Mathematics on a Casio 9860/CG20/CG50

Volume 3: Advanced

Chapters 16 – 21

Peter McIntyre

School of Science

UNSW Canberra

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Contents

Mathematics on a Casio 9860/CG20/CG50

Volume 1 of this book contains the basic topics: *Graphics Calculators and Mathematics*; *Getting Started*; *Coordinate Geometry*; *Inequalities and Linear Programming*; *Fitting Curves to Data – Calculator Functions*; *Population Modelling 1 – Exponential Growth*; *Financial Mathematics 1 – Compound Interest*; and *Probability and Statistics 1 – Descriptive Statistics.*

Volume 1 Supplement: Activities for Years 9 and 10 contains extra activities for *Coordinate Geometry* and *Probability and Statistics 1.*

Volume 2 of this book contains topics directly relevant to Calculus and its applications, although *Functions and their Graphs* is of more general relevance and also contains details of how to capture screenshots from your calculator, crop them if desired and insert them into documents. The topics in Volume 2 are: *Functions and their Graphs*; *Graph and Calculus Operations*; *Numerical Integration*; *Taylor Series*; *Differential Equations*; *Population Modelling 2 – Logistic and Epidemic Models*; and *Programming and Program Information.*

Program Information gives a list of the programs cited in Volumes 2 and 3 of the book, and full information on copying and using these programs.

Calculator versions

The Casio graphics calculator models CG20 AU and CG50 AU are basically the same as the 9860 used here (except for a higher-resolution colour screen). This is probably true of all Casio graphics calculators one level below the ClassPad. There may be minor differences in how the screen looks and in the menus but they all do the same calculations. There are some extras on the CG50 (e.g. colour) but these are not used here.

Calculations, screenshots and figures were done on a Casio fx-9860G AU PLUS. The calculator programs were also written on this calculator, and converted to run on the other calculators. The programs are available at *www.canberramaths.org.au* under *Resources.*

Reference

Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS by Barry Kissane.[1] This book is a real bible on everything a graphics calculator can do and how to do it. A must-have for teachers of Years 10 – 12 using Casio calculators. Still very relevant but sadly now hard to find. Happily, a new electronic version is on the horizon.

Meanwhile, a shorter version (no Finance) is in *Learning Mathematics with Graphics Calculators* by Barry Kissane and Marian Kemp, available at *www.canberramaths.org.au Resources Graphics Calculators* in the *Articles* folder.

[1] *Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS* by Barry Kissane. The Mathematical Association of Western Australia, 2003, ISBN 1 876583 24 X.

16 Sequences and Series

16.1 Introduction

These notes provide a comprehensive review of generating, displaying and graphing sequences and series on a Casio 9860/CG20/CG50 graphics calculator.[2] An arithmetic progression, a geometric progression and the Fibonacci numbers are used as examples. A number of questions (with solutions) illustrate the use of the calculator. Finally there are two topics that could be used as a basis for group investigation or a small project.

The calculators can generate sequences, sum series, and display sequence terms in a table or graph. However, we should first ask whether it makes sense to use a graphics calculator at all for sequences and series.

Certainly the first few lessons on sequences should be pencil and paper, until some of the concepts and calculations are understood, although a class activity such as that on page 7 can add variety to the early learning stages. However, having to work out terms of a sequence or series by hand eventually becomes tedious, especially those terms that are not very simple and require a calculator anyway. This becomes an impediment to further learning and exploration.

The calculator automates the process of calculating terms in a sequence or series *once it is given an appropriate definition*. It is in finding an appropriate definition that most of the thought goes — the calculator can't do this. With automatic calculation comes the ability to explore particular sequences and series, to conjecture and test, and to look at ideas such as the convergence of an infinite sequence or series.[3]

Some of the questions and investigations in Sections 16.4 and 16.5 demonstrate this extra capability when using a graphics calculator.

At this stage, it is perhaps useful for the reader to review the use of the calculator graphics by graphing, say $y=x^3$ ($\boxed{\text{MENU}}$ $\boxed{5}$) and generating a table of function values ($\boxed{\text{MENU}}$ $\boxed{7}$).

- Set a suitable $\boxed{\text{V-Wind}}$ manually first and graph $\text{Y1}=\text{X}^3$.
- Then use $\boxed{\text{Zoom}}$ $\boxed{\text{Auto}}$ to carry out the process of finding a suitable Y scale automatically.
- Review line styles ($\boxed{\text{STYL}}$), particularly solid and dotted lines.
- Generate a table of values of Y1.

