}$ becomes $\left[\begin{array}{cc} 0 & 0.9 \\ 0.9 & 0.8-h \end{array}\right]$.

If we now choose h so that $\underline{\underline{T}}$ has an eigenvalue $\lambda_1 = 1$, and also choose for the initial relative distribution of young and old the corresponding eigenvector $\underset{\sim}{v_1}$, we know we will have a steady-state situation.

For the eigenvalues, $|\underline{\underline{T}} - \lambda \underline{\underline{I}}| = \left|\begin{array}{cc} -\lambda & 0.9 \\ 0.9 & 0.8-h-\lambda \end{array}\right| = 0$, giving $-\lambda(0.8-h-\lambda) - 0.81 = 0$.

If $\lambda = 1$ is to be a solution, we must have $-(0.8-h-1) - 0.81 = 0$, so that $h = 0.61$.

For $h = 0.61$, $\underline{\underline{T}} = \left[\begin{array}{cc} 0 & 0.9 \\ 0.9 & 0.19 \end{array}\right]$, and $\lambda = 1$ gives eigenvector $\underset{\sim}{v} = t \left[\begin{array}{c} 9 \\ 10 \end{array}\right]$ (*check*).

Therefore, a population of 9 young for every 10 adults will allow 61% of adults to be harvested each year while maintaining a stable population.

If we wanted to harvest 122 adults each year, we normalise this vector by choosing $ha_n = 0.61 \times 10t = 122$, i.e. by setting $t = 20$. The initial population of young is then 180, adults 200.

What happens if, say, $h = 0.55$ or 0.66, slight variations on the harvesting rate? We can recalculate the eigenvalue corresponding to 1 and the corresponding eigenvector to see how they change but this doesn't give us much idea of how the populations then vary over time (the new eigenvalue does give the growth rate). Plotting the populations shows us this.

Changing h in $\underline{\underline{T}} = \left[\begin{array}{cc} 0 & 0.9 \\ 0.9 & 0.8-h \end{array}\right]$ and using the POPMX4/POPMX4CE program does this quickly.
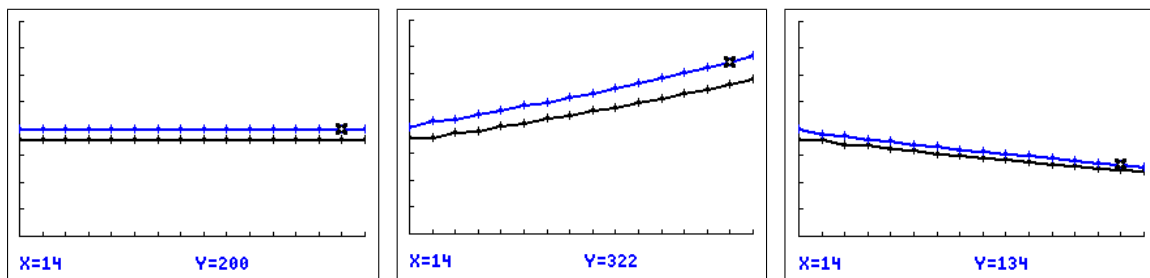
Adults (top curve) and young, 15 cycles, with $h = 0.61$, $h = 0.55$ and $h = 0.66$, respectively.

window $[0, 15, 1] \times [0, 400, 50]$

20.2   Matrix-modelling programs

Available at *canberramaths.org.au* under *Resources*.

POP/POPCE multiplies an $n \times 1$ column vector $\underset{\sim}{\boldsymbol{v}}$ (n age classes) by an $n \times n$ matrix $\underset{=}{\boldsymbol{T}}$ (transition matrix), displays the new $\underset{\sim}{\boldsymbol{v}}$ and the sum of the components of $\underset{\sim}{\boldsymbol{v}}$ (total population).

Use: Store $\underset{=}{\boldsymbol{T}}$ in matrix [A] and $\underset{\sim}{\boldsymbol{v}}$ in matrix [B]. Run the program for the first step; press
$\boxed{\text{enter}}$ repeatedly for subsequent steps. Press $\boxed{\text{on}}$ $\boxed{1}$ (Quit) to stop the program.

POPMX4/POPMX4CE calculates and plots as a function of cycle number (time) the numbers in *from 2 to 4 age classes* of a population which is modelled using a projection matrix.

WHALES/WHALESCE is a version specifically for the problem on killer whales on page 155. **POPCLEAR/POPCLRCE** deletes all the lists used in WHALES/WHALESCE and resets defaults. Run when you have finished using a program.

POPMXM/POPMXMCE calculates and displays (no plots) as a function of cycle number (time) the numbers *in M age classes* of a population and gives the ratios of the individual populations.

Use: Put your 2×2, 3×3, 4×4 or $m \times m$ projection matrix $\underset{=}{\boldsymbol{P}}$ in matrix [A] and the initial populations in the different age classes in the 2×1, 3×1, 4×1 or $m \times 1$ matrix [B].

Run the relevant program and input the number of cycles for which you wish to calculate the populations. The program will do the calculations.

POPMX4/POPMX4CE pauses in $\boxed{\text{trace}}$ mode so that you can analyse the graphs. The arrow keys allow you to move along a graph or from graph to graph. Which graph the cursor is on is indicated at the top left of the screen.

Pressing $\boxed{\text{enter}}$ then gives a plot of the total population versus cycle number. Again you can $\boxed{\text{trace}}$ this graph.

Pressing $\boxed{\text{enter}}$ again gives you a menu from which you can either quit, keeping the matrices, lists and other settings, or quit, deleting everything. The former allows you to replot data or look at the numbers in $\boxed{\text{stat}}$ Edit. The latter is a good option when you have finished.

Once POPMX4/POPMX4CE has finished (and you keep all the data), you can replot and $\boxed{\text{trace}}$ any or all of the graphs of the individual populations versus cycle by selecting the appropriate plot in the $\boxed{\text{y=}}$ menu.

You can change the window either manually in the $\boxed{\text{window}}$ menu or using $\boxed{\text{zoom}}$ $\boxed{9}$ (Zoom-Stat).

To plot the graph of the fourth population class, you will have to change the Ylist to ʟCLS4 in one of the plots using $\boxed{\text{statplot}}$ $\left(\boxed{\text{2nd}}\ \boxed{\text{y=}}\right)$.

The data generated in these programs are stored in lists which can be accessed by pressing $\boxed{\text{stat}}$ Edit. Scrolling across will show the lists not initially on the screen.

20.3 Solutions

Exercise: Leslie matrices and beetles 1

(a) Writing out the equations for the three beetle age classes,

$$
\begin{aligned}
y_1 &= 0y_0 + 2.75a_0 + 2.5s_0 \\
a_1 &= 0.25y_0 + 0a_0 + 0s_0 \\
s_1 &= 0y_0 + 0.5a_0 + 0s_0
\end{aligned}
\qquad \text{or} \qquad
\begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix}
=
\begin{bmatrix} 0 & 2.75 & 2.5 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}
\begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix}.
$$

Therefore, $\underset{=}{\boldsymbol{T}} = \begin{bmatrix} 0 & 2.75 & 2.5 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$

(b) $\underset{\sim}{\boldsymbol{v}}_1 = \underset{=}{\boldsymbol{T}} \begin{bmatrix} 40 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}.$

(c)

After cycle	$\underset{\sim}{\boldsymbol{v}}$	Total pop'n
1	$\begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$	10
2	$\begin{bmatrix} 27.5 \\ 0 \\ 5 \end{bmatrix}$	32.5
$\vdots$	$\vdots$	$\vdots$
11	$\begin{bmatrix} 17.320 \\ 4.305 \\ 2.184 \end{bmatrix}$	23.809
12	$\begin{bmatrix} 17.299 \\ 4.330 \\ 2.153 \end{bmatrix}$	23.781
13	$\begin{bmatrix} 17.289 \\ 4.325 \\ 2.165 \end{bmatrix}$	23.779
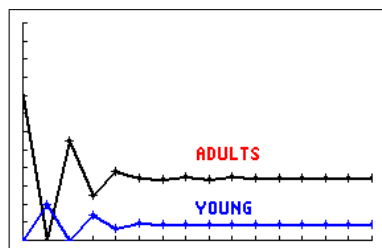
Young and adults, 15 cycles   Total population
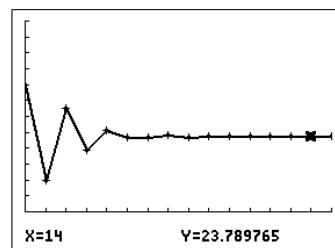
window $[0, 15, 1] \times [0, 60, 5]$

The total population seems to be stabilising at a little under 24, and the ratios of the populations in the 3 classes at about $8:2:1$.

Divide the first two numbers by the third (smallest) number to see this.

Exercise: Leslie matrices and beetles 2

In the previous exercise we found that a certain population of beetles was described by the equations $\underset{\sim}{v}_{n+1} = \underset{=}{T}\underset{\sim}{v}_n$, where

$$\underset{=}{T} = \begin{bmatrix} 0 & 2.75 & 2.5 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

An orchardist has beetles in her trees and knows the beetle population evolves as described by this transition matrix,

At the start of last year she employed a pest-control company to reduce the beetle numbers; they did that (they claim) by using a spray that killed off the senior beetles.

This year the orchardist did a survey over 50 trees and found 800 young beetles, 200 adults and 100 seniors.

She now says the pest-control company failed and she wants her money back.

Does she have a case?

To go backwards, we use $\underset{=}{T}^{-1}$. The number of beetles the year before, supposedly after the pest-control company killed off all the seniors, is

$$\underset{=}{T}^{-1} \begin{bmatrix} 800 \\ 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 800 \\ 200 \\ 100 \end{bmatrix}.$$

Two things come out here. One is that the pest-control company could not have killed off all the seniors the year before. Clearly the orchardist has a good case.

The second point is that $\begin{bmatrix} 800 \\ 200 \\ 100 \end{bmatrix} = 100 \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$ is a steady-state vector for the population with this transition matrix.

Exercise: Populations and oscillations

(a) After 1 cycle: $\mathbf{v}_1 = \begin{bmatrix} 0 & 0 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.01 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.08 \\ 0.5 \\ 0.0025 \end{bmatrix}$, so $P = 0.5825$.

From successive cycles, we build up a table.

Cycle	0	1	2	3	4	5	6	$\cdots$
Pop'n	1.02	0.5825	0.185	1.02	0.5825	0.185	1.02	$\cdots$
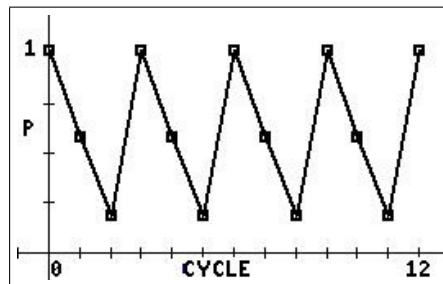
The population is oscillating or going in waves, with no overall growth or decline.

A group of young must first become adults (survival rate 0.5), then seniors (survival rate 0.25) before producing a new group of 8 young; the process then repeats itself.

The overall survival rate between young and seniors of $0.5 \times 0.25 = 1/8$ is balanced by a birth rate of 8, so that the overall population is not growing or declining.

(b) The populations are now 1.02, 0.755, 0.41, 1.785, 1.321, 0.718, 3.124, 2.312, 1.256, 5.467, 4.046, 2.197, 9.566, 7.081, 3.846, 16,741, ...

The population is oscillating, but growing overall. The birth rate of 5 and the survival rate of $0.7 \times 0.5 = 0.35$ gives an overall growth rate of 1.75 (dashed line).

Exercise: A bank workforce

The manager of a bank has in his workforce t_n trainee tellers and s_n senior tellers at the start of Year n. At the end of each year:

— the number of new trainee tellers recruited is set equal to the number of senior tellers in the bank that year;

— 36% of trainee tellers graduate to become seniors; 24% are allowed to break their contract and leave the bank, while the rest remain as trainees;

— 40% of senior tellers are promoted to higher positions; 20% of senior tellers leave; the remainder stay as senior tellers.

(a) Write down the equations for the teller workforce at the start of Year $n+1$ in terms of the workforce at the start of Year n.

$$t_{n+1} = 0.4t_n + s_n$$
$$s_{n+1} = 0.36t_n + 0.4s_n$$

(b) Convert the equations in (a) to matrix form.

$$\begin{bmatrix} t_{n+1} \\ s_{n+1} \end{bmatrix} = \begin{bmatrix} 0.4 & 1 \\ 0.36 & 0.4 \end{bmatrix} \begin{bmatrix} t_n \\ s_n \end{bmatrix}.$$

(c) If there are 125 trainees and 125 senior tellers at the start of Year 1, how many of each are there at the start of Year 3?

We are given that $\begin{bmatrix} t_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 125 \\ 125 \end{bmatrix}$.

Therefore, at the start of Year 2, we have

$$\begin{bmatrix} t_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0.4 & 1 \\ 0.36 & 0.4 \end{bmatrix} \begin{bmatrix} 125 \\ 125 \end{bmatrix} = \begin{bmatrix} 175 \\ 95 \end{bmatrix}.$$

At the start of Year 3, we have

$$\begin{bmatrix} t_3 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0.4 & 1 \\ 0.36 & 0.4 \end{bmatrix} \begin{bmatrix} 175 \\ 95 \end{bmatrix} = \begin{bmatrix} 165 \\ 101 \end{bmatrix}.$$

There are 165 trainee tellers and 101 senior tellers at the start of Year 3.

With your graphics calculator, you can calculate directly

$$\begin{bmatrix} t_3 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0.4 & 1 \\ 0.36 & 0.4 \end{bmatrix}^2 \begin{bmatrix} 125 \\ 125 \end{bmatrix} = \begin{bmatrix} 165 \\ 101 \end{bmatrix}.$$

(d) Is there a steady state here? If so, what is it?

If we continue to multiply by the transition matrix $\begin{bmatrix} 0.4 & 1 \\ 0.36 & 0.4 \end{bmatrix}$, we find the steady-state workforce is

$$\begin{bmatrix} 166.\dot{6} \\ 100 \end{bmatrix} = \begin{bmatrix} \frac{500}{3} \\ 100 \end{bmatrix} = 100 \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix},$$

that is 166 or 167 trainee tellers and 100 senior tellers.

(e) Find the eigenvalues and eigenvectors of the transition matrix and reconcile them with your answer in (d).

The eigenvalues are -0.2 and 1. The larger of these indicates a steady state of no growth.

The corresponding eigenvector is $\begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}$, in agreement with the calculations above.

20.3.1   Exercise: Disease modelling

A population of mice in a medical experiment is specified at the beginning of Week n by the number h_n of healthy mice and the number i_n of mice infected by the virus under investigation.

During each week:

40% of the healthy mice contract the virus;

30% of the infected mice recover to become healthy again;

50% of the infected mice die.

(a) Find the linear equations that give the number of healthy mice h_{n+1} and the number of infected mice i_{n+1} at the beginning of Week $n+1$ in terms of the numbers h_n, i_n at the beginning of Week n.

$$h_{n+1} = 0.6h_n + 0.3i_n$$

$$i_{n+1} = 0.4h_n + 0.2i_n$$

(b) Define the mouse population vector $\underset{\sim}{v}_n = \begin{bmatrix} h_n \\ i_n \end{bmatrix}$ and write your equations in matrix form.

$$\underset{\sim}{v}_{n+1} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.2 \end{bmatrix} \underset{\sim}{v}_n = \underset{\underset{\sim}{=}}{T} \underset{\sim}{v}_n.$$

PTO

(c) If the experiment starts at the beginning of Week 1 with 1000 healthy mice, use your matrix equation to find the populations of healthy and infected mice at the beginning of Week 2.

We have $\boldsymbol{v}_1 = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$. Therefore, at the beginning of Week 2, the population is

$$\boldsymbol{v}_2 = \boldsymbol{T}\boldsymbol{v}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 1000 \\ 0 \end{bmatrix} = \begin{bmatrix} 600 \\ 400 \end{bmatrix}.$$

There are 600 healthy mice and 400 infected mice at the beginning of Week 2.

(d) Find the respective populations at the beginning of Week 8.

Using the calculator to work through 7 cycles or calculating $\boldsymbol{v}_8 = \boldsymbol{T}^7 \boldsymbol{v}_1$ gives

$$\boldsymbol{v}_8 = \begin{bmatrix} 157 \\ 105 \end{bmatrix},$$

both rounded to the nearest integer.

Therefore, there are 157 healthy and 105 infected mice at the beginning of Week 8.

(e) Given 1000 mice to start with, how many would have to be healthy and how many infected to start with the 'steady state'?

From (d), we observe that the 'steady state' contains healthy and infected mice in the ratio 1.5 : 1. You could verify this by finding the eigenvector corresponding to the larger eigenvalue of $\boldsymbol{T}$, $\lambda = 0.8$.

Therefore, we should start with 600 healthy mice and 400 infected mice. This is actually achieved after the first cycle.

What is the form of 'steady state' here?

The 'steady state' contains healthy and infected mice in the ratio 1.5 : 1, but the total population is reduced by 20% each cycle.

What do you predict for the long-term future of this population?

Extinction.

Exercise: Killer whales

1. A project you are involved in wants to re-introduce killer whales into an area of ocean from which they have disappeared. The project leader wants to know what is the best combination of juveniles and mature females to re-introduce, assuming an overall total of 50 (plus an appropriate number of adult males).

   You model three options, with the resulting plots shown below. The left-hand plots show the populations of juvenile females (squares), mature females (crosses) and post-reproductive females (PRF: thin line); the right-hand plots show the total population (including yearlings), both as functions of the number of cycles (1 cycle = 1 year).

   (a) 50 juveniles.
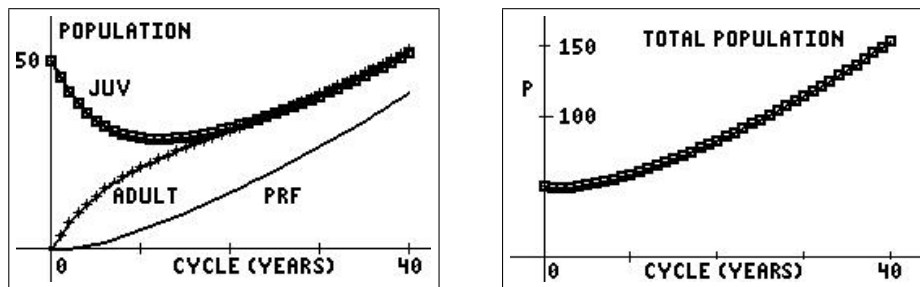
   (b) 40 juveniles and 10 mature females.
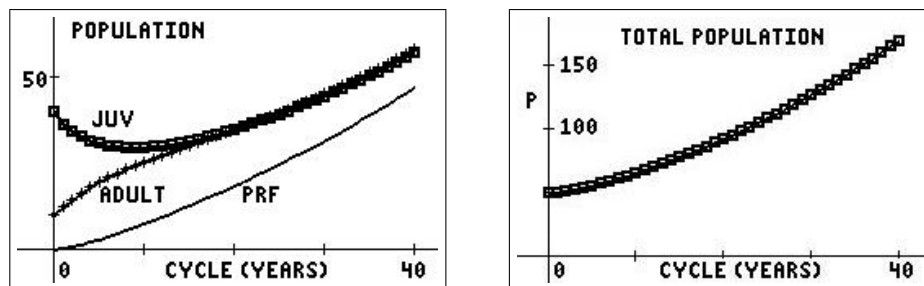
   (c) 50 adult females.
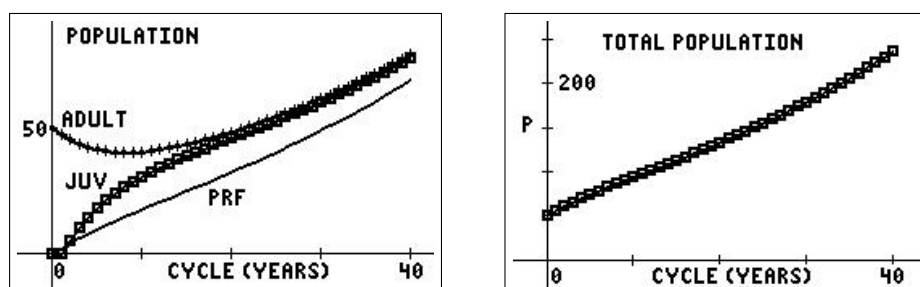
   PTO

What happens to the different population classes of female killer whales over time in each of these options (according to this model)? What is the best strategy for re-establishing the killer-whale population?