[2]See also *Financial Mathematics 1* in Volume 1 for the use of sequences in simple financial calculations.
[3]These are not esoteric beasts — the humble AP and GP continue on indefinitely.

16.2 Sequences

A sequence is an ordered set of numbers, usually with the numbers or terms in the sequence determined by some sort of formula.[4]

For example

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \ldots \qquad 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots \qquad 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots$$

are sequences for which we can find a formula to determine each term.

In the usual notation, a general sequence is written as

$$a_1, a_2, \ldots, a_n, \ldots,$$

where each term a_1, a_2, $a_3, \ldots$ is a number. The subscript gives the position of the term in the sequence.

There are two ways to give a formula for each term.

- **Recursively:** write the nth term in terms of the previous term or terms. Here we also need to know a value for the first term (or the first few terms) in the sequence.

  Examples

  1. $a_2 = a_1+2$, $a_3 = a_2+2$, $\ldots$ or in general, $a_{n+1} = a_n+2$.
  With $a_1 = 1$, this recursive formula gives the first sequence above, an arithmetic progression (AP).

  2. $a_{n+1} = a_n/2$: $a_1 = 1$ gives the second sequence above, a geometric progression (GP).

  3. $a_{n+2} = a_{n+1} + a_n$: $a_1 = 1$, $a_2 = 1$ gives the third sequence above, the famous Fibonacci numbers.

- **Explicitly:** specify the nth term as a function of n, where n takes integer values.

  Examples

  1. $a_n = 2n-1$, $n=1, 2, 3, \ldots$, which again gives the first sequence above.

  2. $a_n = 0.5^{n-1}$, $n=1, 2, 3, \ldots$, which again gives the second sequence above.

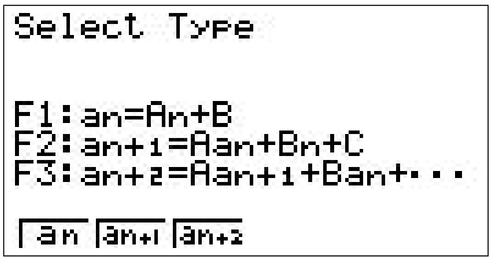
  The Fibonacci sequence can also be defined explicitly — see Section 16.5.

There are basically two ways on the 9860 to generate and display terms of a sequence — using the RECUR mode and using LIST commands in RUN mode. Which method you use will depend on how you teach the topic. Here we'll look at both, using the above examples to illustrate the methods. There are questions to practise on in Section 16.4.

[4]although we can have sequences of random numbers.

16.2.1 Using RECUR(SION) mode

Sequences can be defined either recursively or explicitly, displayed as a table and graphed in RECUR mode. Press MENU 8 or move the cursor to the RECUR icon and press EXE.



If your screen does not look like the one above left, press F3 (TYPE) and F2 (above right).

A. Arithmetic progression

In an arithmetic progression, there is a constant difference between successive terms. The recursive definition is $a_{n+1}=a_n+d$, where d is a constant called the common difference.

Example: $a_{n+1} = a_n + 2$, with $a_1=1$.

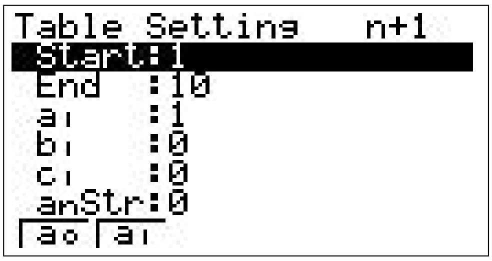
Set this sequence up on your calculator as shown below. Press F4 to show the keys for n and a_n. Press EXE to store the definition. The first term a_1 is input when we set up the table below.

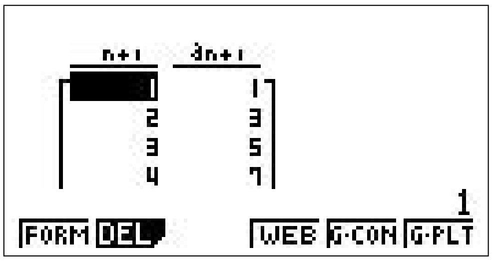
Displaying the sequence

Press F5 (SET) and set your screen up like the one below left, pressing EXE after each entry. F1 and F2 allow you to choose whether to start the sequence at a_0 or a_1.

Press EXIT when you have finished and F6 (TABL) to display the table.

F1 (FORM) takes you back to the formula for a_{n+1} if you need to change it.





Graphing the sequence

There are several ways to graph sequences, selected by F4, F5 or F6 under the table.

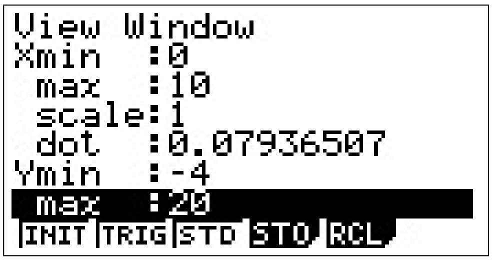
- F4 (WEB) plots a cobweb plot, a_{n+1} versus a_n. We'll look at this in Section 16.5.2.
- F5 (G·CON) plots a so-called time plot, a_n versus n, connecting the points.
- F6 (G·PLT) also plots a_n versus n, but doesn't connect the points. This is what we will use here after setting the View Window.

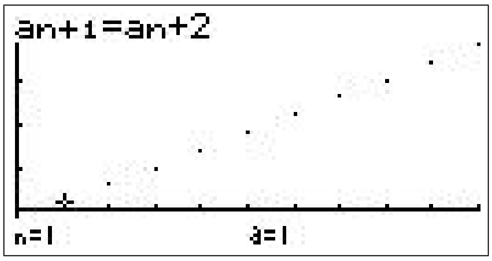
To set the View Window, press SHIFT F3. In our time plot, n is plotted along the X axis and a_n along the Y axis.

Set up your View Window as shown below left; the value of -4 for Ymin is to allow space for the coordinates at the bottom of the screen when using Trace. Yscale $= 5$.

Press EXIT, F6 (TABL) and F6 again (G·PLT) to plot the sequence.

Press F1 (Trace) and use the arrow keys to move along the points of the sequence.





Defining the AP explicitly

The nth term for a general AP is $a_n = a_1+(n-1)d$, or using the calculator notation, $a_{n+1} = a_1+nd$, where a_1 is the first term and d is the common difference.

For our example, we have $a_1 = 1$ and $d = 2$, so that

$$a_{n+1} = 1+2n.$$

We can compare the two definitions of the sequence by putting the nth term in sequence b_n, as shown below.[5]

Press SET and set $b_1 = 1$.[6] Press EXIT, then TABL and compare the sequences a_n and b_n. You should see that the two sequences are identical. If the first terms are different, go back to SET and make sure $b_1 = 1$.

[5] We could have chosen F1 from the TYPE menu to enter the formula for a_n rather than a_{n+1}, but we couldn't then compare the two sequence definitions.

[6] The table value comes from this number, not from setting $n = 0$ in the formula for b_{n+1}!

B. Geometric progression

In a geometric progression, each term is a constant multiple of the previous term. In calculator notation, the recursive definition is

$$a_{n+1} = ra_n,$$

where r is a constant called the common ratio or common multiplier.

Exercise: Display a table and graph the first 10 terms of the geometric sequence $a_{n+1} = 0.5a_n$, with $a_1 = 1$.

The nth term of a geometric progression is given *explicitly* by

$$a_{n+1} = cr^n,$$

where c is a constant and r the common multiplier.

For the sequence in the exercise above, the nth term is given by

$$a_{n+1} = 0.5^n.$$

Exercise: Put this explicit definition in sequence b_{n+1} and compare with the recursive definition. Make sure the value for b_1 in $\boxed{\text{SET}}$ is correct.

C. Fibonacci sequence

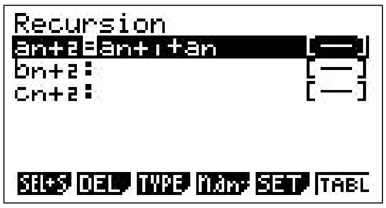
The Fibonacci sequence is defined, in calculator terms, by

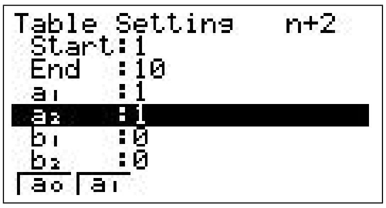
$$a_{n+2} = a_{n+1} + a_n,$$

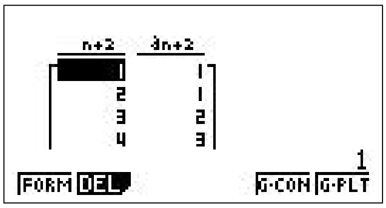
with $a_1 = 1$ and $a_2 = 1$.