After about 15 years, in all three options, there are approximately equal numbers of juvenile and mature females, and this remains the case: both populations are increasing. The greater the inital number of mature females, the greater the numbers of both juvenile and mature females at any given cycle. The number of post-reproductive females follows a similar trend in all three options.[38]

In terms of the best strategy for re-introducing killer whales, it probably comes down to cost. Clearly the best option, according to the model, is to introduce all mature females but it could take quite a long time (and be expensive) to breed up a sufficient number, given it takes at least 14 years to maturity. Releasing all juveniles, a much cheaper option, runs the risk that not enough of the inexperienced juveniles survive long enough to establish a sustainable population. Some combination of juvenile and mature females may be the best option.

2. Does the trend in the total population depend on which option you choose?

In the three options with different initial populations, the trend is the same: the overall female population increases. However, the greater the inital number of mature females, the greater the total population at any given cycle. The fact of an increase in total population (and the eventual ratios of the populations in the four classes) does not depend on the initial populations in each of the classes.

3. The value of 0.1138 in the top row of $\underline{\underline{T}}$ gives the number of live births per mature female per cycle (year). To what value could this birth rate fall before the total population starts to decrease?

This birth rate is clearly important for the overall survival of killer whales.

Experimenting with different values of the birth rate for mature females, either manually or using POPMT4/POPMT4CE, shows that the population becomes steady when the value is about 0.055 ($\lambda_p \approx 1$), i.e. about 48% of the observed value. For birth rates less than 0.055, the total population will decline.

4. How sensitive is the population to the survival rates of yearlings (0.9775), juveniles (0.9111), mature females (0.9534) and post-reproductive females (0.9804)?

The female population is more sensitive to the survival rates than to the birth rates, especially that of the mature females. Reducing the survival rate of 0.9534 for mature females to 0.905 (a reduction of only 5%) is enough to stop the population growing.

For the other classes, the corresponding values are: yearlings 0.9775 down to 0.49 (50%); juveniles 0.9111 down to 0.82 (10%). The growth or otherwise of the population is unaffected by the survival rate of post-reproductive females in this model.

---

[38]The ratios of the populations in the different classes are those of the components of the eigenvector of the population-projection matrix corresponding to λ_p.

Exercise: Age distribution of trees in a forest

1. If the population of trees in time interval n is $N=b_n+y_n+m_n+o_n$, show that the population stays the same size after one more time step, and so by induction the population of trees is a constant N.

Adding the four equations gives

$$
\begin{aligned}
b_{n+1} + y_{n+1} + m_{n+1} + o_{n+1} &= \alpha b_n + \beta y_n + \gamma m_n + \delta o_n + (1-\alpha)b_n \\
&\quad + (1-\beta)y_n + (1-\gamma)m_n + (1-\delta)o_n \\
&= b_n + y_n + m_n + o_n \\
&= N.
\end{aligned}
$$

Therefore, the population of trees stays the same size after one more time step, and so by induction the population of trees is a constant N.

2. Equations $(1)-(4)$ are linear in the dependent variables so that matrices are a way of solving the problem. Define $\mathbf{v}_n = (b_n, y_n, m_n, o_n)^{\mathrm{T}}$ (a column vector) and write down the matrix $\underline{\underline{T}}$ that you would use in the matrix model $\mathbf{v}_{n+1} = \underline{\underline{T}}\mathbf{v}_n$ of these equations.

Equations $(1)-(4)$ can be written in matrix form as

$$
\mathbf{v}_{n+1} =
\begin{bmatrix}
b_{n+1} \\
y_{n+1} \\
m_{n+1} \\
o_{n+1}
\end{bmatrix}
=
\begin{bmatrix}
\alpha & \beta & \gamma & \delta \\
1-\alpha & 0 & 0 & 0 \\
0 & 1-\beta & 0 & 0 \\
0 & 0 & 1-\gamma & 1-\delta
\end{bmatrix}
\begin{bmatrix}
b_n \\
y_n \\
m_n \\
o_n
\end{bmatrix},
$$

so that

$$
\underline{\underline{T}} =
\begin{bmatrix}
\alpha & \beta & \gamma & \delta \\
1-\alpha & 0 & 0 & 0 \\
0 & 1-\beta & 0 & 0 \\
0 & 0 & 1-\gamma & 1-\delta
\end{bmatrix}.
$$

3. Take $\alpha=0.2$, $\beta=0.5$, $\gamma=0.3$, $\delta=0.2$ and $N=1000$. Write down the matrix $\underline{\underline{T}}$ with these values.

$$
\underline{\underline{T}} =
\begin{bmatrix}
0.2 & 0.5 & 0.3 & 0.2 \\
0.8 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 \\
0 & 0 & 0.7 & 0.8
\end{bmatrix}.
$$

PTO

4. Four initial conditions are needed in order to fully solve this matrix equation. Assume all baby trees initially.

Run the matrix model in Problems 2 and 3 through 10 cycles (150 years). Do the individual populations appear to be stabilising?

Run the model $\underset{\sim}{v}_{n+1} = \underset{=}{T}\underset{\sim}{v}_n$ through 10 cycles with $\underset{\sim}{v}_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, that is all baby

trees initially: with $\underset{=}{T}$ in matrix [A] and $\underset{\sim}{v}_0$ in matrix [B], execute the command [A] [B] → [B], then press $\boxed{\text{enter}}$ nine times.

Alternatively, with $\underset{=}{T}$ in matrix [A] and $\underset{\sim}{v}_0$ in matrix [B], run the POPMX4/ POPMX4CE program. This produces the following graph of the first three populations versus cycle number. The window is $[-0.2, 10.2, 1] \times [0, 1100, 100]$.
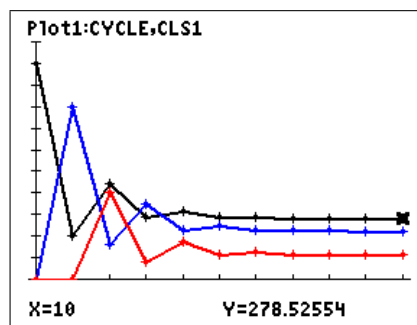
Here, the graph of b_n is in black, y_n in blue and m_n in red.

The number in each population appears to be stabilising: b_n at around 279; y_n at around 223; and m_n around 112. The corresponding value of o_n is 386.

5. Use the EIGENV5/EIGNV5CE program to find the eigenvalues of $\underset{=}{T}$.

The eigenvalues are -0.358, -0.2, $0,558$ and 1.

6. Find the eigenvector of $\underset{=}{T}$ corresponding to $\lambda = 1$ (*explain why we use this value*) and relate this to the behaviour of the model that you found in Problem 4.

An eigenvector of 1 means the total population does not change, the case here. The corresponding eigenvector from EIGENV5/EIGNV5CE, normalised to a maximum component of 1, is

$$\begin{bmatrix} 0.7143 \\ 0.5714 \\ 0.2857 \\ 1 \end{bmatrix},$$

giving a total population of 2.5714.

To relate this to the problem here, we multiply the eigenvector by $1000/2.5714$ to give a total population of 1000. The new eigenvector, giving the four populations after a large (actually infinite) number of cycles, is (values rounded to integers)

$$\begin{bmatrix} 278 \\ 222 \\ 111 \\ 389 \end{bmatrix},$$

in good agreement with our results after 10 cycles in Problem 4.

21   Fitting Curves to Data 2

21.1   Introduction

Much scientific and other research involves data. Fitting a function to the data is a way of summarising the data; if the fit is good, the fitted curve can be used instead of the data in further calculations, especially useful if Calculus is involved. Sometimes the function chosen is guided by the theory involved, for example motion under gravity. At other times, the choice is empirical: the function which gives the best fit is used.

The material here is especially relevant to the second case. A range of functions is given, together with an example of each in fitting a given dataset. The functions available on the TI-84/CE to fit data are dealt with in *Fitting Curves to Data 1* in the first volume of this book, Here, we look at fitting polynomials to data, then work through the functions available using the CURVEFIT/CRVFITCE program[39] In most cases, an indication of the goodness of the fit is given, usually the correlation coefficient r and r^2 or the coefficient of determination R^2. Finally, we look at time averaging of data.

21.2   Exact fitting of polynomials to data

One very commonly used class of functions used to fit data is polynomials. Examples have already been encountered in *Fitting Curves to Data 1*; the fits there are exact when the number of points fitted is the same as the number of coefficients in the equation to be fitted. If there are more points than coefficients, the curve of best fit (regression) is obtained.

A general polynomial of degree m is of the form

$$y \;=\; a_0 \,+\, a_1 x \,+\, a_2 x^2 \,+\, a_x^3 \,+\, a_4 x^4 \,+\, \cdots \,+\, a_m x^m. \tag{1}$$

The data points to be fitted are then used to find the coefficients a_0, a_1, a_2, …, a_m, using the fact that the x and y values of each point must satisfy the polynomial. A polynomial of degree m has $m+1$ coefficients, and therefore will fit $m+1$ data points exactly. This is what we do here. The method applies to polynomials of any degree.

Note: If critical points ($y'=0$) are given, differentiate Eq. (1) and substitute the coordinates of the point into the derivative (a polynomial). Similarly with points of inflection ($y''=0$).

21.2.1   Manual method

Find a polynomial passing through the 10 points shown in the screen below.

X	Y
0	1
1	2
2	6
3	5
4	6
5	3
6	3
7	8
8	9
9	8

There are 10 data points, so we can fit a unique 9th-degree polynomial

$$y \;=\; a_0 \,+\, a_1 x \,+\, a_2 x^2 \,+\, \ldots \,+\, a_9 x^9, \tag{2}$$

which has the 10 unknowns a_0, a_1, a_2, …, a_9.

---
[39]available at *canberramaths.org.au* under *Resources*

Substituting the x and y values of each data point into Eq. (2) gives 10 linear equations:

$$a_0 = 1$$

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 2$$

$$a_0 + 2a_1x + 4a_2 + 8a_3 + 16a_4 + 32a_5 + 64a_6 + 128a_7 + 256a_8 + 512a_9 = 6$$

$$a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4 + 243a_5 + a_63^6 + a_73^7 + a_83^8 + a_93^9 = 5$$

$$a_0 + 4a_1 + 16a_2 + 64a_3 + 256a_4 + a_54^5 + a_64^6 + a_74^7 + a_84^8 + a_94^9 = 6$$

$$a_0 + 5a_1 + 25a_2 + 125a_3 + 625a_4 + a_55^5 + a_65^6 + a_75^7 + a_85^8 + a_95^9 = 3$$

$$a_0 + 6a_1 + 36a_2 + 216a_3 + a_46^4 + a_56^5 + a_66^6 + a_76^7 + a_86^8 + a_96^9 = 3$$

$$a_0 + 7a_1 + 49a_2 + 343a_3 + a_47^4 + a_57^5 + a_67^6 + a_77^7 + a_87^8 + a_97^9 = 8$$

$$a_0 + 8a_1 + 64a_2 + 512a_3 + a_48^4 + a_58^5 + a_68^6 + a_78^7 + a_88^8 + a_98^9 = 9$$

$$a_0 + 9a_1 + 81a_2 + 729a_3 + a_49^4 + a_59^5 + a_69^6 + a_79^7 + a_89^8 + a_99^9 = 8$$

Solving these equations[40] gives (to 4 significant digits) $a_0 = 1$, $a_1 = -76.32$, $a_2 = 187.4$, $a_3 = -175.0$, $a_4 = 85.45$, $a_5 = -24.25$, $a_6 = 4.138$, $a_7 = -0.4183$, $a_8 = 0.02307$ and $a_9 = -5.346 \times 10^{-4}$.
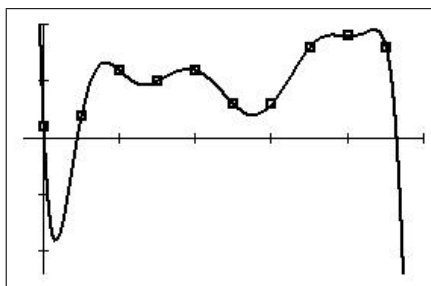
window $[-0.5, 10, 2] \times [-12, 10, 5]$

Once we find a formula for the polynomial, we can use it to give y values for x values between the data points — this is called *interpolation*.

We can also use the formula to find y values when x is outside the set of x values for which data are available. This is called *extrapolation*, and can be dangerous, as the curve might in practice behave quite differently beyond the measured data points, as it does here.

21.2.2 Using a program

This process is automated in the polynomial-fit program POLYFIT/POLYFTCE. This fits a polynomial of degree N to $N+1$ points, then plots the polynomial, displays the coefficients and calculates points on the polynomial. The polynomial is stored in Y7.

The program is available at *canberramaths.org.au* under *Resources*.

---

[40]by Gauss Elimination or the inverse-matrix method— see *Matrix and Vector Operations*, Chapter 19

Use

Run the program. Select INPUT POINTS and enter the points on the prompts. The points are copied to lists ʟX and ʟY.[41] Press CALCULATE FIT to do the calculations.

```
POLYNOMIAL FIT
1:INPUT POINTS
2:CALCULATE FIT
3:GRAPH POLYNOMIAL
4:CHANGE WINDOW
5:DISPLAY COEFFICIENTS
6:CALCULATE POINT
7:QUIT
```

Select GRAPH to graph the polynomial and the points.

CHANGE WINDOW allows you to change the X range for the graph, with the calculator automatically choosing the Y range.

DISPLAY COEFFICIENTS displays the coefficients of the polynomial in ascending powers of x.

CALCULATE POINT allows you to calculate a point on the polynomial.

QUIT does just that: not deleting the data leaves everything as it is; deleting the data removes everything, but you can still plot the polynomial in Y7.[42]

Try it out on the 10 points above. You'll find that the polynomial is not defined at $x = 0$ because the calculator cannot calculate 0^0 in Y7. The curve disappears near this point but clearly passes through the point $(0, 1)$ if you expand the window out in both directions.

Exercise: Fit a polynomial to the points $(0, 1)$, $(1, 6)$, $(-1, 0)$, $(2, 63)$, $(-2, -21)$ and $(-3, -182)$.

Answer: $y = x^5 + x^4 + x^3 + x^2 + x + 1$.

PTO

---

[41]If you need to change a point, Quit out of the program without deleting the data. Press ⬚stat⬚ Edit, change the point(s), then rerun the program and recalculate if necessary.

[42]the coefficient list ʟCOEF is not deleted, as it is used by Y7

21.3   Using the CURVEFIT/CRVFITCE program

Reference: *Curve Fitting for Programmable Calculators* by William M. Colb, Syntec, 1984.

The CURVEFIT/CRVFITCE program[43] fits 19 different functions (curves) to the data in lists L_1 (x) and L_2 (y), and displays the correlation coefficient (coefficient of determination) R^2 and conditional correlation coefficient[44] R_c^2 (CE only) for the fit. The screens below show use of the program up to the selection of the function.

The second option in the main menu, FIND BEST FIT, fits all 19 functions to the data, then tells you the function of best fit, the one with the largest value of R^2 (Section 21.3.20).

```
     MULTIPLE CURVE FIT

        DATA IN L₁,L₂

 CURVES 9-19 USE LOGS:
 NEGATIVE DATA CANNOT BE
 USED.

                  [ENTER]
```

```
 MAIN MENU
 1:SELECT A CURVE
 2:FIND BEST FIT
 3:QUIT
```

```
 CURVES *1
 1:Y=A+BX
 2:Y=1/(A+BX)
 3:Y=A+BX+C/X
 4:Y=A+B/X
 5:Y=X/(AX+B)
 6:Y=A+B/X+C/X²
 7:Y=A+BX+CX²
 8:Y=1/(A(X+B)+C)
 9:MORE...
```

```
 CURVES *2   (ADD 8)
 1:Y=A+Bln(X)
 2:Y=1/(A+Bln(X))
 3:Y=AX^B
 4:Y=AX^(BX)
 5:Y=AX^(B/X)
 6:Y=AB^X*X^C
 7:Y=AB^(1/X)X^C
 8:Y=Ae^((B-ln(X))²/C)
 9:MORE...
```

```
 CURVES *3   (ADD 16)
 1:Y=AB^X
 2:Y=AB^(1/X)
 3:Y=Ae^((X-B)²/C)
 4:BACK TO START
 5:MAIN MENU
 6:QUIT
```

The origin and application of some of these functions are often somewhat esoteric but all can be used to fit data. Which one to use depends on the characteristics of your data (e.g. maxima and/or minima) and the behaviour of the different functions. Clearly, there's no point in fitting a quadratic function to data with an obvious maximum and minimum. Ultimately, you can use the correlation coefficient (coefficient of determination) R^2 to give you some indication, and the best-fit option to pick out the mathematically best fit.

We use the data shown below[45] for all the fits. The window for all the plots showing the function fitted to the data is $[0, 10, 2] \times [0, 10, 2]$. Also given in many cases is a plot showing the general shape of the function when this is not obvious from the first plot.

L₁	L₂
1	2
2	6
4	5
4	6
4	3
6	3
6	8
7	9
7	8
9	4

---

[43]Both programs and their three subprograms, MCFSUB0, MCFSUB1 and MCFSUB2, are available individually or in the respective group files, CURVE.8xg and CURVECE.8xg, at *canberramaths.org.au* under *Resources*.

[44]I don't understand what this is, even after looking it up. If you think it is important for your application, you probably already know much more than I do. Otherwise, look it up or ask a statistician.

[45]the data used in *Fitting Curves to Data 1* in Volume 1 of this book

21.3.1   Linear: Y=A+BX

Same as *LinReg(ax+b)* in *Fitting Curves to Data 1* in Volume 1 of this book.
See the discussion there.

```
1. Y=A+BX
   LINEAR

A=3.455555556
B=0.3888888889

R²=0.1559
R²c=0.0503
```

```
Y1=A+BX




X=5              Y=5.4
```
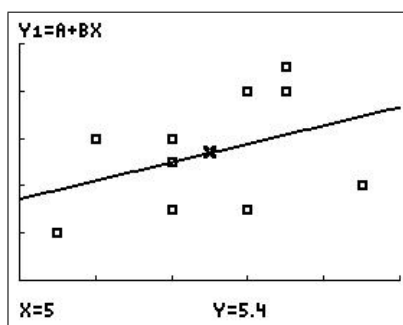
21.3.2   Reciprocal: Y=1/(A+BX)

The reciprocal function is one of the simplest functions having an infinite discontinuity (when
$A+BX=0$), not seen in the graphs here. Such discontinuities in modelling usually represent a
system moving beyond physical bounds (and usually self-destructing). Resonance in a system
is one such situation.

```
2. Y=1/(A+BX)
   RECIPROCAL

A=0.3486522634
B=-0.0235082305

R²=0.2135
R²c=0.1152
```

```
Y1=1/(A+BX)




X=5              Y=4.3269231
```

```
            B<0




            B>0
```
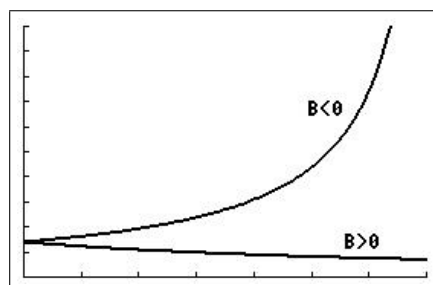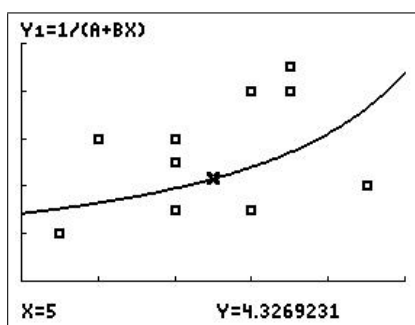
21.3.3   Linear-hyperbolic: Y=A+BX+C/X

This function becomes linear for large enough x, and hence tends to $\pm\infty$, but starts with an
infinite discontinuity at $x=0$. With appropriate choices for A and B, the curve can have a
local maximum or minimum, or can be monotonic.

```
3. Y=A+BX+C/X
   LINEAR-HYPERBOLIC

A=6.911763162
B=-0.0283398015
C=-4.597285845
R²=0.2419
R²c=0.0253
```

```
Y1=A+BX+C/X




X=5              Y=5.850607
```

```
               C>0



               C<0
```
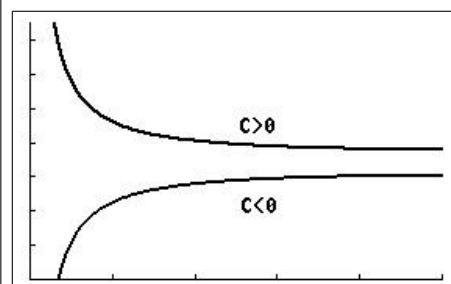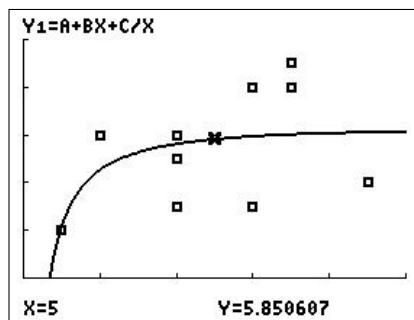
21.3.4   Hyperbolic: Y=A+B/X