Press $\boxed{\text{TYPE}}$ and $\boxed{\text{F3}}$ to select the appropriate form. Enter the Fibonacci formula.

Set the two starting values $a_1 = 1$ and $a_2 = 1$ in $\boxed{\text{SET}}$. Generate a table of values.







16.2.2 Using LIST commands in RUN mode

Press $\boxed{\text{MENU}}$ $\boxed{\text{1}}$ to return to RUN mode. Press $\boxed{\text{OPTN}}$ to bring up menus at the bottom of the screen, the first of which is $\boxed{\text{LIST}}$. Select this.

The *Seq* command $\left(\boxed{\text{F5}}\right)$ in the LIST menu generates a sequence (list) specified by an explicit general term. The syntax is

Seq (general term, variable, start, end, step),

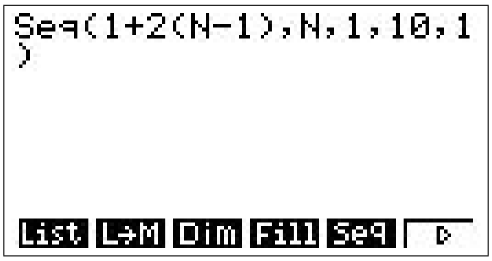
where variable can be any letter.[7] The total number of terms must be less than 999.

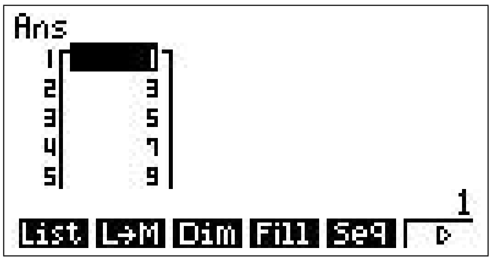
[7]The variable in a *Seq* command doesn't have to take integer values, e.g. Seq (sin(X), X, 0, 1, 0.05) generates a sequence of values of the sine function.

A. Arithmetic progression

In the following sections, there is an implicit $\boxed{\text{EXE}}$ after each command.

Seq (1+2(N−1), N, 1, 10, 1)





Scroll down/up the sequence using the arrow keys. Press $\boxed{\text{EXIT}}$ to return to the RUN screen.

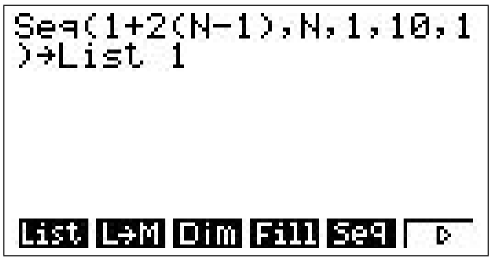
Storing to a list

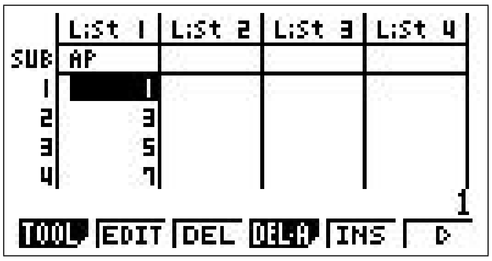
For analysis and graphing, it is often convenient to store a sequence in a named list, which is then stored in memory. The 9860 has six built-in lists: List 1 – List 6.

To store a sequence to List 1:

Seq (2N−1, N, 1, 10, 1) → List 1,

where *List* is $\boxed{\text{F1}}$ in the LIST menu.





Now press $\boxed{\text{MENU}}$ $\boxed{\text{2}}$ (STAT) to enter the List Editor. You can scroll down, edit and carry out other operations here. I've given List 1 the subtitle AP.

B. Geometric progression

Seq (0.5^(N−1), N, 1, 10, 1)

Exercise: On the RUN screen, store 1 in A, 0.5 in R, type in seq (AR^(N−1), N, 1, 10, 1) and press $\boxed{\text{EXE}}$.

Then use the left or right arrow to recall the previous command and edit it to produce terms 11 to 20 in the sequence.

PTO

Class Activity

Give students one calculator between two. Use the OHP screen or calculator emulator if you have one or write on the board.