Has an infinite discontinuity at $x=0$, then tends asymptotically to the constant value A.
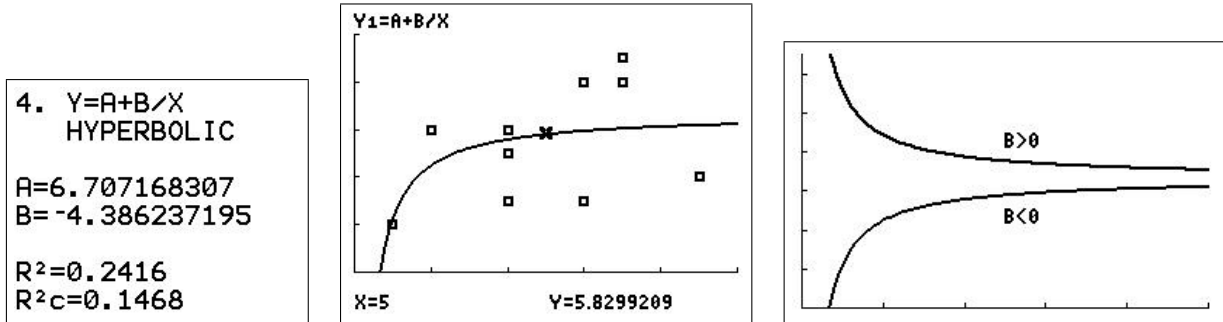
21.3.5   Reciprocal hyperbolic: Y=X/(AX+B)

Is zero at $X=0$ (unless $B=0$) and has an infinite discontinuity when $A+BX=0$.
The right-hand graph shows its behaviour when $A>0$.
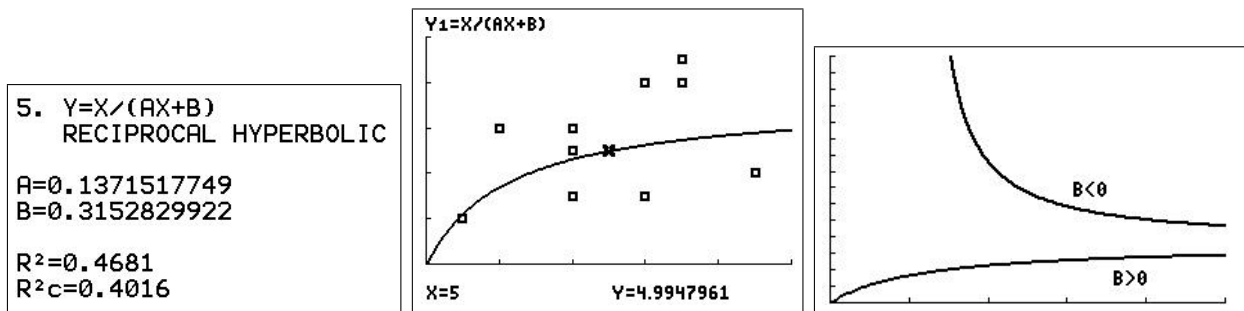
21.3.6   Second-order hyperbolic: Y=A+B/X+C/X²

Similar behaviour to the hyperbolic function (Section 21.3.4).
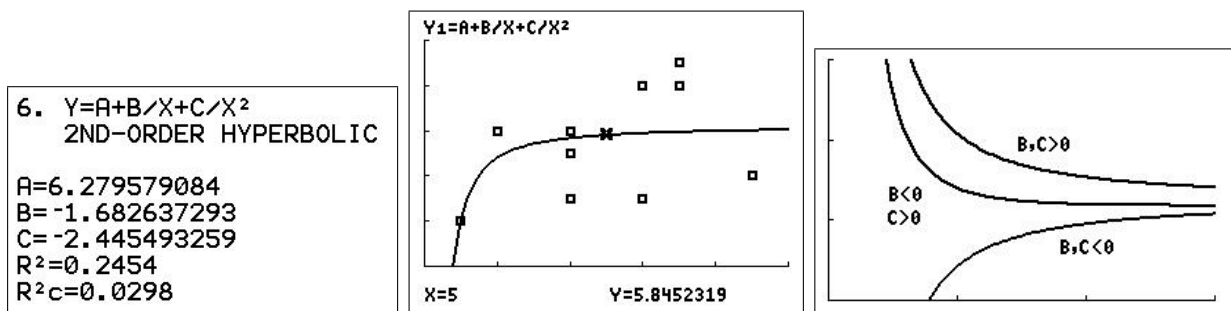
21.3.7 Parabolic: $Y=A+BX+CX^2$

Same as *QuadReg* in *Fitting Curves to Data 1* in Volume 1 of this book.
See the discussion there.

```
 7. Y=A+BX+CX²
    PARABOLIC

A=1.036139793
B=1.668387609
C=-0.1308578237
R²=0.2656
R²c=0.0557
```
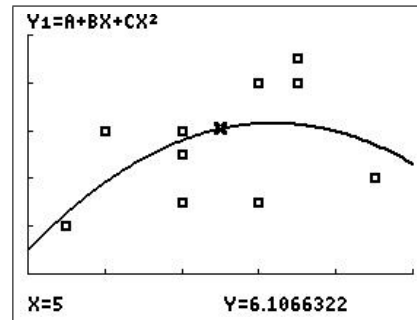

21.3.8 Cauchy Distribution: $Y=1/(A(X+B)^2+C)$

A well-known continuous probability distribution, also known to physicists as the Lorentz distribution. Can have an infinite discontinuity for positive X if A and C have opposite signs.

```
 8. Y=1/(A(X+B)²+C)
    CAUCHY DISTRIBUTION

A=0.008461117
B=-6.278080793
C=0.171599925
R²=0.3855
R²c=0.21
```
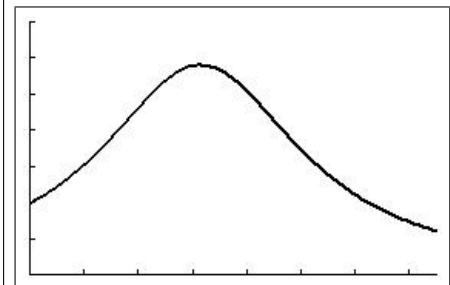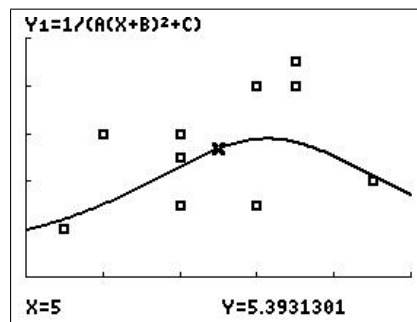

21.3.9 Logarithmic: $Y=A+B\ln(X)$

Same as *LnReg* in *Fitting Curves to Data 1* in Volume 1 of this book.
See the discussion there.

```
 9. Y=A+Bln(X)
    LOGARITHMIC

A=2.94478461
B=1.690384858

R²=0.2166
R²c=0.1186
```
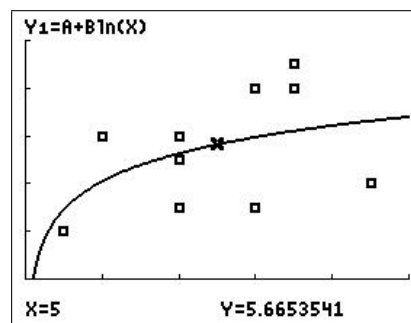
21.3.10   Reciprocal logarithmic: Y=1/(A+Bln(X))

This function is only defined for $X > 0$ because of the $\ln(X)$. $\ln(X)$ tends to $-\infty$ as $x \to 0$ and ∞ as $x \to \infty$, so eventually the function tends to 0 in both limits. There is also an infinite discontinuity when $A+B\ln(X) = 0$: if A and B have the same sign, this will occur when $0 < X < 1$ $\big(\ln(X) < 0\big)$; otherwise at some $X > 1$ ($X \approx 34$ here).

```
10.  Y=1/(A+Bln(X))
     RECIPROCAL LOGARITHMIC

A=0.3923621937
B=-0.1110193385

R²=0.3502
R²c=0.269
```

```
Y1=1/(A+Bln(X))




X=5                Y=4.6798194
```
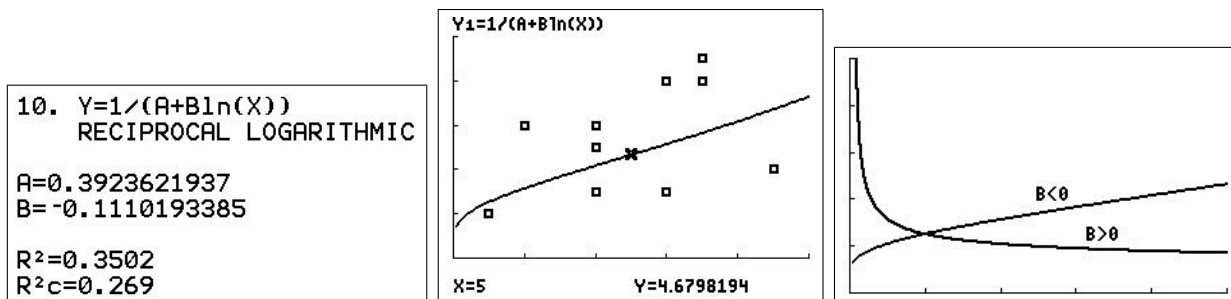
21.3.11   Power: Y=AX^B

Same as *PwrReg* in *Fitting Curves to Data 1* in Volume 1 of this book.
See the discussion there.

```
11.  Y=AX^B
     POWER

A=2.755057327
B=0.3918421467

R²=0.2701
R²c=0.1788
```

```
Y1=AX^B




X=5                Y=5.1762655
```
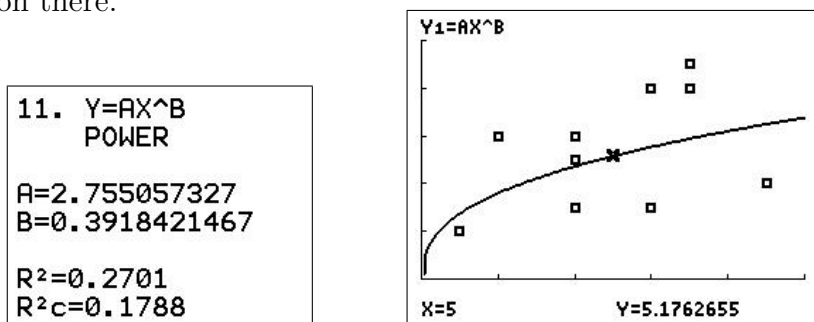
21.3.12   Super exponential: Y=AX^(BX)

Called Super geometric in the original publication but has no connection to the Geometric Distribution. Defined for $X > 0$, tends to 0 or ∞ as $X \to \infty$, depending on the sign of B.

```
12.  Y=AX^(BX)
     SUPER EXPONENTIAL
A=3.73941059
B=0.0304648621

R²=0.1392
R²c=0.0315
```

```
Y1=AX^(BX)




X=5                Y=4.7782985
```

177
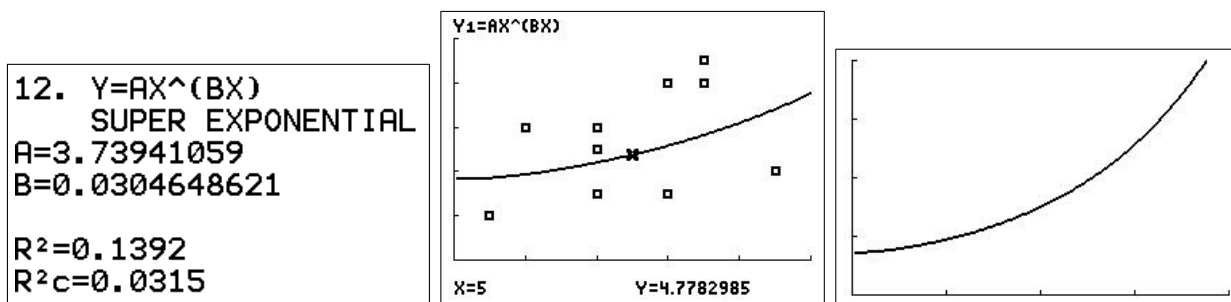
21.3.13   Modified exponential: $Y=AX^{\wedge}(B/X)$

Called Modified geometric in the original publication but also has no connection to the Geometric Distribution. Defined for $X > 0$, tends to A as $X \to \infty$. Has a local maximum if A and B are positive, a local minimum if A is negative, B positive. Is monotonic if B is negative.

```
Y1=AX^(B/X)

13. Y=AX^(B/X)
    MODIFIED EXPONENTIAL
A=2.384139076
B=2.564018285

R²=0.2856
R²c=0.1963

X=5              Y=5.442093
```
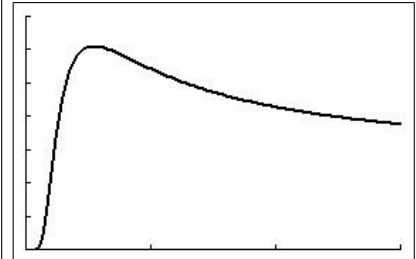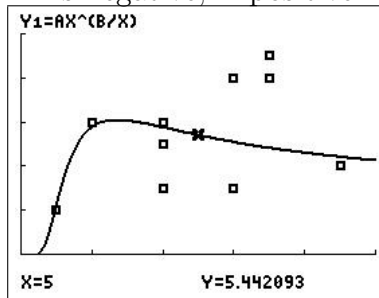
21.3.14   Hoerl Function: $Y=AB^{\wedge}X*X^{\wedge}C$

The exponential dominates for large enough X, so the function tends asymptotically to 0 if $|B| < 1$, with a local maximum or minimum if C is positive. If $|B| > 1$, the function tends to $\pm\infty$.

```
Y1=AB^X*X^C

14. Y=AB^X*X^C
    HOERL FUNCTION

A=2.716042844
B=0.8539540148
C=0.9451463763
R²=0.3277
R²c=0.1356

X=5              Y=5.6459627
```
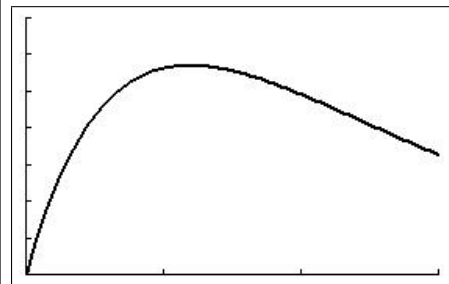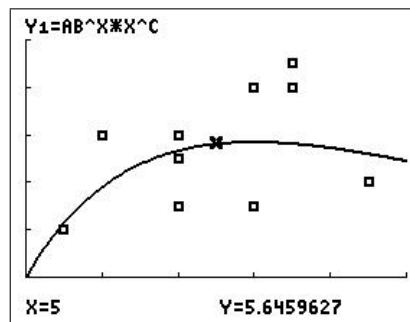
21.3.15   Modified Hoerl: $Y=AB^{\wedge}(1/X)X^{\wedge}C$

The exponential dominates for large enough X, so the behaviour of the function then depends on whether $|B| < 1$ (tends to 0) or $|B| > 1$ (tends to $\pm\infty$).

```
Y1=AB^(1/X)X^C

15. Y=AB^(1/X)X^C
    MODIFIED HOERL

A=12.94248275
B=0.1665336571
C=-0.3055034655
R²=0.3513
R²c=0.1659

X=5              Y=5.5307018
```

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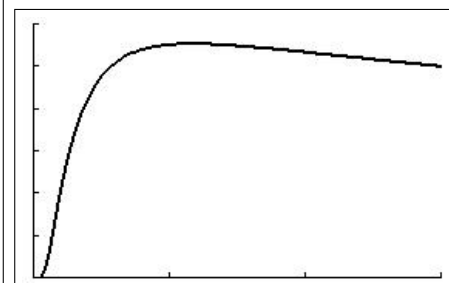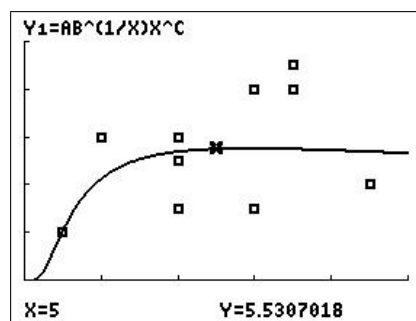
21.3.16   Log-Normal: $Y=Ae^{\wedge}((B-\ln(X))^2/C)$

A well-known continuous probability distribution, defined for $X > 0$.


```
16. Y=Ae^((B-ln(X))²/C)
    LOG-NORMAL

A=5.634936053
B=1.769175932
C=-3.397472118
R²=0.3362
R²c=0.1466
```
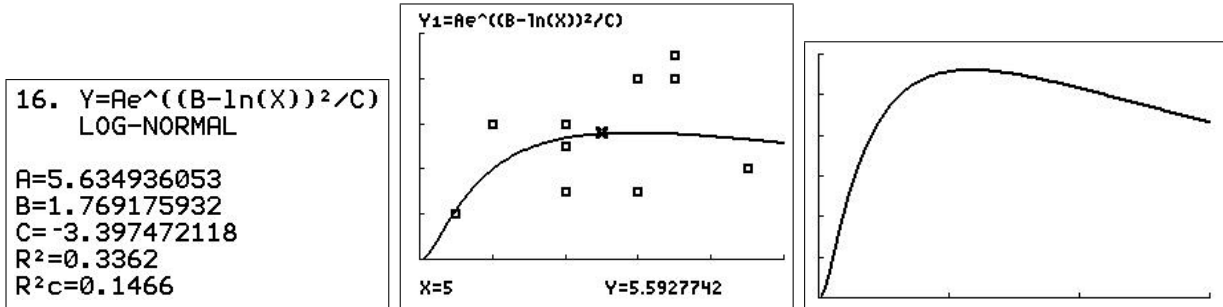
21.3.17   Exponential: $Y=AB^{\wedge}X$

Same as *ExpReg* in *Fitting Curves to Data 1* in Volume 1 of this book.
See the discussion there.


```
17. Y=AB^X
    EXPONENTIAL

A=3.170763598
B=1.089501361

R²=0.1757
R²c=0.0727
```
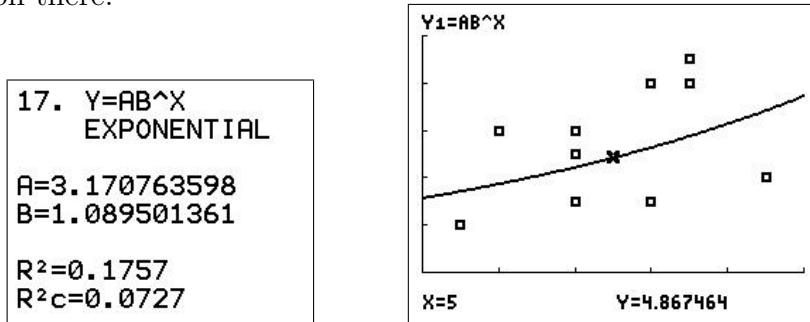
21.3.18   Root: $Y=AB^{\wedge}(1/X)$

Similar to the logarithmic distribution but starts at the origin.


```
18. Y=AB^(1/X)
    ROOT

A=6.706308889
B=0.3411734262

R²=0.3371
R²c=0.2542
```
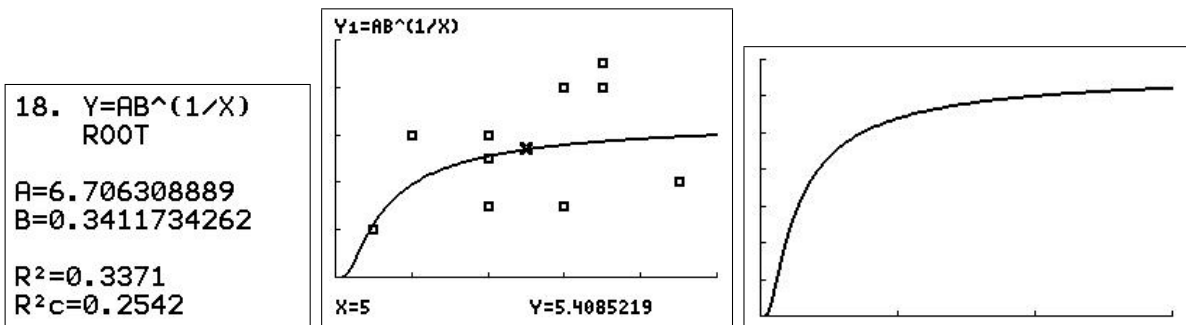
21.3.19   Normal Distribution: Y=Ae$^\wedge$((X$-$B)^2/C)

A very well-known continuous probability distribution with a bell-shaped graph.

```
19.  Y=Ae^((X-B)²/C)
     NORMAL DISTRIBUTION

A=6.040256374
B=6.292901531
C=-32.7580636
R²=0.3143
R²c=0.1184
```


21.3.20   Best fit

```
CURVE OF BEST FIT IS
CURVE 5

Y=X/(AX+B)
RECIPROCAL HYPERBOLIC

R²=0.4016
```
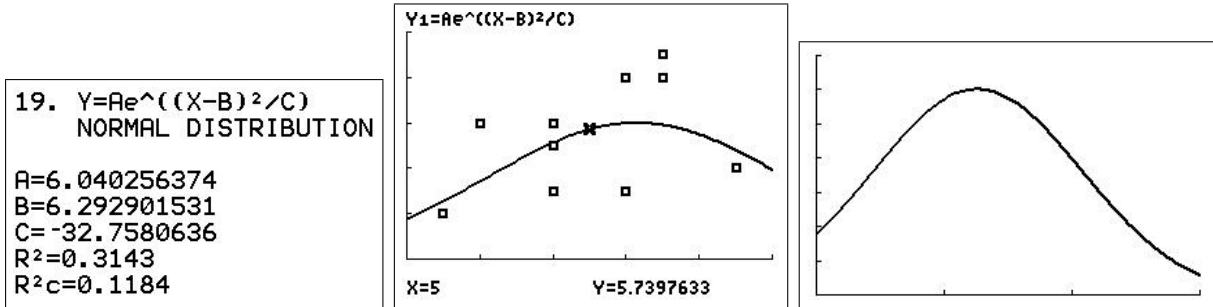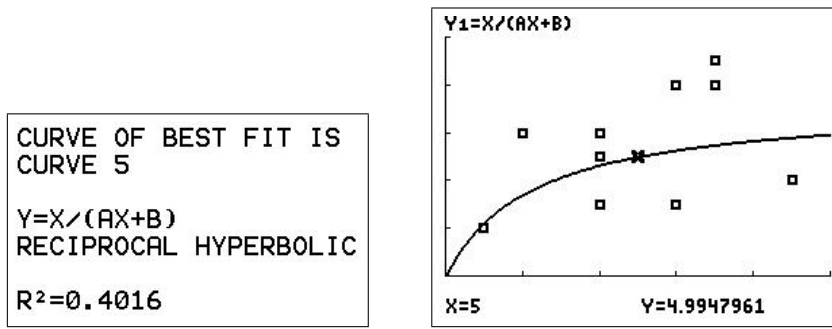
21.4   Time averaging of data

Some data can vary quite rapidly over short time scales such as the numbers of daily confirmed Covid cases in Sydney over 40 days, starting on March 1, 2020, in the table below.[46]

It can be difficult to discern trends in such data, so often the data are averaged over several days, moving-average data, to smooth out the bumps.

Day	Cases	Day	Cases
1	2	21	119
2	3	22	121
3	7	23	176
4	6	24	192
5	4	25	205
6	7	26	185
7	2	27	312
8	7	28	142
9	13	29	124
10	6	30	90
11	16	31	131
12	14	32	125
13	19	33	92
14	20	34	106
15	38	35	73
16	29	36	63
17	64	37	40
18	45	38	44
19	56	39	44
20	88	40	41

The MOVEAVGE/MVEAVGCE program[47] calculates and plots the data in lists L_1 and L_2 averaged over a period of C days, where you specify C; the first two screens of the program are shown below.

```
       MOVING AVERAGE

  ASSUMES...

      TIME IN L₁
      DATA IN L₂


          [ENTER]
```

```
  AVERAGE OVER A CYCLE
  OF C PERIODS

  C=?7
```

PTO

---

[46]https://data.nsw.gov.au/data/dataset/aefcde60-3b0c-4bc0-9af... f52778048b29/download/confirmed_cases_table1_location.csv
[47]available at *canberramaths.org.au* under *Resources.*

For example, if $C = 7$, the first data point in the average is at $t = 4$; the corresponding number of Covid cases is the average of the first 7 values: the three before $t = 4$; the value at $t = 4$; and the three after $t = 4$, and so on.
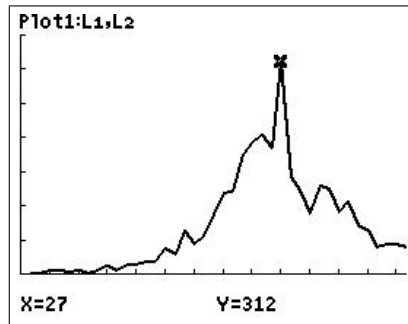
raw data: window $[0, 40, 5] \times [0, 350, 50]$
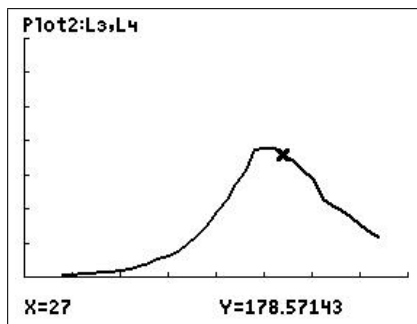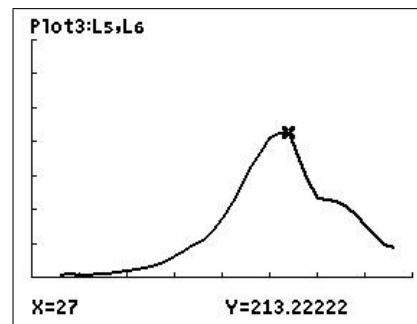
7-day (single) averaged data   3-day (single) averaged data

The program also calculates a double average, the average of the single-average values, for even greater smoothing.
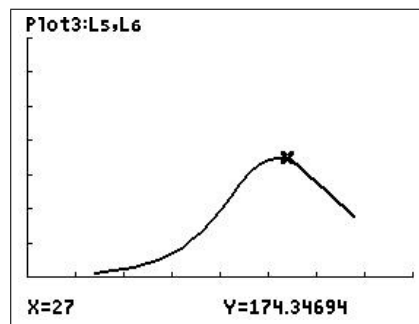
7-day double-averaged data

22   Financial Mathematics 2: TVM Calculations

22.1   Introduction

Financial Mathematics: Part 1 dealt with the basic calculations of compound interest, including the use of sequences on the calculator. Here, to be able to do some useful financial modelling, such as comparing loans, etc, we use the TVM (Time Value of Money) Solver.

The TVM notes are closely based on Chapter 11 of *Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS* by Barry Kissane. This book is a real bible on everything a graphics calculator can do and how to do it on a Casio graphics calculator. Still very relevant but sadly now hard to find.

Texas Instruments also has a book, *Time, Value, Money: Applications on the TI-83*, which covers all the financial calculations you are ever going to want to do, and then more. If you are using calculators from the TI-84 family, this is an invaluable resource.

Section 22.2 here covers calculations which involve compound interest, such as savings with or without regular payments, loans, discounting, effective rate of interest, annuities, sinking funds, leasing and bonds. Other calculations, such as days and dates, cost price/selling price/margins and cash-flow analysis, you will find in the TI book.

The good news is that all the compound-interest calculations are essentially the same: you specify the values of all the variables except one, then find the one you don't know. This is much simplified by using the TVM Solver on the calculator.

There are plenty of TVM exercises and activities with solutions in Sections 22.4 and 22.5.

References

Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS by Barry Kissane,
The Mathematical Association of Western Australia, 2003, ISBN 1 876583 24 X.

Time, Value, Money: Applications on the TI-83 by Roseanne and Charles Hoffman,
Texas Instruments, 1997, ISBN 1 886309 09 4.

PTO

22.2   Using the TVM Solver

The TVM Solver solves the same basic equations as those encountered in *Financial Mathematics 1* in Volume 1, with the addition of regular payments or repayments; it just makes solving the equations easy.

22.2.1   Selecting the TVM Solver

First press $\boxed{\text{mode}}$ and select 2 $\big($third line; use the arrow keys and $\boxed{\text{enter}}$ $\big)$ to set 2 decimal places for answers in dollar and cents.

```
MATHPRINT CLASSIC
NORMAL  SCI   ENG
FLOAT   0 1 2 3 4 5 6 7 8 9
RADIAN  DEGREE
FUNCTION  PARAMETRIC  POLAR  SEQ
THICK DOT-THICK THIN DOT-THIN
SEQUENTIAL  SIMUL
REAL   a+bi  re^(θi)
FULL   HORIZONTAL  GRAPH-TABLE
FRACTION TYPE: n/d  Un/d
ANSWERS: AUTO  DEC
STAT DIAGNOSTICS: OFF  ON
STAT WIZARDS: ON  OFF
SET CLOCK 01/04/15 09:07 PM
LANGUAGE:        ENGLISH
```

The TVM Solver is in the Finance App: press the $\boxed{\text{apps}}$ key and select Finance. Most of the other options are variables to use in programs (but see Section 22.2.4).

```
APPLICATIONS
1: Finance…
2: Conics
3: Hub
4: SmartPad
5: Transfrm
```

```
CALC VARS
1: TVM Solver…
2: tvm_Pmt
3: tvm_I%
4: tvm_PV
5: tvm_N
6: tvm_FV
7: npv(
8: irr(
9↓bal(
```

Press $\boxed{1}$ to select the TVM Solver.

```
N=0.00
I%=0.00
PV=0.00
PMT=0.00
FV=0.00
P/Y=1.00
C/Y=1.00
PMT: END  BEGIN
```

The TVM screen shows that a number of variables are possibly involved in calculations involving compound interest. To use the TVM module, you need to specify the value of each of the variables except one; the calculator works out the missing one for you.

Variables

N: the number of payment periods; if no regular payments, the number of years involved.

I%: the annual rate of interest, given as a percentage.

PV: the present value of the loan or investment. In the case of a loan, the amount borrowed. In the case of an investment, the original deposit.

PMT: the regular payment — either repaying a loan, depositing money for investment or a regular payment from a superannuation or investment account.

FV: the future value of the loan or investment. For a loan, this is usually zero, since the loan must be all paid off.

P/Y: the number of payments made per year. Set to 1 if there are no payments made.

C/Y: the number of compounding periods per year (in case it is different from P/Y).

For each variable, enter the necessary value, followed by $\boxed{\text{enter}}$ or the up/down-arrow key.

The calculator will set P/Y and C/Y to be the same as soon as you enter P/Y, as these two are usually the same. If they are not the same, you must enter P/Y first, then C/Y.

PMT on the bottom line should be set to END — use the arrow keys and $\boxed{\text{enter}}$.

Important Note: Payments made to you (from the bank) are positive, payments made by you (to the bank) are negative.

22.2.2   Compound interest

Most interest calculations involve compound interest, in which interest is paid on the amount of money still owing on a loan or earned on the amount of money in the account (not just on what was originally deposited).

The simplest case of compound interest is depositing some money into an account and leaving it there for some time to accrue interest.

Example 1

An inheritance of \$5000 is placed into an investment account paying 11% interest per annum, compounded annually. How much will there be in the account after 5 years?

As shown in the figure below, N, the number of years, is set to 5, I% to 11, PMT to 0 as there are no regular payments, and P/Y and C/Y to 1. Note that 1 is the minimum value permitted by the calculator for P/Y, even though there are no payments.

PV, the present value, is set to -5000, negative because this amount is paid to the bank.

To find FV, the future value of this investment (the amount after 5 years), *move the cursor to FV* (it doesn't matter what number is there) and press $\boxed{\text{solve}}$ ($\boxed{\text{alpha}}$ $\boxed{\text{enter}}$).

```
N=5.00
 I%=11.00
 PV=-5000.00
 PMT=0.00
∎FV=8425.29
 P/Y=1.00
 C/Y=1.00
 PMT:END BEGIN
```

After 5 years, the investment will have grown to \$8425.29, positive because it is paid to you.

Compound interest can also be calculated without using the TVM Solver. The variables here are related by the standard formula[1]

$$\text{FV} \;=\; \text{PV}\left(1 + \frac{\text{I\%}}{100}\right)^{\text{N}}. \tag{1}$$

Evaluating this formula with the parameter values of Example 1, shown below, gives the same answer as the TVM Solver.

```
5000(1+11/100)^5
                 8425.29
```

Other TVM calculations can also be done directly using the appropriate formula, but it is usually easier (and sometimes a lot easier) to let the calculator to do them for you. However, you should understand what the calculator is doing.

Example 2

How much is in the account in Example 1 if the interest is now compounded every three months (quarterly)?

Here we just change C/Y to 4. Again, evaluate FV by moving the cursor to FV and pressing solve (alpha enter).

```
N=5.00
I%=11.00
PV=-5000.00
PMT=0.00
▪FV=8602.14
P/Y=1.00
C/Y=4.00
PMT:END BEGIN
```

The future value has increased to \$8602.14, so only an extra \$176.85 over the 5 years.

22.2.3   Discounting

Compound-interest calculations can be used to find the future value of an investment at a certain rate of interest. However, sometimes we are interested in the opposite problem: finding the present value of a specific sum of money some time in the future.

One good reason for doing this is to compare various kinds of investments. If all are expressed in present values, a direct comparison can be made. We could also compare investments by calculating their future values, but the use of present values allows for today's money values to be used directly.

The process of finding the present value of a guaranteed future sum of money is called *discounting*, and the interest rate involved called the *discount rate*.

The relationship between PV and FV in Eq. (1) can be rearranged to show how PV can be calculated:

$$\text{PV} \;=\; \frac{\text{FV}}{\left(1 + \frac{\text{I\%}}{100}\right)^{\text{N}}}. \tag{2}$$

---

[1]If this formula is not familiar to your students, you need to work through *Financial Mathematics 1* in Volume 1.

In the TVM module, discounting is performed by entering the appropriate values for FV, I% and N, then calculating PV.

Example 3

What is the present value of $10,000 paid to you in 10 years' time, assuming a discount rate of 8% per annum?

The TVM set up is shown below after setting the cursor on PV and pressing $\boxed{\text{solve}}$.

Again, it doesn't matter what value is in PV when you press $\boxed{\text{solve}}$, as this is the value being calculated.

```
N=10.00
 I%=8.00
∎PV=-4631.93
 PMT=0.00
 FV=10000.00
 P/Y=1.00
 C/Y=1.00
 PMT:END BEGIN
```

The present value is $4631.93, negative in the TVM result because you would have to pay this amount now to have the $10,000 (FV positive) paid to you in 10 years' time, given an annual interest rate of 8%.

You can check this result using Eq. (2) or by calculating that an investment of $4631.93 at 8% per annum for 10 years will yield $10,000.

22.2.4   Effective rate of interest

As seen above, if interest is compounded more often than annually, the amount of interest involved increases, as there is more interest on the interest. To compare interest rates, you therefore need to know how often the interest is compounded. Most interest rates are given as a rate per annum, sometimes called the *nominal rate* (annual compounding).

The *effective rate* of interest takes into account the fact that the interest can be compounded more than once a year. The effective rate of interest is useful for comparing different financial offerings.

Essentially we first calculate the total amount of interest earned on an amount of say $1000 (the amount does not matter) for 1 year under whatever compounding scheme is used. We then calculate the annual interest rate that would have generated this amount if there were annual compounding: this is the effective rate.

Example 4

Consider an investment of $1000 for 1 year at 9% per annum. After 1 year, if there is annual compounding, the total interest is 9% of $1000 or $90, and the future value is $1090.

If the interest were compounded monthly instead of annually, the interest rate would be 0.75% per month (9%÷12), and the future value can be determined from Eq. (1). The interest rate here must be the monthly interest rate and the number of payment periods 12N:

$$\text{FV} \;=\; \text{PV}\left(1 + \frac{\text{I\%}}{1200}\right)^{12\text{N}} \;=\; 1000\big(1 + 0.0075\big)^{12} \;=\; \$1093.81.$$

PTO

The screens below show these calculations both from the formula and using the TVM Solver. Note that, for monthly compounding for a year, N = 1, P/Y = 1 (no regular payments), but C/Y = 12.

```
1000*1.09
                1090.00
1000(1+0.09/12)^12
                1093.81
```

```
 N=1.00
 I%=9.00
 PV=-1000.00
 PMT=0.00
▪FV=1093.81
 P/Y=1.00
 C/Y=12.00
 PMT:END BEGIN
```

The investment in this case has earned interest of $93.81, the amount of interest that would be earned with a (nominal) rate of interest of 9.38% compounded annually. This is the effective rate of interest.

You can calculate the effective rate of interest directly using the command ▶Eff in the Finance menu. Scroll down this menu (it's actually faster scrolling up) until you come to this command, press ⌐enter⌐ to select it, enter the nominal interest rate and the number of compounding periods in a year, separated by a comma, and press ⌐enter⌐ to evaluate the command.

For the present case, evaluate ▶Eff (9, 12) to give the answer 9.38% (to 2 decimal places).

```
▶Eff(9,12)
                   9.38
▶Nom(9.38,12)
                   9.00
```

If you now use the (inverse) command ▶Nom (9.38, 12), the calculator will give the nominal rate corresponding to an effective rate of 9.38%, as shown above.

22.2.5   Analysing a loan

Most loans involve both compound interest and regular payments. Using the calculator we ask questions like: *What if I pay fortnightly rather than monthly? What if the interest rate changes? What if pay a bit more each payment?*, and so on. Here we explore some of those possibilities in an example.

Example 5

I borrow $9000 to buy a car. The interest rate is 18% per annum, compounded monthly, and I plan to make monthly payments of $350. How long will it take to pay off the loan?

Here PV is positive because the bank has paid the money to me, PMT is negative because I pay this to the bank and FV is zero because I will pay off the loan. P/Y and C/Y are both 12 because the payment and compounding periods are monthly.

```
▪N=32.73
 I%=18.00
 PV=9000.00
 PMT=-350.00
 FV=0.00
 P/Y=12.00
 C/Y=12.00
 PMT:END BEGIN
```

Evaluating N, the length of the loan, gives 32.73. The interpretation of this is that 32 payments of $350 are needed, and that the last payment will then be less than this. The loan will certainly be paid off within 33 months or 2 years 9 months.

Details (and graphs) of the payment-by-payment values — the balance, how much of each payment goes to interest and how much to paying off the principal, the total interest paid and total paid off the principal — can be found by running the AMORTTBL/AMRTBLCE program, which generates an amortisation table — see the Appendix on page 193.

In calculating the total interest paid, we can use the amortisation table or we can just do the calculation

$$total\ interest = number\ of\ payments \times payment\ amount - loan\ amount.$$

Here we have total interest $= 32.73 \times 350 - 9000 = \$2455.50$.

Let's now look at three alternatives to the original loan, shown in the three screens below. In each case, we have calculated N.

```
·N=28.20          ·N=27.65          ·N=23.96
 I%=18.00          I%=18.00          I%=18.00
 PV=8000.00        PV=9000.00        PV=8000.00
 PMT=-350.00       PMT=-400.00       PMT=-400.00
 FV=0.00           FV=0.00           FV=0.00
 P/Y=12.00         P/Y=12.00         P/Y=12.00
 C/Y=12.00         C/Y=12.00         C/Y=12.00
 PMT:END BEGIN     PMT:END BEGIN     PMT:END BEGIN
```

The first screen shows that if only $8000 is borrowed, perhaps after paying a deposit of $1000, the loan is paid off in a little over 28 months.

The second screen shows that increasing the payment from $350 to $400 each month on the original loan of $9000 will reduce the length of the loan to a little less than 28 months.

The third screen shows that making both these changes will reduce the length of the loan to less than 2 years. The total interest paid here, $23.96 \times 400 - 8000 = \$1584$, is a saving of $871 on the interest paid on the original loan.

Alternatively, we can ask what the payment needs to be to pay off the loan in say 2 years or 18 months.

```
 N=24.00           N=18.00
 I%=18.00          I%=18.00
 PV=9000.00        PV=9000.00
·PMT=-449.32      ·PMT=-574.25
 FV=0.00           FV=0.00
 P/Y=12.00         P/Y=12.00
 C/Y=12.00         C/Y=12.00
 PMT:END BEGIN     PMT:END BEGIN
```

The loan can be paid off in 2 years with monthly payments of $449.32, or in 18 months with monthly payments of $574.25.

The total interest paid is about $1784 and $1337, respectively.

22.2.6   Annuities

An annuity involves the payment of a regular sum of money over a period of time at a fixed rate of interest. Although you can regard payments on a loan as a kind of annuity, the term usually refers to other financial arrangements, such as a superannuation scheme (which has regular payouts) or a savings scheme (which has regular deposits). All these essentially involve compound interest, and so can be investigated using the TVM Solver.

Example 6

A family has won a prize of \$200,000 in a lottery and decided to use it as an annuity to provide them with a regular amount of money every 3 months over the next 10 years. They invest the money in an account paying 8.6% interest per annum, compounded annually. How much money will they get every 3 months?