Set up the OHP calculator to evaluate sequences of the form $a_n = An+B$ ($\boxed{\text{F1}}$ in $\boxed{\text{TYPE}}$ in RECUR mode).

On the OHP calculator, with the OHP turned off, set $a_n = 2n+1$. Set $\boxed{\text{SET}}$ to generate 10 terms in the sequence, press $\boxed{\text{EXIT}}$ then $\boxed{\text{TABL}}$. Turn the OHP on and ask the students: *What's the rule?*

When they have worked it out as a class, show them how the rule is entered and $\boxed{\text{SET}}$ is set. Have them enter the rule and generate the table. Write on the board that the rules to follow are all of the form $\_\_n + \_\_$.

Now, again with the OHP off, enter a different rule (remember to press $\boxed{\text{EXE}}$) and show them the table. Ask them to make the table on their calculators the same. Remember $\boxed{\text{FORM}}$ and $\boxed{\text{TABL}}$ are the two keys to move between table and formula.

Give them various rules to find, moving eventually to negative numbers for the coefficients. Ask them to summarise their findings regarding the two numbers in the rules.

16.3 Series

A series is the sum of the terms in a sequence, that is a sequence with + signs between successive terms. However, we also say 'the sum of a series' to distinguish the sequence with + signs from the actual value when we carry out the additions. To calculate the sum of a series, we need to find the terms in the corresponding sequence, then add them up.

If a series has a finite number of terms, we just add them up to give the sum. All of what we do below can be applied to finite series. However, the more interesting series are infinite — we can't calculate the sum for these by carrying out the additions because there is an infinite number of them.[8] However, we can work out the sum of a finite number of terms, called a partial sum — the nth partial sum is the sum of the first n terms of the series. The behaviour of the partial sums as n gets bigger tells us something about the convergence of the series — whether the sum may be a finite number or infinite. The sum of an infinite series is defined as the limit of its partial sums as $n \to \infty$.

Both RECUR mode and the LIST commands in RUN mode can be used to sum a series, but only RECUR mode works for terms defined recursively.

16.3.1 Using RECUR(SION) mode

The nth partial sum of a series S_n is the sum of the previous $n-1$ terms, S_{n-1}, plus the nth term; partial sums can therefore be defined recursively. In symbols,

$$S_n = S_{n-1} + a_n \qquad S_1 = a_1.$$

We'd like to generate the sequence of partial sums

$$S_2,\ S_3,\ S_4,\ S_5,\ \ldots.$$

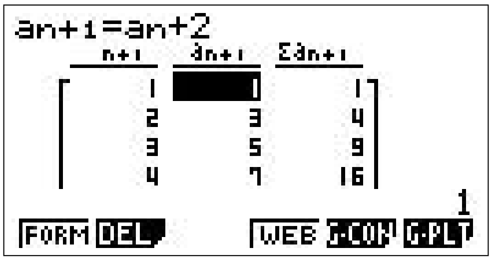
[8]There are algebraic methods for summing some infinite series, one of the triumphs of Calculus.

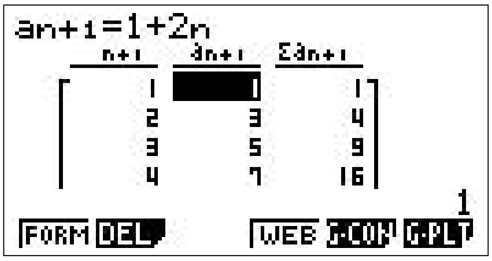
Enter RECUR mode (MENU 8).

The calculator will display the partial sums automatically: press SHIFT MENU and turn on Σ Display. Press EXIT to return to the formula-entry screen.

A. Arithmetic progression

Displaying the table for the AP with general term $a_{n+1} = a_n + 2$ or $a_{n+1} = 1 + 2n$ now also displays the partial sums.





B. Geometric progression

Exercise: Set up a table of partial sums for our GP with $a_{n+1} = 0.5a_n$, $a_1 = 1$, or $a_{n+1} = 0.5^n$. To what value does the (infinite) series appear to converge. Confirm your answer algebraically.

16.3.2 Using LIST commands in RUN mode

The *Sum* command in the LIST menu (press F6 twice) sums a sequence (list). The syntax is

Sum *list*,

where *list* can be a *Seq* command.