Here there are 4 payments per year for 10 years, giving 40 payment periods altogether, so that N $=40$. The present value PV is $-200,000$ (they are paying it to the bank), and the final value FV is 0, since all the money (and interest) will be used. There are 4 payments per year, so that P/Y $=4$ and C/Y $=1$, as the interest is compounded annually. The only missing variable is the payment PMT.

```
N=40.00
 I%=8.60
 PV=-200000.00
▪PMT=7419.22
 FV=0.00
 P/Y=4.00
 C/Y=1.00
 PMT:END BEGIN
```

So this annuity provides the family with \$7419.22 every 3 months for 10 years.

22.2.7   Sinking funds

A sinking fund is a further example of an annuity, as it involves regular payments over a certain period, with a fixed rate of compound interest. The idea of a sinking fund is that money is invested in instalments into a fund for some particular purpose, and accumulates over time.

Example 7

You estimate you will need \$10,000 for a long overseas holiday in 3 years' time. The rate of interest available in an investment account is 7%, compounded monthly. How much should you save each month to reach the goal?

In this case, PV $=0$, FV $=10,000$ and N $=3\times12=36$.

```
N=36.00
 I%=7.00
 PV=0.00
▪PMT=-250.44
 FV=10000.00
 P/Y=12.00
 C/Y=12.00
 PMT:END BEGIN
```

The required payment is \$250.44 per month.

22.2.8   To lease or to buy?

The TVM Solver is a powerful tool for comparing alternative financial arrangements over different times. One example of this involves deciding whether it's financially better to purchase something now or lease it over a period of time.

Example 8

A computer store advertises a computer system for sale at $1999 or offers an alternative involving leasing. The store will lease the same computer system to you for $75 a month over 2 years at an interest rate of 12% per annum (compounded monthly). At the end of the two-year lease, you will be given an option to purchase the system for $400. Should you buy the computer system now or lease it first and buy it in 2 years' time?

A quick comparison suggests that there is not much difference between these two choices, as the total payments for the lease-and-purchase option are $24 \times \$75 + \$400 = \$2200$, a bit more expensive than the cash purchase.

To explore these options more carefully, we need to have a way of comparing their monetary value at the same time. One way of doing this is to find the present value of the leasing option, and compare it with the present value of the cash purchase.

The leasing option comprises two parts: the monthly payments and the residual value of $400. To determine the present value of 24 monthly payments of $75 each, enter the values as shown below left, and solve for PV.

```
N=24.00            N=24.00
I%=12.00           I%=12.00
▪PV=1593.25        ▪PV=315.03
PMT=-75.00         PMT=0.00
FV=0.00            FV=-400.00
P/Y=12.00          P/Y=12.00
C/Y=12.00          C/Y=12.00
PMT:END BEGIN      PMT:END BEGIN
```

The present value of the payments is $1593.25.

To determine the present value of the residual payment of $400, enter the values as shown above right and evaluate PV.

The present value of the leasing option is $\$1593.25 + \$315.03 = \$1908.28$.

The cash purchase has a present value of $1999, almost $100 more than this, so it is a financially better deal to take the lease-and-purchase option in this case.

We could have also compared these two options using their respective future values (in 2 years' time). In the case of the lease, the screen below shows that the monthly payments have a future value of $2023.01. Added to the $400 purchase price in 2 years' time (which of course has a future value of $400), this gives a total future value of $2423.01.

```
N=24.00
I%=12.00
PV=0.00
PMT=-75.00
▪FV=2023.01
P/Y=12.00
C/Y=12.00
PMT:END BEGIN
```

The cash-purchase option has a future value of $2538.20:

```
N=24.00
I%=12.00
PV=-1999.00
PMT=0.00
■FV=2538.20
P/Y=12.00
C/Y=12.00
PMT:END BEGIN
```

Again, leasing is the better option in this case.

22.2.9   Bonds

A bond is an agreement by a debtor to pay a certain amount of money (the *face value*) at a certain future date (the *maturation date*) and also to pay a sum of money periodically, the sum being a fixed percentage (interest rate) of the face value of the bond. Bonds are issued by public authorities, governments and companies as a way of raising money to finance their activities. In many cases, a bond has attached to it small dated coupons that can be used to obtain the periodical payment, so the periodical payments are usually called *coupon payments*, and the interest rate of the coupon payments is called a *coupon rate*. Unlike private bonds issued by a company seeking to raise capital, government bonds may have some associated tax implications (which are not considered here).

After they have been issued, bonds can be bought and sold by investors. Naturally enough then, the main interest in bonds is in determining their present value, deciding how much they should fetch when traded and thus deciding whether or not they are a good investment. Government bonds are usually regarded as more secure investments than private bonds, as there is less risk of default (i.e. there is less risk that the issuing body will not be able to redeem (pay for) the bonds at the maturation date or make the coupon payments). You can use the TVM Solver to analyse bonds.

Example 9

A bond is issued on 1 July 2021 by an electricity authority in order to finance a new power plant. The bond has a face value of $1000, a maturation date of 1 July 2031 (that is, it matures in 10 years) and a coupon rate of 7%, compounded and paid semi-annually.

You want a rate of return of 9% on your money. Do you buy the bond when it is issued?

In this case, the payments are fixed at $7\% \div 2 = 3.5\%$ of $1000 = \$35$ every six months, and the future value is fixed at the face value of $1000. The required rate of return is I% = 9, and both P/Y and C/Y are set to 2 for semi-annual payment and compounding. There are 20 compounding periods, 2 each year for 10 years.

To obtain the present value of the bond, solve for PV.

```
N=20.00
I%=9.00
■PV=-869.92
PMT=35.00
FV=1000.00
P/Y=2.00
C/Y=2.00
PMT:END BEGIN
```

So the bond will return 9% on your money if you pay $869.92 for it when it is issued. You will get a better rate of return than 9% if you pay less than this.

You can use the same module to calculate the Yield to Maturity (YTM) of a bond. The YTM is the equivalent interest rate you will earn if you retain the bond until it matures and you redeem it, as well as collecting the coupon payments up till then.

Example 10

An investor pays \$800 for an electricity-authority bond as described in the previous example, 5 years before it matures (and just after a coupon payment was made). What is the YTM?

To calculate the YTM, enter the relevant values and solve for I%.

```
N=10.00
▪I%=12.50
 PV=-800.00
 PMT=35.00
 FV=1000.00
 P/Y=2.00
 C/Y=2.00
 PMT:END BEGIN
```

In this case, the investment will yield $12\frac{1}{2}\%$ interest per annum, compounded semi-annually (assuming that the electricity authority is able to redeem the bond when it matures).

22.2.10   A cautionary note

The methods here are generally less complicated than those used in practice, and do not include taxes, fees and other charges. You should seek proper financial advice before making any financial decisions, rather than relying on the calculations outlined here.

22.3   Appendix: Amortisation tables

An amortisation table shows, for each payment period: the balance after this payment; how much of this payment goes to interest and how much to repaying the principle; the total interest paid up to and including this payment; and the total paid towards the principal after this payment.

The TI-84/CE program[48] AMORTTBL/AMRTBLCE uses the TVM variables *bal* to find the balance and ΣInt to find the cumulative interest; it then calculates the other values.[49]

All values are stored in lists:

- PAYT   payment number
- BAL   balance after this payment
- INT   amount of interest paid in this payment (negative)
- PR   amount of principal repaid in this payment (negative)
- CINT   cumulative interest paid after this payment (negative)
- CPR   cumulative principal paid after this payment (negative).

Press stat Edit after the program has run to view these lists. Scroll across to see all the lists.

The program also sets up three plots, all with payment number on the x axis:

- balance (Plot1);
- the amount of the payment that goes to paying interest (Plot2);
- the amount of the payment that goes towards repaying the principal (Plot3).

---

[48]available at *canberramaths.org.au* under *Resources*

[49]See *Time, Value, Money: Applications on the TI-83* for more details.

Using the program

First run the TVM Solver to find values for all the variables. Then press $\boxed{\text{prgm}}$, press the number/letter against the AMORTTBL/AMRTBLCE program and press $\boxed{\text{enter}}$ to run it.

The program will ask if you have run the TVM Solver — respond appropriately. After the calculations have been carried out, the program will display all three plots superimposed. Use the arrow keys to move along and between the plots. Press $\boxed{\text{enter}}$ to move to a screen that tells you where to find the table and plots.

You can then either view the lists by pressing $\boxed{\text{stat}}$ Edit or select the particular plot you want by highlighting it in $\boxed{\text{y=}}$ using $\boxed{\text{enter}}$. Make sure all functions and other plots are turned off, again using $\boxed{\text{enter}}$. Press $\boxed{\text{zoom}}$ $\boxed{9}$ to set appropriate axes for that plot. Press $\boxed{\text{trace}}$ and use the arrow keys to move along the graph. The amounts at each payment are shown at the bottom of the screen.

The following screens show the superimposed three plots generated by the program from Example 5 on page 188, the top few rows of the amortisation table and the three individual plots, *balance, amount of payment going to interest* and *amount of payment going to the principal*, all versus *payment number*.

PAYT	BAL	INT	PR	CINT
1.00	8785.0	-135.0	-215.0	-135.0
2.00	8566.8	-131.8	-218.2	-266.8
3.00	8345.3	-128.5	-221.5	-395.3
4.00	8120.5	-125.2	-224.8	-520.5
5.00	7892.3	-121.8	-228.2	-642.3
6.00	7660.7	-118.4	-231.6	-760.7
7.00	7425.6	-114.9	-235.1	-875.6
8.00	7186.9	-111.4	-238.6	-986.9
9.00	6944.7	-107.8	-242.2	-1095
10.00	6698.9	-104.2	-245.8	-1199
11.00	6449.4	-100.5	-249.5	-1299

PAYT(1) = 1

BAL	INT	PR	CINT	CPR
8785.0	-135.0	-215.0	-135.0	-215.0
8566.8	-131.8	-218.2	-266.8	-433.2
8345.3	-128.5	-221.5	-395.3	-654.7
8120.5	-125.2	-224.8	-520.5	-879.5
7892.3	-121.8	-228.2	-642.3	-1108
7660.7	-118.4	-231.6	-760.7	-1339
7425.6	-114.9	-235.1	-875.6	-1574
7186.9	-111.4	-238.6	-986.9	-1813
6944.7	-107.8	-242.2	-1095	-2055
6698.9	-104.2	-245.8	-1199	-2301
6449.4	-100.5	-249.5	-1299	-2551

CPR(1) = -215
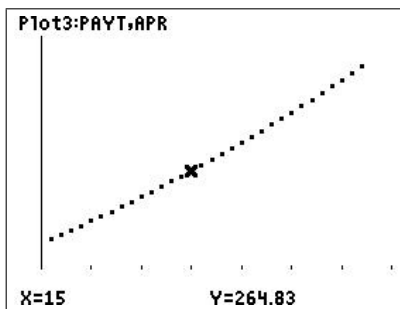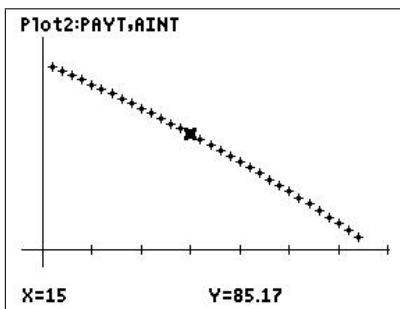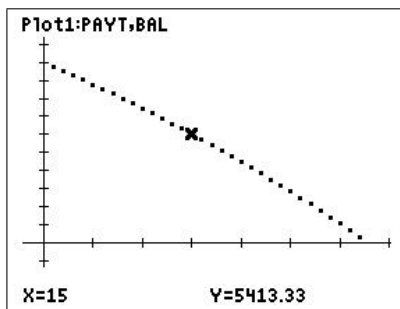
When you have finished your analysis, re-run AMORTTBL/AMRTBLCE and select $\boxed{3}$ to delete all the lists and tidy up.

22.4   TVM exercises and solutions

Note that N in the TVM Solver stands for different things in different types of calculations.
All results are rounded to two decimal places.
Starred questions were not covered here.

*1. Find the simple interest on a 90-day loan of \$1600 at 5.6% per annum.

Solution   *No built-in commands. Use the formula.*

The daily interest rate is 90/365 of the annual rate of interest. Therefore the amount of interest is
$$\frac{90}{365} \times 0.056 \times 1600 = 22.09.$$

The interest is \$22.09.

*2. How many days are there between January 12 2003 and June 3 2003?
Hint: the *dbd* command in the Finance menu.

Solution   Finance menu: *dbd* command

Dates are entered in the form *mm.ddyy.*

dbd (1.1203, 6.0303) = 142

There are 142 days between January 12 2003 and June 3 2003.

*3. A store buys a skateboard for \$45. What selling price will give a 30% margin?

Solution   *No built-in command. Use the formula.*

If S is the selling price, $45 + 0.3S = S$, so that $S = 45/0.7 = 64.29$.

The selling price should be \$64.29.

Note that a 30% margin means the return is 30% of the selling price. A 30% profit means a return of 30% of the cost price.

4. (a) How much compound interest will be earned by an investment of \$1600 for 10 years at $9\frac{3}{4}$% per annum?

Solution   TVM Solver
N = 10   with no regular payments, N is the number of *years*
I% = 9.75   annual percentage interest rate
PV = −1600   negative because paid to bank
PMT = 0   no regular payments
FV = ?   doesn't matter what is here — this is what we have to find
P/Y = 1   payments per year — must be set to 1 if no payments
C/Y = 1   compounding periods per year (assumed to be equal to P/Y)
Then, FV = 4056.63   positive because paid to you

The final amount is \$4056.63.

Interest = \$4056.63 − \$1600 = \$2456.63.

(b) How much more interest will be earned if the interest were compounded quarterly?

> **Solution**   Change C/Y to 4 (compounding periods per year).
> Then, FV = 4192.56.
> The final amount is $4192.56.
> Interest = $4192.56 − $1600 = $2592.56.
> Extra interest = $2592.56 − $2456.63 = $135.93.
>
> You could also just subtract one final amount from the other, as the difference is the extra interest.

5. For how long should you leave $10,000 in the bank at 6% per annum interest, compounded monthly, so that it grows to $12,000?

   Solution   TVM Solver
   N = ?   doesn't matter what is here — this is what we have to find
   I% = 6   annual percentage interest rate
   PV = −10000   negative because paid to bank
   PMT = 0   no regular payments
   FV = 12000   the amount we want to receive
   P/Y = 1   payments per year — must be set to 1 if no payments
   C/Y = 12   compounding periods per year

   Then, N = 3.046: because there are no regular payments, N is in years

   You would need to leave the money in the bank 3.046 years or 3 years 17 days.

   You might want to check that the bank will in fact give you the interest for the 17 days if you withdraw the money then. You may have to leave the money in the bank until the next monthly interest credit. By then you will have $12,026.64.

6. A credit card company charges 16% per annum interest, compounded monthly. What is the effective rate of interest?

   Solution: Here you can use the ▶*Eff* command in the Finance menu (Method A). However, this is a bit of a black box. More intuitively, we can do the calculations on an amount of say $1000 (Method B).

   Method A

   Finance menu: ▶*Eff* command

   ▶Eff (16, 12) = 17.23

   16 is the annual percentage interest rate.
   12 is the number of compounding periods in a year.

   The effective interest rate is 17.23% (to 2 decimal places).

   Method B

   Here say we owe the credit card company an amount of $1000 over a year (the amount does not matter — you always get the same answer).

   If the interest is compounded once a year, the amount of interest to be paid is
   $0.16 \times 1000 = \$160$.

Now we calculate how much interest is to be paid *if the interest is compounded monthly.*

TVM Solver
N = 1   number of *years*, because no regular payments
I% = 16   annual percentage interest rate
PV = 1000   positive because it is a 'loan' paid to us
PMT = 0   no regular payments
FV = ?   doesn't matter what is here — this is what we have to find
P/Y = 1   payments per year — must be set to 1 if no payments
C/Y = 12   compounding periods per year

Then, FV = 1172.27.

The total amount of interest paid this time is $1172.27 − $1000 = $172.27.
If this were the result of compounding once a year, the rate of interest would have to have been $172.27/1000 = 0.17227$ or $17.227\% \approx 17.23\%$. This is the effective rate of interest.

7. **(a)** To finance extensions to her house, Susan borrowed $7000 at 11% per annum. What monthly payments are needed to repay the loan in 3 years?

   Solution   TVM Solver
   N = 12×3 = 36   with regular payments, N is the total number of *payments*
   I% = 11   annual percentage interest rate
   PV = 7000   positive because paid to Susan
   PMT = ?   doesn't matter what is here — this is what we have to find
   FV = 0   loan paid off
   P/Y = 12   payments per year — monthly payments here
   C/Y = 12   compounding periods per year — assumed equal to P/Y

   Then, PMT = −229.17, negative because paid to the bank

   The monthly payment is $229.17.

   (b) How much interest will she pay for this loan?

   Solution: Total interest is equal to the total amount paid minus the loan amount (principal), here $36 \times 229.17 − 7000 = \$1250.12$.

8. Julian's lump-sum superannuation amounts to $350,000. He invests this sum to receive a regular payment each month for the next 20 years. Assuming the interest rate stays at 6.8% per annum, find the size of each payment.

   Solution   TVM Solver
   N = 12×20 = 240   total number of *payments*
   I% = 6.8   annual percentage interest rate
   PV = −350000   negative because paid to bank
   PMT = ?   doesn't matter what is here — this is what we have to find
   FV = 0   capital and interest all used up after 20 years
   P/Y = 12   payments per year
   C/Y = 12   compounding periods per year — assumed equal to P/Y

   Then, PMT = 2671.69

   The monthly payment to Julian is $2671.69.

9. A school establishes a sinking fund to save \$90,000 over the next 2 years for a new computer laboratory. How much should be deposited into the fund each month if the interest rate stays at $7\frac{1}{2}\%$ per annum?

Solution   TVM Solver
N $= 12 \times 2 = 24$ total number of *payments*
I% $= 7.5$ annual percentage interest rate
PV $= 0$ no capital
PMT $= ?$ doesn't matter what is here — this is what we have to find
FV $= 90000$ final amount required
P/Y $= 12$ payments per year
C/Y $= 12$ compounding periods per year — assumed equal to P/Y

Then, PMT $= -3487.46$
The monthly payment is \$3487.46.

10. What is worth more? \$1200 today or \$100 per month for 1 year? Assume the interest rate stays constant at 7.1% per annum.

Solution: Find the present value of \$100 per month for a year at 7.1% per annum.

TVM Solver
N $= 12$ total number of *payments*
I% $= 7.1$ annual percentage interest rate
PV $= ?$ doesn't matter what is here — this is what we have to find
PMT $= -100$ monthly payment to bank
FV $= 0$ you need to put 0 here
P/Y $= 12$ payments per year
C/Y $= 12$ compounding periods per year — assumed equal to P/Y

Then, PV $= 1155.10$

The present value of \$100 per month for 1 year at 7.1% per annum is \$1155.10. This is less than \$1200, so \$1200 today is worth more than \$100 per month for 1 year at 7.1% per annum.

We could have found the future value of both amounts, although this is more work. Proceed as follows.

Set PV $= 0$ and solve for FV to find the future value of \$100 per month for a year at 7.1% per annum to be \$1239.83.

Convert \$1200 today to its future value:
N $= 1$ total number of *years* (no regular payments)
I% $= 7.1$
PV $= -1200$ present value
PMT $= 0$ no regular payments
FV $= ?$ doesn't matter what is here — this is what we have to find
P/Y $= 1$ payments per year — must be set to 1 if no payments
C/Y $= 12$ compounding periods per year

Then, FV $= 1288.03$, so that the future value of the \$1200 is \$1288.03.
We reach the same conclusion as above: take the \$1200!

22.5   TVM activities and solutions

The activities are to help you use your calculator to learn mathematics.

All results rounded to two decimal places.

1. **(a)** Find the number of days between 12 January 2003 and 12 January 2004.

   Solution: Finance menu, *dbd* command
   Dates are entered in the form *mm.ddyy*:   dbd (1.1203, 1.1204) = 365
   There are 365 days between 12 January 2003 and 12 January 2004.

   (b) Find the number of days between 12 January 2004 and 12 January 2005.

   Solution: dbd (1.1204, 1.1205) = 366
   There are 366 days between 12 January 2004 and 12 January 2005.

   (c) Explain why the answers to Questions 1a and 1b are different.

   Solution: 2004 was a leap year.

2. Banks sometimes advertise that interest is compounded daily on customers' investments in order to attract more business. But how much difference does more-frequent compounding make? To find out, consider an investment of $100 in an account earning compound interest at a rate of 8% per annum. Compare the interest obtained by compounding: **(a)** annually; **(b)** quarterly; **(c)** monthly; **(d)** daily.

   Solution   TVM Solver
   N = 1   number of *years* (no regular payments)
   I% = 8   annual percentage interest rate
   PV = −100   the amount invested — negative because paid to the bank
   PMT = 0   no regular payments
   FV = ?   doesn't matter what is here — we calculate this
   P/Y = 1   payments per year — must be set to 1 if no payments
   C/Y =   compounding periods per year: set this value and calculate FV: see table

Compounding	C/Y	FV ($)	Interest ($)
annually	1	108.00	8.00
quarterly	4	108.24	8.24
monthly	12	108.30	8.30
daily	365	108.33	8.33

   You can also do these calculations directly with the command ▶Eff (8, C/Y) because the amount invested is $100: interest rate (%) = interest paid ($).

3. The effective rate of interest associated with more-frequent compounding rapidly approaches $e^r - 1$, where r is the annual rate of interest and e is the base of the exponential function ($e = 2.7182818\ldots$). Investigate this claim by taking some values for r and examining the effect of more-frequent compounding (such as weekly, daily, hourly, etc).

Solution: See Exercise 6 above for details on how to calculate effective rate of interest. Here we use various values for $r = \text{I}\%/100$, and work out the effective rates of interest for different compounding periods. The results are shown in the table below. In the table, N is the number of compounding periods in a year. Interest rates are rounded to 2 decimal places.

The continuously compounded interest rate is $100\bigl(e^r - 1\bigr)$.

I%	r	Effective interest rate % when compounded ...					
		yearly $N=1$	monthly $N=12$	weekly $N=52$	daily $N=365$	hourly $N=8760$	continuously $N \to \infty$
5	0.05	5	5.12	5.13	5.13	5.13	5.13
7.5	0.075	7.5	7.76	7.78	7.79	7.79	7.79
10	0.1	10	10.47	10.51	10.52	10.51	10.52
20	0.2	20	21.94	22.09	22.13	22.14	22.14

Most of the change in the effective interest rate is between annual compounding ($N=1$) and monthly compounding ($N=12$). The change between monthly and continuous compounding $100(e^r - 1)$ is relatively small, but larger for higher interest rates.

4. Jill purchased a new car for \$11,999 by obtaining a 5-year loan from a finance company. The interest on the loan was at a flat rate of 14% per year. Compare the amounts of interest paid on this loan each year with the amounts paid for a loan of the same amount at a rate of 14% reducible compound interest per annum, with monthly payments and compounding.

Solution: The flat rate means she pays 14% of \$11,999, or \$1679.86, each year for 5 years. For the reducing rate:

TVM Solver
N $= 12 \times 5 = 60$ number of *payments*
I% $= 14$ annual percentage interest rate
PV $= 11999$ the loan amount
PMT $= ?$ we need to find this first
FV $= 0$ loan paid off
P/Y $= 12$ payments per year
C/Y $= 12$ compounding periods per year
Then, PMT $= -279.20$
The monthly repayments are \$279.20.

PTO

The simplest way to find the amount of interest paid each year is to run the AMORT-TBL/AMRTBLCE program. You should get the amounts shown in the table below.

Year	Flat-rate Interest ($)	Reducing Interest ($)
1	1679.86	1568.39
2	1679.86	1302.27
3	1679.86	996.40
4	1679.86	644.86
5	1679.86	240.82
Total	8399.30	4752.74

5. A couple are hoping to buy their first home. They expect to obtain a 25-year loan, and their monthly payments cannot exceed $920 if their living expenses are taken into account. If the interest rate is assumed to be constant at 7% per annum, how much money will they be able to borrow?

   Solution   TVM Solver
   N $= 12 \times 25 = 300$ number of *payments*
   I% $= 7$ annual percentage interest rate
   PV $= ?$ doesn't matter what is here — we calculate this
   PMT $= -920$ monthly payments to bank
   FV $= 0$ loan paid off
   P/Y $= 12$ payments per year
   C/Y $= 12$ compounding periods per year — assumed equal to P/Y
   Then, PV $= 130{,}167.95$
   The couple can only borrow $130,167. This is why first-home buyers often struggle.

6. A student purchased a new laptop multimedia computer system with software for $4500 by obtaining a personal loan at 13.4% per annum interest, compounded monthly.

   (a) How much will each payment be if the loan is spread over 3 years?

   Solution   TVM Solver
   N $= 12 \times 3 = 36$ number of *payments*
   I% $= 13.4$ annual percentage interest rate
   PV $= 4500$ amount of loan
   PMT $= ?$ doesn't matter what is here — we calculate this
   FV $= 0$ loan paid off
   P/Y $= 12$ payments per year
   C/Y $= 12$ compounding periods per year
   Then, PMT $= -152.49$
   The monthly repayment is $152.49.

   (b) How much interest will be paid over the 3 years?

   Solution
   Total interest $=$ total payments $-$ amount of loan $= 36 \times 152.49 - 4500 = \$989.64$.

*(c) If she arranged for a three-year loan, as in Parts (a) and (b), but then decided to pay the entire loan balance at the end of 2 years, how much would she save?

Solution: To find out the total cost of this scheme, we need to find the balance owing after 2 years and the total payments in the first 2 years.

Run the AMORTTBL/AMRTBLCE program to find the balance after 2 years ($1703.72), the total interest paid in 2 years ($863.51) and the total principal paid in 2 years ($2796.28). The sum of these, $3659.79, is the total payments made in 2 years. Alternatively calculate $24 \times 152.49 = \$3659.76$ (the difference in the two amounts is due to rounding the payments to 2 decimal places.)

The total payment here is then the total payment for 2 years plus the balance owing after 2 years, that is $3659.79 + \$1703.72 = \$5363.51$.

The total payment for the full three-year loan is principal + interest, giving from (b), $4500 + \$989.68 = \$5489.68$.

The total saving in Part (c) is then $5489.68 - \$5363.51 = \$126.17$.

7. Jeremy decided to invest money in a superannuation account. He deposited $400 per month into an account earning 9% interest per annum, compounded monthly.

(a) How much money will be in the account after 25 years?

Solution   TVM Solver
N $= 12 \times 25 = 300$ number of *payments*
I% $= 9$ annual percentage interest rate
PV $= 0$ nothing in the account to start with
PMT $= -400$ monthly payment to the bank
FV $=$?   doesn't matter what is here — we calculate this
P/Y $= 12$ payments per year
C/Y $= 12$ compounding periods per year
Then, FV $= 448448.77$
The amount in the account after 25 years is $448,448.77.

(b) At the end of that time, Jeremy arranged to use the accumulated funds in the account to receive a monthly payment for the next 25 years. If the interest rate remains the same, find the size of the payment each month.

Solution
N $= 12 \times 25 = 300$ number of *payments*
I% $= 9$ annual percentage interest rate
PV $= -448448.77$ accumulated funds
PMT $=$?   doesn't matter what is here — we calculate this
FV $=$ 0   all funds used up after 25 years
P/Y $= 12$ payments per year
C/Y $= 12$ compounding periods per year
Then, PMT $= 3763.37$
The amount payed to Jeremy each month is $3763.37.

Seems like a good deal! Pay in $400 per month for 25 years, then receive $3763.37 per month for the next 25 years. What's the catch?

8. Carole and Joyce have taken out a 25-year home loan of $120,000 to buy an apartment at an interest rate of $7\frac{1}{2}\%$ per annum, compounded monthly.

(a) Check that their monthly payment for this loan is $886.79.

Solution   TVM Solver
N $= 12 \times 25 = 300$ number of *monthly payments*
I% $= 7.5$ annual percentage interest rate
PV $= 120000$ amount of loan
PMT $= ?$ doesn't matter what is here — we calculate this
FV $= 0$ loan paid off
P/Y $= 12$ payments per year
C/Y $= 12$ compounding periods per year
Then, PMT $= -886.79$
The monthly repayment is $886.79.

(b) What would be the effect on the length of their loan if they were to pay an extra $50 per month?

Solution: The new payment is $886.79 + $50 = $936.79.
Set PMT $= -936.79$.
Then, N $= 258.8$ (months)
The length of the loan would be reduced to 259 months or 21 years 7 months (the last payment would be less than the full monthly payment).

(c) What would be the effect on the length of their loan of a drop in interest rates to 7% (assuming they pay $886.79 per month)?

Solution
I% $= 7$
PMT $= -886.79$

Then, N $= 267.8$ (months)
The length of the loan would be reduced to 268 months or 22 years 4 months (the last payment would be less than the full monthly payment).

(d) Carole suggests paying $443.40 every 2 weeks instead of $886.79 per month, with the interest rate at $7\frac{1}{2}\%$ per annum interest, compounded monthly. How would this plan affect the length of the loan?

Solution   TVM Solver
N $= ?$
I% $= 7.5$
PV $= 120000$
PMT $= -443.40$ fortnightly payment
FV $=$ 0
P/Y $= 26$ fortnightly payments per year
C/Y $= 12$ compounding periods per year
Then, N $= 525.5$ (fortnights)
The length of the loan would be reduced to 526 fortnights or 20 years 12 weeks (the last payment would be less than the full fortnightly payment).

(e) If they were to borrow only \$115,000, yet still pay \$886.79 per month, with interest arrangements still the same, for how long will the loan last?

Solution
N = ?
I% = 7.5
PV = 115000   new loan amount
PMT = −886.79   monthly payment
FV = 0
P/Y = 12   monthly payments per year
C/Y = 12

Then, N = 267.0 (months)

The length of the loan would be reduced to 267 months or 22 years 3 months.

(f) Calculate how much time Carole and Joyce can save off the original repayment time of 25 years if all of the above strategies are adopted: an interest rate of 7% is obtained, compounded monthly; payments of \$468.40 are made every 2 weeks (an extra \$25 per fortnight); and they only borrow \$115,000 instead of \$120,000.

How much interest would they save?

Solution
N = ?
I% = 7
PV = 115000
PMT = −468.40
FV = 0
P/Y = 26
C/Y = 12

Then, N = 401.8 (fortnights)

The length of the loan would be reduced from 25 years to 402 fortnights or 15 years 24 weeks.

They would pay a total of \$73,203.12 in interest, compared to \$146,037 for the original loan, a saving of about \$72,830.

PTO

9. A *perpetuity* is an annuity that lasts forever.

Find the present value of a perpetual annual payment of \$1000, assuming an interest rate of 6% per annum, compounded monthly. (This is the amount you would need to invest today to provide enough funds for an annual payment of \$1000 forever.)

In TVM mode, explore this situation by choosing a large number for the number of compounding periods N, and set FV to be zero.

Solution   TVM Solver
N $=1000$ number of compounding periods (years)
I% $=6$ annual percentage interest rate
PV $=\,?$ doesn't matter what is here — we calculate this
PMT $=1000$ positive because the bank pays this out to the beneficiary
FV $=\,0$ nothing left at the end (of time)
P/Y $=1$ payments per year
C/Y $=12$ compounding periods per year
Then, PV $=-16213.29$
You would need to invest an amount of about \$16,213.
Changing N to 10,000 years also gives PV $=-16213.29$, so we can say the amount invested is enough to make the payment 'perpetual'.

Here, the amount of interest on the principal of \$16,213.29 is just enough to cover the annual \$1000 payment. The principal actually remains the same from year to year.

We can check that this is the case using the TVM Solver.

N $=1$ one payment a year
I% $=6$ annual percentage interest rate
PV $=-16213.29$ the principal in the bank
PMT $=\,?$ doesn't matter what is here — we calculate this
FV $=\,16213.29$ principal remains the same at the end of the year
P/Y $=1$ payments per year
C/Y $=12$ compounding periods per year
Then, PMT $=1000$, as we predicted.

PTO

10. You may have heard of the *Rule of 72*, which provides a rough approximation to the amount of time it will take for an investment to double when interest is compounded annually at a rate of I%. The Rule claims

$$\text{doubling time} \approx \frac{72}{I\%}.$$

So, according to the *Rule of 72*, an investment at a rate of 6%, compounded annually, will take about $72/6 = 12$ years to double in value.

Investigate this Rule using the TVM mode of the calculator. Over what range of values for I% does it provide a useful approximation?

Solution   TVM Solver
N = ?   doesn't matter what is here — we calculate this
I% = see table   annual percentage interest rate
PV = −100   choose any amount
PMT = 0   no regular payments
FV = 200   double the PV
P/Y = 1   payments per year — must be set to 1 if no payments
C/Y = 1   compounding periods per year

Set values of I% and calculate N, the time to double the original amount.

The exact time N, the time according to the Rule of 72 and the percentage error in the Rule of 72 are shown in the table below.

I%	N (yrs)	72/I%	% error
1	69.7	72.0	3.3
2.5	28.1	28.8	2.5
5	14.2	14.4	1.4
7.5	9.6	9.6	0.0
10	7.3	7.2	−1.4
20	3.8	3.6	−5.3

The approximation is good except for I% > 20, but particularly good around I% = 7.5.

23 Complex Numbers

The material here is directly relevant to the topics on Complex Numbers in the Australian Curriculum for Specialist Mathematics.

23.1 Setting complex mode

Press $\boxed{\text{mode}}$ and select the Cartesian form $a+bi$ (below left, 8th line).
Press $\boxed{\text{2nd}}$ $\boxed{\text{mode}}$ (Quit) to return to the Home screen.

We shall use $z_1 = 1+2i$ and $z_2 = 3-i$ in our examples. i is $\boxed{\text{2nd}}$ $\boxed{\cdot}$.

Complex numbers can be stored in the same way as ordinary numbers.

Store z_1 in memory A: $1+2i$ $\boxed{\text{sto}}$ A; store z_2 in memory B: $3-i$ $\boxed{\text{sto}}$ B (above right).

23.2 Basic operations

23.2.1 Addition and subtraction

Just as you would expect.

$1+2i + 3-i = 4+i$ or $A+B = 4+i$
$1+2i - (3-i) = -2+3i$ or $A-B = -2+3i$

23.2.2 Multiplication and division

Again as you would expect. Implied multiplication works too.

$(1+2i)(3-i) = 5+5i$ or $AB = 5+5i$
$(1+2i)/(3-i) = 0.1+0.7i$ or $A/B = 0.1+0.7i$

23.2.3 Conjugation

Finding the complex conjugate.

$\bar{z}_1 = \text{conj}(1+2i) = 1-2i$ \qquad conj is $\boxed{\text{math}}$ CMPLX $\boxed{1}$

23.2.4 Real part

$\text{Re}(z_1) = \text{real}(1+2i) = 1$ \qquad real is $\boxed{\text{math}}$ CMPLX $\boxed{2}$

```
conj(1+2i)
                          1-2i
real(1+2i)
                             1.
imag(1+2i)
                             2.
abs(1+2i)
                   2.236067977
```

23.2.5 Imaginary part

$\text{Im}(z_1) = \text{imag}(1+2i) = 2$ \qquad imag is $\boxed{\text{math}}$ CMPLX $\boxed{3}$

23.2.6 Modulus

Sometimes called length or absolute value.

$|z_1| = \text{abs}(1+2i) = \sqrt{1^2+2^2} = \sqrt{5} \approx 2.236$ \qquad abs is $\boxed{\text{math}}$ CMPLX $\boxed{5}$

23.3 Polar form

The polar form (r, θ) of a complex number is represented on the TI-84/CE in exponential form $re^{i\theta}$, where r is the modulus and θ is the (polar) angle.

This form is equivalent to the form $r\big(\cos(\theta)+i\sin(\theta)\big)$, often abbreviated as $r\text{cis}(\theta)$.

23.3.1 Modulus and angle

Use the $\boxed{\text{math}}$ CMPLX operations *abs* and *angle* to extract the modulus and angle, respectively, of a complex number in either Cartesian ($a+ib$) or polar form.

Similarly, *real* and *imag* can be used to find the real and imaginary parts respectively of a number in polar form, and *conj* its complex conjugate.

```
abs(1+2i)
                   2.236067977
angle(1+2i)
                   1.107148718
real(e^(iπ/4)
                  0.7071067812
imag(e^(iπ/4)
                  0.7071067812
```

23.3.2   Forms of output

Cartesian ($a+bi$) mode

Set $a+bi$ in $\boxed{\text{mode}}$.

Input $\sqrt{(2)}e^{i\pi/4}$ using the $\boxed{e^x}$ key and press $\boxed{\text{enter}}$.
$\sqrt{(2)}e^{i\pi/4} \rightarrow 1+1i$.

Try the famous calculation $e^{i\pi}$.

To input an angle in degrees, use the degree symbol
$\boxed{\text{angle}}\ \boxed{1}$. $\sqrt{(2)}e^{i45°} \rightarrow 1+1i$.

```
∫(2)e^(iπ/4)
                            1+1i
e^(iπ)
                           -1+0i
∫(2)e^(i45°)
                            1+1i
```

Polar ($re^{\theta i}$) mode

Set $re^{\theta i}$ in $\boxed{\text{mode}}$. Make sure you are in Radians.

Enter the number $1+i$ and press $\boxed{\text{enter}}$.
$1+i \rightarrow 1.414213562e^{0.7853981634i}\ (=\sqrt{2}e^{i\pi/4})$;
use the right arrow to scroll across.

If you repeat this in Degree mode,
$1+i \rightarrow 1.414213562e^{45i}$,
i.e. the angle is displayed in degrees.

```
1+i
1.414213562e^(0.785398163…
1+i
        1.414213562e^(45i)
```

However, **if you do not change back to Radians to input polar form, any input with an angle in degrees will be wrong** (see Section 23.6.1).

A better alternative to changing back and forth between Degree and Radian mode when you are using angles in degrees is to stay in Radian mode and use the CMPXANGD/ CXANGDCE program[50] to convert your output to degrees; it rounds all numbers to 5 decimal places. As the program gives both the Cartesian and polar forms, it can also be used to convert one form to the other.

If your calculations produce the decimal equivalent of $\sqrt{(2)}e^{i\pi/5}$, running the program gives

Cartesian form $1.14412 + i0.83125$

modulus R $= 1.41421$ (stored in memory R)

angle θ (rad) $= 0.62832$ (stored in memory T)

angle θ (°) $= 36$ (stored in memory θ).

```
A+iB: (A,B)=
  {1.14412,0.83125}

R: 1.41421
θr: 0.62832
θ°: 36
```

---

[50]available at *canberramaths.org.au* under *Resources*

23.3.3   Forms of input

In carrying out calculations using complex numbers (page 207), you can input the numbers in any form, Cartesian or polar or mixed. The mode setting only determines the output form of the answer.

$$1+i + \sqrt{(2)}e^{i45°} = 2.828427125e^{0.7853981634i} \quad \text{polar form}$$
$$= 2+2i \quad \text{Cartesian form.}$$

When you are inputting in polar form, it is VITAL that you are in Radian mode (see Section 23.6.1), even if you wish to input an angle θ in degrees.

23.3.4   Conversion between forms

To convert complex numbers from one form to the other, use the conversions in the $\boxed{\text{math}}$ CMPLX menu. This works whether you are in Cartesian or polar mode.

You can also use the CMPXANGD/CXANGDCE program (Section 23.3.2).

$1+i$ $\boxed{\blacktriangleright\text{Polar}}$ $\boxed{\text{enter}}$ gives $1.414213562e^{0.7853981634i}$ in Radian mode and $1.414213562e^{i45}$ in Degree mode.

$\sqrt{(2)}e^{i\pi/4}$ $\boxed{\blacktriangleright\text{Rect}}$ $\boxed{\text{enter}}$ gives $1+1i$ in Radian mode and an error in Degree mode[a] (see Section 23.6.1).

```
1+i▶Polar
1.414213562e^(0.785398163…
1+i▶Polar
            1.414213562e^(45i)
√(2)e^(iπ/4)▶Rect
                          1+1i
√(2)e^(iπ/4)▶Rect
                         Error
```

---
[a]or the wrong answer on a TI-84Plus

23.4   Powers and roots

23.4.1   Powers

Integer powers work as you would expect.

$(1+2i)^4 = -7-24i$.

$(1+2i)^{-3} = -0.088+0.016i = -\frac{11}{125} + \frac{2}{125}i$.

$\left(\sqrt{2}e^{i\pi/4}\right)^4 = 4e^{i\pi} = -4$.

23.4.2   Roots

Unfortunately you only get one root of a complex number when you use square root, cube root, nth root and fractional powers. To find all the roots,[51] use the CMPXROOT/CXROOTCE program (see Section 23.6.2), which calculates and plots the n nth roots of a given complex number.

---
[51]A complex number has n nth roots (n an integer).