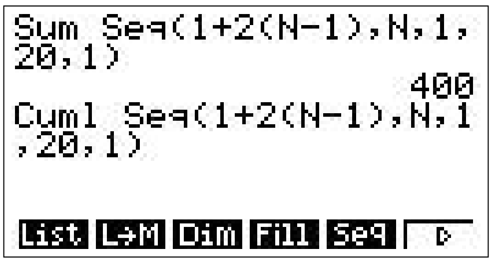
The *Cuml* command in the LIST menu generates a sequence of partial sums. The syntax is

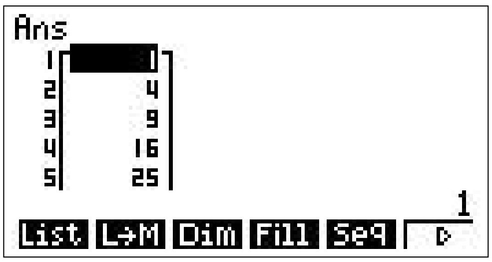
Cuml *list*.

A. Arithmetic progression

Sum Seq (1+2(N−1), N, 1, 20, 1) finds the sum of the first 20 terms in our AP.

Cuml Seq (1+2(N−1), N, 1, 20, 1) generates the first 20 partial sums of our AP: the ith entry is the sum of the first i terms.





The sum of an infinite series is defined as the limit as $n \to \infty$ of the nth partial sum. Scrolling down a list of cumulative sums like the one above can give an idea of what that limit might be. Here, of course, there is no limit — the nth partial sum $\to \infty$ as $n \to \infty$.

B. Geometric progression

Sum Seq (0.5^(N−1), N, 1, 20, 1)

Cuml Seq (0.5^(N−1), N, 1, 20, 1)

Exercise: Use List commands to make a conjecture on the sum of the GP

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots.$$

You might want to recall the command you entered (left or right arrow) and increase the end value of N to be (almost) sure of the answer. We can't *prove* this is the answer numerically, but we can be reasonably sure it is the answer.

16.4 Questions on sequences and series

1. Generate a sequence of the cubes of first ten positive integers using a LIST command. Store the sequence in a list.

Use the *Sum* command to evaluate $1^3 + 2^3 + \cdots + 10^3$.

2. Find the sum of the first ten terms of the series $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots$.

3. **(a)** Use a single command to find the sum of the first 100 positive integers. Store the answer in memory S for use in (c).

(b) Edit your command in (a) (left or right arrow) to find the sum of the *cubes* of the first 100 positive integers.

(c) Which is bigger: the sum of the cubes of the first 100 positive integers or the cube of the sum of the first 100 positive integers?

4. The half-life of a certain radioactive substance is 1 week. This means that, of the amount present at a particular time, only half will be left 1 week later. Suppose 1000 grams of the substance exists today, the beginning of Week 1.

(a) Write down the amount left at the beginning of Week 2, Week 3, . . . , Week 10.

(b) Determine an infinite geometric sequence (recursive or explicit) that is a model of the amount of the substance at the beginning of Week n, where $n = 1, 2, 3, \dots$. What is the common ratio of this sequence?

(c) When will there be only 0.005 grams remaining?

(d) How much of the substance was there a week ago (beginning of Week 0)?

(e) When will the substance be reduced to nothing *according to this model*?

5. The height of a particular fast-growing plant increases at the rate of 2.5% per month. Assume the plant is 30 cm high today and that it dies after 12 months.

(a) Determine a finite geometric sequence that is a model of the height of the plant after n months. Write out all the terms of the sequence. What is the common ratio?

(b) How long would the plant have to live to double in height??

6. Sue had \$1250 in a savings account 3 years ago. What will be the value of her account 2 years from now, assuming no deposits or withdrawals are made and the account earns 6.5% interest compounded annually?

7. Frank has \$12,876 in a savings account today. He made no deposits or withdrawals during the last 6 years. What was the value of his account 6 years ago? Assume that the account earned 5.75% interest compounded monthly.

8. Generating sequences recursively is equivalent to another mathematical process called iteration, in which we do the same operation over and over. Try some of the following sequences/iterations.

Generate some terms in each sequence. What happens to a_n in each sequence as n becomes large?

(a) $a_{n+1} = a_n^2$. Try $a_1 < -1$; $a_1 = \pm 1$; $-1 < a_1 < 1$; $a_1 > 1$.