23.5   Exercises

$$z_1 = 3+4i \qquad z_2 = 2+3i \qquad z_3 = \sqrt{(2)}e^{i\pi/4} \qquad z_4 = e^{i\pi/2}$$

Find

1. $z_1 + z_2$

2. $2z_1 + 3z_2$

3. $z_1 - z_2$

4. $4z_1 - 2z_2$

5. $z_1 z_2$

6. z_1 / z_2

7. $\bar{z}_1$

8. $z_1 \bar{z}_1$

9. $|z_1|^2$

10. $\text{Re}(z_1)$

11. $\text{Im}(z_2)$

12. z_1^2

13. z_1^4

14. $\sqrt{z_1}$

15. $z_3 z_4$ in polar form

16. z_3 / z_4 in polar form

17. z_1 in polar form

18. z_3 in Cartesian form

19. z_4 in Cartesian form

20. $\sqrt{z_4}$ in polar and Cartesian forms

Answers in Section 23.7.

23.6   Appendix

23.6.1   Input/output of complex numbers

TI-84CE

Mode		Input	Output*	Correct?
$a + bi$	Radian	$\sqrt{(2)}e^{i\pi/4}$	$1+1i$	YES
		$\sqrt{(2)}e^{i45°}$	$1+1i$	YES
	Degree	$\sqrt{(2)}e^{i45}$	ERROR	NO
		$\sqrt{(2)}e^{i45°}$	ERROR	NO
		$\sqrt{(2)}e^{i\pi/4}$	$1+1i$	YES
$re^{i\theta}$	Radian	$1+i$	$\sqrt{(2)}e^{i\pi/4}$	YES
		$\sqrt{(2)}e^{i45°}$	$\sqrt{(2)}e^{i\pi/4}$	YES
	Degree	$1+i$	$\sqrt{(2)}e^{i45}$	YES
		$\sqrt{(2)}e^{i45}$	ERROR	NO
		$\sqrt{(2)}e^{i45°}$	ERROR	NO

*given in decimal form

TI-84Plus

Where the TI-84CE gives ERROR, the TI-84Plus (OS 2.55MP) gives a **wrong answer**.

To input a complex number, **always use Radian mode**.

Use the degree symbol ($\boxed{\text{angle}}\,\boxed{1}$) if the angle is in degrees.

Alternatively, use the CMPXANGD/CXANGDCE program to convert the polar form in radians to the Cartesian form and the polar form in degrees (Section 23.3.2).

23.6.2 CMPXROOT/CXROOTCE program

CMPXROOT/CXROOTCE[52] displays x and y of the rectangular (Cartesian) form $x + iy$ and plots the N roots of the complex number $Z = A + iB$ on the Argand diagram. The roots all lie on a circle (their moduli are the same) which is drawn.

Lines are drawn from the origin to each root to show where the root is and to make the symmetry of the roots more obvious. Note that if a root lies on a co-ordinate axis, you won't see it, but it should be obvious from symmetry.

Use: Run the program. Input Z and N (positive integer) when prompted. The program displays each root. Press ⌑enter⌑ to continue at each step.

The roots are also stored in list L_6 for later use.

Move the cursor around the final plot using the arrow keys to see rectangular (x, y) values if RectGC is set in ⌑format⌑ or polar (r, θ) values if PolarGC is set.

Example: The three cube roots (N = 3) of $1 + 2i$ are $1.220 + 0.472i$, $-1.018 + 0.820i$ and $-0.201 - 1.292i$, all rounded to three decimal places.

```
      NTH ROOTS OF A+IB

A=?1
B=?2
N (POS INT): 3

              1.219616508
              0.4717112678
[ENTER]
```

```
N (POS INT): 3

              1.219616508
              0.4717112678
[ENTER]

             -1.018322195
              0.8203632449
[ENTER]
```

```
[ENTER]

             -1.018322195
              0.8203632449
[ENTER]

             -0.2012943128
             -1.292074513
```


---
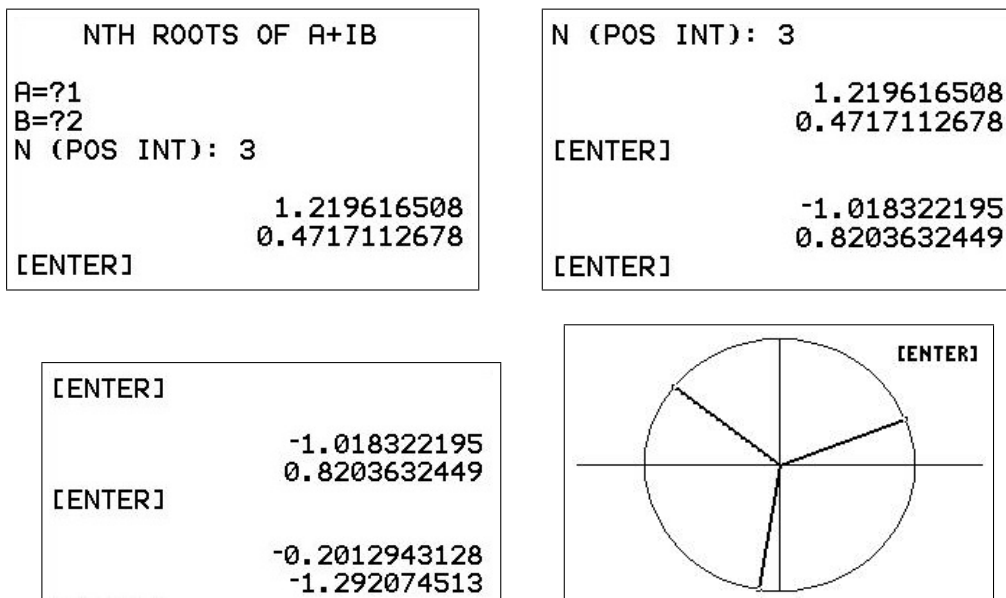
[52]available at *canberramaths.org.au* under *Resources*

23.7 Answers to exercises

1. $5+7i$

2. $12+17i$

3. $1+i$

4. $8+10i$

5. $-6+17i$

6. $18/13 - i/13$
   ▶Frac $\left(\boxed{\text{math}}\ \boxed{1}\right)$ is useful here

7. $3-4i$

8. 25

9. 25

10. 3

11. 3

12. $-7+24i$

13. $-527-336i$

14. $2+i$ and $-2-i$

15. $\sqrt{2}e^{i3\pi/4}$

16. $\sqrt{2}e^{-i\pi/4}$

17. $5e^{0.927295218i}$

18. $1+i$

19. i

20. $\pm e^{i\pi/4} \equiv \pm(1+i)/\sqrt{2}$

24   Programming

24.1   Introduction

A basic program structure is $\boxed{\text{INPUT}} \rightarrow \boxed{\text{CALCULATION}} \rightarrow \boxed{\text{OUTPUT}}$. The input is usually numbers (it might also be a function), the output can be text (e.g. 'COMPLEX ROOTS'), numbers, a graph or a table.

The programming language of the TI-84/CE is a subset of the BASIC programming language, containing most of the useful features of that language together with the calculator graphics commands and a large library of scientific functions.

Most of the commands have to be selected from a menu — you cannot type them in letter by letter. $\boxed{\text{prgm}}$ selected from the Home screen allows you to run, edit and create programs; $\boxed{\text{prgm}}$ selected while editing a program gives access to most of the programming commands.

The hardest thing to start with is finding the appropriate commands on the keyboard and in menus.[53] *On pages 232 and 233 of these notes is a list of key strokes for the commands used in the programs here.* The Table of Functions at the back of the TI-84 Guidebook is also very useful.

24.2   Basics

We start with a simple example and successively add features.

Input the radius of a circle, calculate and output its area.

24.2.1   Writing the basic program

Press the $\boxed{\text{prgm}}$ key and use the left arrow to move to NEW. Press $\boxed{\text{enter}}$.

Enter the name of the program (up to 8 characters) — CIRCLE. Note that the calculator is in alpha mode.

Press $\boxed{\text{enter}}$ and you are now ready to enter the program.

Press $\boxed{\text{prgm}}$ and have a look at the commands in each of the sub-menus CTL (Control), I/O (Input/Output) and COLOR on the CE.

Here is our basic program. Press $\boxed{\text{enter}}$ at the end of each line to go to the next line. Press $\boxed{\text{clear}}$ in a menu to return to the program without selecting a menu item.

Prompt R $\boxed{\text{prgm}}$ I/O $\boxed{2}$ $\boxed{\text{alpha}}$ $\boxed{\text{R}}$

$\pi R^2 \rightarrow A$ $\boxed{\text{2nd}}$ $\boxed{\wedge}$ $\boxed{\text{alpha}}$ $\boxed{\text{R}}$ $\boxed{x^2}$ $\boxed{\text{sto}}$ $\boxed{\text{alpha}}$ $\boxed{\text{A}}$

Disp A $\boxed{\text{prgm}}$ I/O $\boxed{3}$ $\boxed{\text{alpha}}$ $\boxed{\text{A}}$

```
PROGRAM:CIRCLE
:Prompt R
:πR²→A
:Disp A
```

When you are happy with the program, press $\boxed{\text{2nd}}$ $\boxed{\text{quit}}$ to return to the Home screen.

Press $\boxed{\text{prgm}}$, select the CIRCLE program by typing its number or with the cursor and $\boxed{\text{enter}}$. This displays *prgmCIRCLE* on the Home screen.

---

[53]If you're really stumped, all the commands are in the $\boxed{\text{catalog}}$ menu ($\boxed{\text{2nd}}$ $\boxed{0}$): press a letter key (*don't* press $\boxed{\text{alpha}}$ first) to go to commands starting with that letter. You can scroll upwards from the top or down to the bottom to get to the commands and characters that don't contain letters.

Press ⬚enter to run the program. Try some R values (R = 1 should give a recognisable area) to make sure the program is running correctly.

```
PrgmCIRCLE
R=?1
                    3.141592654
```

If you press ⬚enter after the program has finished, you will rerun the program — the calculator executes the last command again, making it easy to enter successive R values.

When a program is running, you will see continuously moving vertical bars (84) or a moving yellow circle (CE) in the top right corner of the screen.

24.2.2   Making corrections

If you have made a mistake in entering the code, the calculator may detect the error and stop with an error message ERR: *message* and a choice of quitting (returning to the Home screen) or going to the error. Select the second option by pressing ⬚2 and the calculator will take you to the problem in the program.

If the program runs OK, but gives the wrong output, press ⬚prgm, select EDIT, then the program. This allows you to edit the program.

Use the ⬚del and ⬚ins keys to edit within a program line. ⬚clear clears a whole line but leaves the blank line. ⬚del on a blank line deletes the line.

24.2.3   Making the program more user-friendly

It makes the program easier to use if there isn't clutter on the screen and if the program gives some idea of what it does, what input it requires and what it outputs.

Commands that are new or changed from the previous program are highlighted.

The ⬚ins key allows you to insert characters in a line.

⬚ins ⬚enter at the beginning or end of a line gives a blank line.

⬚2nd ⬚alpha gives alpha LOCK (for letters), ⬚alpha turns it off.

ClrHome   ⬚prgm I/O ⬚8   *clear the Home screen*

Disp "AREA␣OF␣A␣CIRCLE"   ⬚prgm I/O ⬚3 ⬚2nd ⬚alpha ⬚+ ⬚A ⬚R ...

                                space ␣ is ⬚alpha ⬚0

Input "RADIUS:␣", R   ⬚prgm I/O ⬚1; colon : is ⬚alpha ⬚.

$\pi R^2 \rightarrow A$

Disp "AREA", A

Run this version.

```
PROGRAM:CIRCLE
:ClrHome
:Disp "AREA OF A CIRCLE"
:Input "RADIUS: ",R
:πR²→A
:Disp "AREA",A
```

```
AREA OF A CIRCLE
RADIUS: 1
AREA
                 3.141592654
```

24.3 Looping

24.3.1 Looping for input

Maybe we'd like to enter a number of values for R without having to rerun the program each time. We do this with a simple loop using a *Label* and a *Goto*.

ClrHome

Disp "AREA OF A CIRCLE"

Lbl 1   ⌞prgm⌝ ⌞9⌝ ⌞1⌝

Input "RADIUS:␣", R

$\pi R^2 \rightarrow A$

Disp "AREA", A

Goto 1   ⌞prgm⌝ ⌞0⌝ ⌞1⌝

```
PROGRAM:CIRCLE
:ClrHome
:Disp "AREA OF A CIRCLE"
:Lbl 1
:Input "RADIUS: ",R
:πR²→A
:Disp "AREA",A
:Goto 1
```

```
AREA OF A CIRCLE
RADIUS: 1
AREA
              3.141592654
RADIUS: 2
AREA
              12.56637061
RADIUS:
```

This makes entering a number of radius values easier, but how do we stop the program? Press ⌞on⌝ and select *Quit*. This stops any program and returns to the Home screen.

Use ⌞on⌝ *Goto* to go back to edit the program — useful during its development.

24.3.2 Looping to check input

Another place for a loop is to test whether an input value falls within a specified range. In our example, we expect $R \geqslant 0$ (although it doesn't matter in this case because we square R). Let's use a loop to test for this. We also introduce the *If* statement here.

ClrHome

Disp "AREA OF A CIRCLE"

Lbl 1

Input "RADIUS:␣ ", R

If R < 0   *If* is ⌞prgm⌝ ⌞1⌝; < is ⌞2nd⌝ ⌞test⌝ (⌞math⌝ key) ⌞5⌝

Goto 1   *Do this step if the* If *is true, otherwise skip this step*

$\pi R^2 \rightarrow A$

Disp "AREA", A

Goto 1

Check your program with suitable inputs, i.e. at least one negative R.

See Extension 1 on page 226 for another way to use such a loop and Extension 2 on page 226 for a more elegant error check. Make a copy of your program here (see the last chapter in Volume 2 of this book) so you can come back to these extensions later.

```
PROGRAM:CIRCLE
:ClrHome
:Disp "AREA OF A CIRCLE"
:Lbl 1
:Input "RADIUS: ",R
:If R<0
:Goto 1
:πR²→A
:Disp "AREA",A
:Goto 1
```

24.3.3 Including an error message

Let's say we want to tell the user that he or she has made a mistake in the input. Such diagnostics are always useful, especially for students.

Here we embellish the *If* statement with a *Then* and *End*, and introduce *Pause*, which stops the program until the user presses enter .

That a program has Paused is indicated by shimmering dots (84) or a rotating white circle (CE) in the top right corner of the screen.

Note that anything in inverted commas is text, and can be whatever you want.

ClrHome

Disp "AREA OF A CIRCLE"

Lbl 1

Input "RADIUS: = ", R

If R < 0

Then   prgm 2

Disp "∗ ∗ ∗ MAKE R≥0 ∗ ∗ ∗"   ∗ is ×   *used for emphasis here*

Pause   prgm 8

Goto 1

End   prgm 7   *ends the If Then*

$\pi R^2 \to A$

Disp "AREA", A

Goto 1

Don't forget suitable tests for this version of the program.

```
PROGRAM:CIRCLE
:Input "RADIUS: ",R
:If R<0
:Then
:Disp "***MAKE R≥0***"
:Pause
:Goto 1
:End
:πR²→A
:Disp "AREA",A
```

```
AREA OF A CIRCLE
RADIUS: 1
AREA
              3.141592654
RADIUS: -1
***MAKE R≥0***
RADIUS: 2
AREA
              12.56637061
RADIUS:
```

24.3.4 Looping through a range of input values

Here we use a *For* loop to carry out the calculations for a specified range of (equally spaced) input values. The *For* command has the format

For (R, F, L, S)

commands

End

Variable R starts with a value of F and *commands* are executed. R is then increased by S and *commands* are executed again. This continues until $R > L$.[54]

Remove the lines (use ⌐clear⌐ ⌐del⌐) *that test for $R < 0$ to keep the program simple.*

ClrHome

Disp "AREA OF A CIRCLE"

Input "FIRST␣R:␣", F   *prompt for the initial value of R*

Input "LAST␣R:␣", L   *prompt for the final value of R*

Input "R␣STEP:␣", S   *prompt for the increment in R*

For (R, F, L, S) ⌐prgm⌐ ⌐4⌐

$\pi R^2 \to A$

Disp "R,␣AREA", R, A

Pause   *pause to read answer*

End   *end For loop*

Run the program. Press ⌐enter⌐ after each output to continue.

```
PROGRAM:CIRCLE
:Disp "AREA OF A CIRCLE"
:Input "FIRST R: ",F
:Input "LAST R: ",L
:Input "R STEP: ",S
:For(R,F,L,S)
:πR²→A
:Disp "R, AREA",R,A
:Pause
:End
```

```
FIRST R: 1
LAST R: 5
R STEP: 1
R, AREA
                     1
           3.141592654
R, AREA
                     2
           12.56637061
```

---

[54]S can be negative, so that R is decreased. In this case, L must be less than F, and the program stops when $R < L$.

24.4   Displaying results as a table

Rather than displaying the areas one by one as we did in the previous example, we can put them in a table. Then we can scroll up and down to see all the answers.

To make up a table, we need to put the R and A values in lists. Here we use list L_1 for R and list L_2 for A.

Lists $L_1 - L_6$ are built in to the calculator and can be accessed from the keyboard. You can also give lists names of up to five characters.

The variable I in the following program is the list index or counter — we have to increment it by 1 each time we store a value in a list.

ClrHome

SetUpEditor   stat 5   *ensures that lists $L_1 - L_6$ are shown when we press* stat *EDIT after running the program*

ClrList L_1, L_2 stat 4 2nd 1 , 2nd 2   *clears the lists*

$1 \rightarrow I$ *initialise the list index*

Disp "AREA OF A CIRCLE"

Input "FIRST R: ", F

Input "LAST R: ", L

Input "R STEP: ", S

For (R, F, L, S)

$R \rightarrow L_1 (I)$

$\pi R^2 \rightarrow L_2 (I)$

$I+1 \rightarrow I$ *increment the list index by 1*

End   *end of For loop*

Disp "", "PRESS␣STAT␣EDIT"   *tell the user how to view the answers; the inverted commas with no enclosed characters give a blank line in the output*

Run the program with R varying from 0 to 10 in steps of 1 and scroll down the L_2 column. Press 2nd quit to return to the Home screen from the table.

```
PROGRAM:CIRCLE
:Input "FIRST R: ",F
:Input "LAST R: ",L
:Input "R STEP: ",S
:For(R,F,L,S)
:R→L₁(I)
:πR²→L₂(I)
:I+1→I
:End
:Disp "","PRESS STAT EDIT"
```

```
AREA OF A CIRCLE
FIRST R: 0
LAST R: 10
R STEP: 1

PRESS STAT EDIT
```

L₁	L₂
0	0
1	3.1416
2	12.566
3	28.274
4	50.265
5	78.54
6	113.1
7	153.94
8	201.06
9	254.47
10	314.16

24.5   Displaying results as a graph

24.5.1   Graphing data points from a program

Once we have the results in lists, it is a simple step to graph them. We use $\boxed{\text{statplot}}$ ($\boxed{\text{2nd}}$ $\boxed{\text{y=}}$) to define the plot, i.e. tell the calculator where the data are, what sort of graph to plot and what marker to use for the points. On a CE, you can also choose the colour.

We also use *ZoomStat* to set suitable axes for the graph and $\boxed{\text{trace}}$ to allow us to move the cursor along the plotted values. The table of values is still created.

ClrHome

FnOff	$\boxed{\text{vars}}$ Y-VARS $\boxed{4}$ $\boxed{2}$	*turn off any functions in* $\boxed{\text{y=}}$
PlotsOff	$\boxed{\text{2nd}}$ $\boxed{\text{statplot}}$ $\boxed{4}$	*turn off any plots*
ClrDraw	$\boxed{\text{2nd}}$ $\boxed{\text{prgm}}$ (draw) $\boxed{1}$	*clear the graphics screen*
$1 \rightarrow$ Xscl	Xscl is $\boxed{\text{vars}}$ $\boxed{1}$ $\boxed{3}$	*tick marks on the X axis*
$50 \rightarrow$ Yscl	Yscl is $\boxed{\text{vars}}$ $\boxed{1}$ $\boxed{6}$	*tick marks on the Y axis*

ClrList L_1, L_2

$1 \rightarrow I$

Disp "AREA OF A CIRCLE"

Input "FIRST R: ", F

Input "LAST R: ", L

Input "R STEP: ", S

For (R, F, L, S)

$R \rightarrow L_1 (I)$

$\pi R^2 \rightarrow L_2 (I)$

$I+1 \rightarrow I$

End

```
PROGRAM:CIRCLE
:Input "R STEP: ",S
:For(R,F,L,S)
:R→L₁(I)
:πR²→L₂(I)
:I+1→I
:End
:Plot1(Scatter,L₁,L₂,□)
:ZoomStat
:Trace
```

Plot1 (Scatter, L_1, L_2, □)	$\boxed{\text{statplot}}$ $\boxed{1}$ $\boxed{\text{statplot}}$ TYPE $\boxed{1}$ $\boxed{,}$	
	$\boxed{\text{2nd}}$ $\boxed{1}$ $\boxed{,}$ $\boxed{\text{2nd}}$ $\boxed{2}$ $\boxed{,}$ $\boxed{\text{statplot}}$ MARK $\boxed{1}$	
ZoomStat	$\boxed{\text{zoom}}$ $\boxed{9}$	
Trace	Trace *key* $\boxed{\text{trace}}$ on the CE	

Disp "", "PRESS STAT EDIT"

Run the program with R varying from 0 to 10 in steps of 1 to test it. Pressing the left/right arrow keys will move the cursor along the plotted points. Press $\boxed{\text{enter}}$ to exit from the graph.
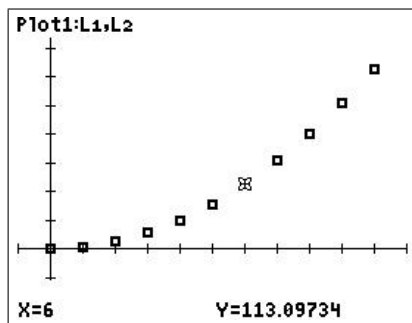
24.5.2   Graphing functions from a program

Graphing a function from a program is simple — the function has to be defined in $\boxed{\text{y=}}$, either before the program is run or from within the program.