(b) $a_{n+1} = a_n^2 - 1$ for $-2 < a_1 < 2$.
Note that $\frac{1}{2}(1+\sqrt{5}) = 1.618\ldots$ and $\frac{1}{2}(1-\sqrt{5}) = 0.618\ldots$.
What's special about these values for a_1?

(c) $a_{n+1} = \sqrt{a_n}$.

(d) $a_{n+1} = \cos(a_n)$.

(e) $a_{n+1} = \tan(a_n)$.

9. What value does the sequence $\left(1 + \dfrac{1}{n}\right)^n$ approach as n gets larger and larger?

Hint: Use a sequence command with a step of at least 1000 and at least 50 terms.

10. What is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$?

factorial ! is in the $\boxed{\text{OPTN}}$ $\boxed{\text{PRB}}$ menu.

The *Cuml Seq* combination works well here, as we don't need too many terms to see the convergence of the partial sums. Note that the series starts at $n = 0$.

What is $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$? $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, where x is any number?

11. Generate a table of values for each of the following recursive sequences. What happens to the terms as n gets larger? Do you recognise the number?

Hint: The answers for the first two sequences are square roots of particular numbers. Can you make and test a conjecture here? Write down a sequence whose limit is 2; $\sqrt{5}$.

The answers for the last two sequences should be obvious. Can you make and test a conjecture here? Write down a sequence whose limit is 4; 5.

(a) $a_1 = 1 \quad a_{n+1} = \dfrac{1}{1 + a_n} + 1$ **(b)** $b_1 = 1 \quad b_{n+1} = \dfrac{2}{1 + b_n} + 1$

(c) $s_1 = 1 \quad s_{n+1} = \dfrac{1}{2}\left(s_n + \dfrac{4}{s_n}\right)$ **(d)** $t_1 = 1 \quad t_{n+1} = \dfrac{1}{2}\left(t_n + \dfrac{9}{t_n}\right)$

12. A rubber ball is dropped from a height of 8 metres and returns to three-quarters of its previous height on each bounce.

(a) How high does the ball bounce after hitting the floor for the third time? for the tenth time?

(b) How far has the ball travelled vertically when it hits the floor for the fourth time? for the twentieth time?

13. According to legend, a man who had pleased the Persian king asked for the following reward. The man was to receive a single grain of wheat for the first square of a chess-board, two grains for the second square, four grains for the third square, and so on, doubling the amount for each square up to the 64th square. How many grains would he receive in all. (Fortunately the king had a good sense of humour.)

14. Find the sum of all integral multiples of 6 between

(a) 10 and 100.

(b) between 1 and 10,000.

PTO

16.5 Further investigations

16.5.1 The Fibonacci sequence and the Golden Ratio

Here is one illustration of the many interesting properties of the Fibonacci sequence. For more, see *www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html*.

1. Generate a table of values of the sequence (be very careful with brackets)

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Do you recognise this sequence?

2. One of the many interesting things about the Fibonacci sequence becomes apparent when we look at ratios of successive terms. Using RECUR mode, set

$$a_{n+2} = a_{n+1} + a_n \qquad a_1 = a_2 = 1$$

to generate the Fibonacci sequence. Set

$$b_{n+2} = a_{n+1}/a_n$$

to calculate the ratios of successive terms of the Fibonacci sequence.

Generate a table of values of 30 terms.

Scroll down and look at the b_{n+2} column containing the ratios of successive terms of a_{n+2}. Does this sequence appear to be approaching a particular value? What value? Find this value accurate to 6 decimal places.

Now find a value for the Golden Ratio or Golden Section, $\frac{1+\sqrt{5}}{2}$.

What conclusion do you reach?

Solutions on page 22.

PTO

16.5.2 The logistic sequence

The logistic sequence or logistic map has become famous because it is one of the simplest sequences that exhibits chaotic behaviour. It also turns up in a number of areas such as population modelling.[9]

The logistic sequence is defined by

$$a_{n+1} = Aa_n(1-a_n),$$

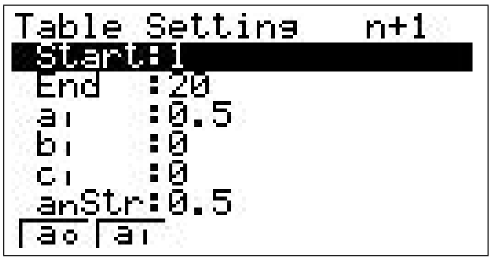
where A is a constant.