Be careful in a program that the correct function is turned on — use FnOn and FnOff to do this. If you define a function within a program, it is automatically turned on.

There are several commands you can use in a program to display a graph. All of these will cause any functions *and any plots* turned on to be graphed.

- Trace — displays the graph with a cursor you can move along the functions/points plotted, just as you would using $\boxed{\text{trace}}$ manually. *Trace* pauses the program — press $\boxed{\text{enter}}$ to continue.

- Input — $\boxed{\text{prgm}}$ I/O $\boxed{1}$ with no arguments. *Input* displays the graph with a cursor that you can move around the screen. *Input* pauses the program — press $\boxed{\text{enter}}$ to continue.

- DispGraph — $\boxed{\text{prgm}}$ I/O $\boxed{4}$. *DispGraph* displays the graph without a cursor. *DispGraph* does not pause the program — if it is not the last statement in the program, you will have to follow it with a *Pause*.

- DrawF — $\boxed{\text{draw}}$ $\boxed{6}$. *DrawF* followed by a function definition either explicitly in terms of X (e.g. 3X+4) or one of $Y_1 - Y_0$, draws that function. Use *Input* or *Pause* to view the graph. A function drawn with *DrawF* cannot be Traced. An advantage of *DrawF* is that it allows successive graphs to be superimposed — see Exercise 1.

To continue our example and do some curve fitting, let's fit a quadratic function to our data points and graph it together with the data.

The data-fitting commands are in the $\boxed{\text{stat}}$ CALC menu. The *QuadReg* command determines the equation of the quadratic function that best fits the data (here a perfect fit), stores it in and turns on Y_1. The *Trace* command causes Y_1 to be graphed and pauses the program.

ClrHome

FnOff

PlotsOff

ClrDraw

$1 \rightarrow$ Xscl

$50 \rightarrow$ Yscl

ClrList L_1, L_2

$1 \rightarrow$ I

Disp "AREA OF A CIRCLE"

Input "FIRST R: ", F

Input "LAST R: ", L

Input "R STEP: ", S

For (R, F, L, S)

$R \rightarrow L_1(I)$

$\pi R^2 \rightarrow L_2(I)$

$I+1 \rightarrow I$

End

Plot1 (Scatter, L_1, L_2, $\square$)

ZoomStat

Trace   *view the data points*

QuadReg L_1, L_2, Y_1 boxed(stat) CALC boxed(5); Y_1 is boxed(vars) Y-VARS boxed(1) boxed(1)

Trace   *view the data points and the fitted curve*

Disp "", "PRESS STAT EDIT"

```
PROGRAM:CIRCLE
:πR²→L₂(I)
:I+1→I
:End
:Plot1(Scatter,L₁,L₂,□)
:ZoomStat
:Trace
:QuadReg L₁,L₂,Y₁
:Trace
:Disp "","PRESS STAT EDIT"
```

Run this program with R varying from 0 to 10 in steps of 1 and see if you get the expected fit to the data points.

Press ⟦enter⟧ after the points are plotted to see the fitted curve. Move along the curve with the left and right arrows.

Press ⟦enter⟧ to finish the program and look at Y₁ in ⟦y=⟧ to see the equation of the fitted curve.
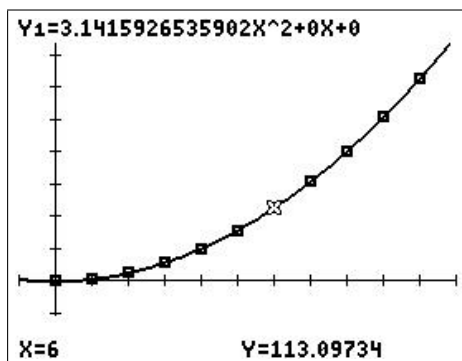
24.5.3   Putting text on a graph

Sometimes it is useful to annotate graphs so that the user can understand what is being plotted or be prompted for the next key stroke(s).

As an example, in the graph of our last two versions of the program, the calculator pauses in Trace. Possible keys to press are the arrow keys to move along or between the graphs and ⟦enter⟧ to finish the program. We use the *Text* command to indicate this on the graph.

The *Text* command is of the form *Text (row, column, "text" or variable)*. The coordinates *row* and *column* are pixel coordinates rather than the coordinates given by the window.

On a TI-84, *row* varies from 0 (top of screen) to 57 (bottom of screen), while *column* varies from 1 (left of screen) to 94 (right of screen).

On a TI-84CE, *row* varies from 0 (top of screen) to 150 (bottom of screen), while *column* varies from 0 (left of screen) to 255 (right of screen).

It is usually necessary to experiment with the pixel coordinates to get the text in the right place.

Text can also be used to display the value of a variable — put the variable name in the command without inverted commas.

Any combination of text and variable values can be put together in a single *Text* command — just put them one after the other, separated by commas. This is useful if you want to display the name of a parameter and its value, e.g. Text $(0, 2, "N=", N)$.

Note the use of a colon to join commands in the program below.

TI-84

ClrHome: ClrDraw

FnOff : PlotsOff

$1 \to$ Xscl: $50 \to$ Yscl

ClrList L₁, L₂

$1 \to$ I

Disp "AREA OF A CIRCLE"

Input "FIRST R: ", F

Input "LAST R: ", L

Input "R STEP: ", S

For (R, F, L, S)

$R \to L_1\,(I)$

$\pi R^2 \to L_2\,(I)$

$I + 1 \to I$

End

Plot1 (Scatter, L₁, L₂, □)

ZoomStat

Text(8, 13, "[ARROWS]␣␣␣[ENTER]")   Text is draw   0 ; [is 2nd   × ,] 2nd   −
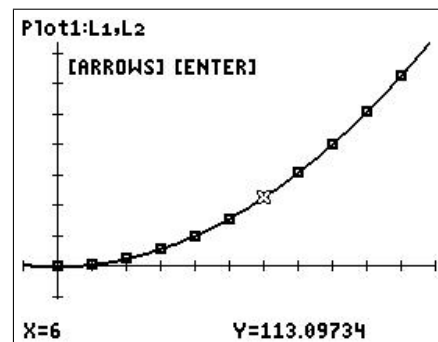
Trace

QuadReg L₁, L₂, Y₁

Text(8, 13, "[ARROWS]␣␣␣[ENTER]")

Trace

Disp "", "PRESS STAT EDIT"

TI-84CE: Text(8, 30, "[ARROWS]␣␣␣[ENTER]") .

```
PROGRAM:CIRCLE
:ZoomStat
:Text(8,30,"[ARROWS]  [ENT
ER]")
:Trace
:QuadReg L₁,L₂,Y₁
:Text(8,30,"[ARROWS]  [ENT
ER]")
:Trace
:Disp "","PRESS STAT EDIT"
```

```
Plot1:L₁,L₂
   [ARROWS] [ENTER]




X=6              Y=113.09734
```

24.6   Simple animation

Animation is achieved by creating the separate frames, storing them in the picture memories Pic1, Pic2, ..., Pic0, then recalling them one by one. Because of the way the CE recalls Pics, the following program gives a better result on the standard 84 than on the CE.

This program also illustrates the use of *getkey* to record a key press while a program (loop) is running — here we use the Q key to stop the program.[55]

Program ANIMATE

FnOff: PlotsOff: ClrDraw

AxesOff   2nd   zoom (format) ; *use the cursor and* enter *to select*

ZStandard   zoom   6   *standard axes*

ZSquare   zoom   5   *same scale on each axis: circles are circular*

Shade $(-\sqrt{(64-X^2)}, \sqrt{(64-X^2)})$ draw   *shade between the two semicircles*

Text $(1, 1, $ "Q:␣QUIT")   *on-screen prompt to tell user how to stop the program*

StorePic 1   draw STO *menu*   *store the first screen in Pic1*

ClrDraw

Circle $(0, 0, 8)$ draw *menu*   *circle centre* $(0, 0)$*, radius 8*

Text $(1, 1, $ "Q:␣QUIT")

StorePic 2   *store the second screen in Pic2*

Lbl 1   *start animation loop*

ClrDraw

RecallPic 1   draw STO *menu*   *recall the first screen*

For $(I, 1, 150)$ *delay loop to view screen*

End   replace these two lines with Wait 0.15 on a CE

ClrDraw   clear the graphics screen

RecallPic 2   *recall the second screen*

For $(I, 1, 120)$ *shorter delay loop to allow for the time taken for getkey test*

End   replace these two lines with Wait 0.12 on a CE

If getKey $= 74$ prgm I/O *menu*   getkey *detects if a key has been pressed*

Goto 2   *if the Q key (key 74) has been pressed, go to Label 2 (tidy up and quit)*

Goto 1   *if the Q key hasn't been pressed, go back to Label 1 (start of loop)*

Lbl 2   *tidy up*   **PTO**

---

[55] on Quit would stop it too, but there would be no 'tidying up' as here.

AxesOn

DelVar Pic1   ⎡prgm⎤ (scroll up)   ⎡vars⎤ ⎡4⎤ ⎡1⎤

DelVar Pic2

ClrHome

24.7   Extensions

Extension 1

Follows on from the example on page 216. We could also use $R < 0$ as a way to stop the program, rather than using ⎡on⎤ Quit.

Note the message telling the user this after the main heading.

ClrHome

Disp "AREA OF A CIRCLE" ,"R<0␣TO␣STOP"

Lbl 1

Input "RADIUS: ", R

If R < 0

Stop   ⎡prgm⎤ ⎡F⎤ *or scroll up and select*

$\pi R^2 \to A$

Disp "AREA", A

Goto 1

Extension 2

Follows on from the example on page 216. For a simpler, more elegant check on the input, we can use a *Repeat* loop.

ClrHome

Disp "AREA OF A CIRCLE"

Lbl 1

Repeat R ≥ 0   ⎡prgm⎤ ⎡6⎤   *repeat down to End until the condition $R \geq 0$ is true*

Input "RADIUS: ", R

End

$\pi R^2 \to A$

Disp "AREA", A

Goto 1

Extension 3

Here we introduce a custom *Menu* in a program that calculates the area of a triangle, rectangle or circle. Only an outline of the program is given here — details are to be completed in Exercise 2.

Menu is in the $\boxed{\text{prgm}}$ CTL menu.

Output, in the $\boxed{\text{prgm}}$ I/O menu, is like *Text*, but writes on the Home screen rather than the graphics screen.

Note the use of a suggestive letter for each label.

Program AREA

Lbl M   *main menu*

ClrHome

Menu ("␣AREA␣OF␣A . . .␣", "TRIANGLE", T, "RECTANGLE", R,
                                                "CIRCLE", C, "QUIT", Q)

Lbl T   *triangle*

Insert heading, prompts, inputs and calculations to calculate the area of a triangle. Store the area in A.

Goto D

Lbl R   *rectangle*

Insert heading, prompts, inputs and calculations to calculate area of a rectangle. Store the area in A.

Goto D

Lbl C   *circle*

Insert heading, prompts, inputs and calculations to calculate area of a circle. Store the area in A.

Lbl D   *display area A*

Disp "AREA", A

Output (8, 10, "[ENTER]") *   *tells the user to press* $\boxed{\text{enter}}$ *to continue*

Pause

Goto M

Lbl Q   *quit*

ClrHome

* Output (10, 20, "[ENTER]") on a CE

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Extension 4

A program to move a cursor around the screen according to which arrow key is pressed.

StoreGDB 0   *stores current graphics settings and functions*
                       for later restoration: $\boxed{\text{draw}}$ STO *menu*

FnOff: PlotsOff: ClrDraw

AxesOff   *turns off the axes:* $\boxed{\text{format}}$ *menu*

CoordOff   *turns off the display of the cursor coordinates:* $\boxed{\text{format}}$ *menu*

$28 \to$ A   *vertical coordinate of initial cursor position*   **CE**: $75 \to$ A

$45 \to$ B   *horizontal coordinate of initial cursor position*   **CE**: $127 \to$ B

$2 \to$ S   *distance the cursor will move at each step*   **CE**: $5 \to$ S

Text(A,B, "∗")   *draw the cursor at the location* (A,B): ∗ *is* $\boxed{\times}$

Repeat K $= 74$ *outer Repeat loop: loop until the Q key is pressed*

Repeat K $\neq 0$ *inner Repeat loop: determines when a key has been pressed*

getKey $\to$ K   *store the key coordinate in K*

End   *end of inner Repeat loop*

Text(A,B,"␣␣␣␣␣")   *erase the cursor at previous position*   **CE**: $4_\sqcup$ (spaces)

If K $= 24$: B $- $ S $\to$ B   *subtract S from B if key 24 (left arrow) has been pressed*

If K $= 25$: A $- $ S $\to$ A   *subtract S from A if key 25 (up arrow) has been pressed*

If K $= 26$: B $+ $ S $\to$ B   *add S to B if key 26 (right arrow) has been pressed*

If K $= 34$: A $+ $ S $\to$ A   *add S to A if key 34 (down arrow) has been pressed*

Text(A,B,"∗")   *draw the cursor at the current location given by A and B*

Game commands go here, for example, to test whether the cursor is now in some previously specified position where something is to be gained or lost, or has reached the edge of the screen.

End   *end of outer Repeat loop*

ClrDraw   *tidy up and stop*

RecallGDB 0   *restores the original graphics settings:* $\boxed{\text{draw}}$ STO *menu*

DelVar GDB0   *delete GDB0:* $\boxed{\text{prgm}}$ CTRL *menu;* $\boxed{\text{vars}}$ GDB *menu*

ClrHome

Disp "GAME␣OVER"

24.8   Exercises

Possible solutions in Section 24.9.

If these exercises seem too hard, think up a suitable calculation and program it.

Exercise 1: Area of a geometric shape

Write a user-friendly program that calculates the area of a triangle, rectangle or circle, given appropriate inputs. *Hint*: see Extension 3.

Exercise 2: Family of curves

Assume that $Y_1 = AX^2 + B$ and that a suitable window has been set.

Your program called FAMILY should

- clear the Home screen

- turn off all functions

- turn off stat plots

- clear the graphics screen

- display a header on the Home screen

- start a loop (*Lbl*)

- input values for A and B

- draw the graph (DrawF Y_1)

- pause to allow you to look at the graph

- return for another set of values for A and B (end of loop)

FAMILY allows you to see the effect on the function of changing the parameters A and B.

Use a window of $[-5, 5, 1] \times [-10, 10, 2]$. Try A = 1, −1, 2, 0.5 with B = 0. Use $\boxed{\text{on}}$ Quit to stop the program and rerun with A = 1 and B = 0, ±2, ±5.

You can put in Y_1 any function containing two parameters A and B, and plot the corresponding family of curves using FAMILY. Try, for example, $Y_1 = A(X-B)^2$.

Exercise 3: Roots of a quadratic equation

Find the real roots of the quadratic equation $Ax^2 + Bx + C = 0$, where A, B and C are inputs. Display a message if the roots are complex (later you might like to extend the program to complex roots). Finally plot the quadratic: specify the x range in the program (Xmin, Xmax) — the roots should help here — and use ZoomFit, which calculates a suitable y range and plots the graph.

To put the quadratic function in Y_1: "$AX^2 + BX + C$" $\rightarrow Y_1$. This works both within a program and from the Home screen (although entering the function using $\boxed{\text{y=}}$ is clearly an easier option in the second case).

24.9   Possible solutions to the exercises

Exercise 1: Area of a geometric shape

Program AREA

Lbl M
ClrHome
Menu ("␣AREA␣OF␣A...␣","TRIANGLE",T,
       "RECTANGLE",R,"CIRCLE",C,"QUIT",Q)
Lbl T
Disp "␣␣␣␣TRIANGLE"   **CE:** $9_{␣}$ (spaces)
Input "BASE:␣", B
Input "HEIGHT:␣", H
0.5BH $\rightarrow$ A
Goto D
Lbl R
Disp "␣␣␣␣RECTANGLE"   **CE:** $8_{␣}$
Input "LENGTH:␣", L
Input "WIDTH:␣", W
LW $\rightarrow$ A
Goto D
Lbl C
Disp "␣␣␣␣␣CIRCLE"   **CE:** $10_{␣}$
Input "RADIUS:␣", R
$\pi R^2 \rightarrow$ A
Lbl D
Disp "AREA", A
Output $(8, 10, $ "[ENTER]")   **CE:** Output $(10, 20, $ "[ENTER]")
Pause
Goto M
Lbl Q
ClrHome

Exercise 2: Family of curves

Program FAMILY

ClrHome
ClrDraw
FnOff
PlotsOff
Disp "FAMILY␣PROGRAM", "PARAMETERS␣A,B"
Lbl 1
Prompt A, B
DrawF Y₁
Input
Goto 1

Exercise 3: Roots of a quadratic equation

Program QUAD

ClrHome
ClrDraw
FnOff
PlotsOff
"AX^2+BX+C" $\rightarrow$ Y₁
Disp "␣␣QUADRATIC␣EQN", "␣␣␣AX^2 + BX + C = 0"   **CE:** 6 ␣; 8 ␣
Prompt A, B, C
B^2 − 4AC $\rightarrow$ D
If D < 0
Then
Disp "COMPLEX␣ROOTS"
Stop
End
(−B + $\sqrt{}$(D))/(2A) $\rightarrow$ R
(−B − $\sqrt{}$(D))/(2A) $\rightarrow$ S
Disp "ROOTS", R, S
Pause
If R < S
Then
R−2 $\rightarrow$ Xmin
S+2 $\rightarrow$ Xmax
Else
S−2 $\rightarrow$ Xmin
R+2 $\rightarrow$ Xmax
End
0 $\rightarrow$ Xscl
ZoomFit

Once the program has finished, you can press $\boxed{\text{trace}}$ or change the window and re-graph.

24.10   Keystrokes used in the programs

□   2nd statplot (y= key) MARK 1

[2nd ×

]   2nd −

:   alpha .

"   alpha +

=   2nd test (math key) 1

<   2nd test (math key) 5

≤   2nd test (math key) 6

π   2nd ∧

␣ (space)   alpha 0

√(2nd x²

→   sto

AxesOff   2nd format (zoom key) — use the cursor and enter to select

Circle(2nd draw (prgm key) 9

ClrDraw   2nd draw (prgm key) 1

ClrHome   prgm I/O 8

ClrList   stat 4

DelVar   prgm alpha G or scroll up and select

Disp   prgm I/O 3

DrawF   2nd draw (prgm key) 6

Else   prgm 3

End   prgm 7

FnOff   vars Y-VARS 4 2

For(prgm 4

getKey   prgm I/O 7

Goto   prgm 0

If   prgm 1

Input   prgm I/O 1

L₁   2nd 1

L₂   2nd 2

Lbl	prgm 9
Menu(	prgm alpha C or scroll up and select
Output(	prgm I/O 6
Pause	prgm 8
PlotsOff	2nd statplot (y= key) 4
Prompt	prgm I/O 2
Plot1(	2nd statplot (y= key) 1
QuadReg	stat CALC 5
RecallPic	2nd draw (prgm key) STO 2
Repeat	prgm 6
Scatter	2nd statplot (y= key) TYPE 1
SetUpEditor	stat 5
Shade(	2nd draw (prgm key) 7
space	alpha 0
Stop	prgm F or scroll up and select
StorePic	2nd draw (prgm key) STO 1
Text(	2nd draw (prgm key) 0
Then	prgm 2
Xscl	vars 1 3
Y_1	vars Y-VARS 1 1 or alpha f4 1
Yscl	vars 1 6
ZoomFit	zoom 0
ZoomStat	zoom 9
ZSquare	zoom 5
ZStandard	zoom 6