The following figures show the set up for *time plots*, a_n vs n, on the left and *cobweb plots*, a_{n+1} vs a_n, on the right. In both sets of plots $a_1 = 0.5$; for web plots, we must set anStr in $\boxed{\text{SET}}$ (table window) to the same value. Put the value for A in the formula for a_{n+1}.

Set the V-Window $\left(\boxed{\text{SHIFT}}\ \boxed{\text{F3}}\right)$ appropriate to the type of graph (see over the page), press $\boxed{\text{EXIT}}$ then $\boxed{\text{TABL}}$ and either $\boxed{\text{F5}}$ or $\boxed{\text{F4}}$ to generate the graph.

For a time plot, press $\boxed{\text{Trace}}$ so you can trace the graph.

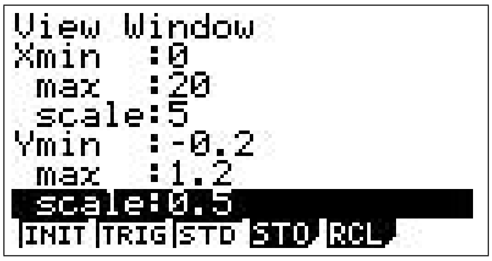
For a cobweb plot, press $\boxed{\text{TABL}}$ $\boxed{\text{WEB}}$, and keep pressing $\boxed{\text{EXE}}$ to generate the plot. If nothing happens, you may not have set anStr in $\boxed{\text{SET}}$ to the correct value. In a cobweb plot, the curves $y=Ax(1-x)$ and $y=x$ are also plotted. The cobweb lines move between these two curves.



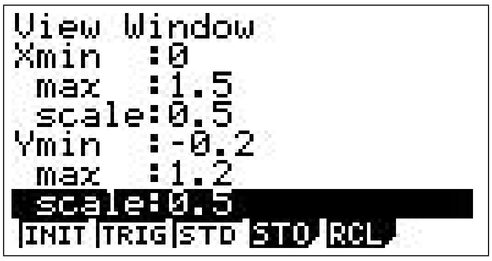
PTO

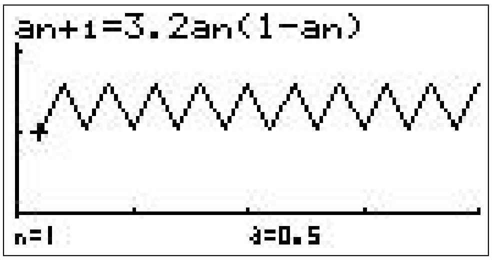
[9]See *Population Modelling 2* in Volume 2 for examples.

TIME | **WEB**

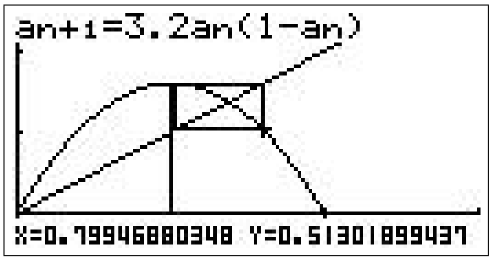


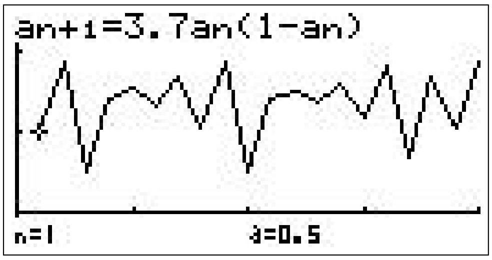
V-Window



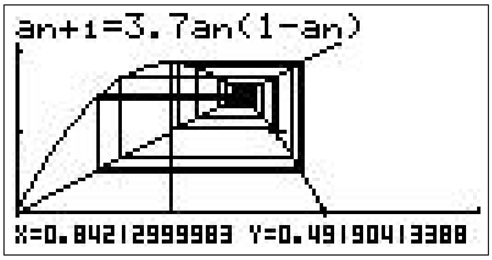


A = 3.2





A = 3.7



Exercise: Confirm graphically (by choosing appropriate values for A) that the logistic sequence:

- converges to 0 for $0 < A < 1$;
- converges to $1 - 1/A$ for $1 < A < 3$;
- oscillates between 2 values, then 4 values, 8 values and so on, as A is increased above 3;
- becomes chaotic for values of A greater than about 3.568;
- diverges rapidly to $\pm\infty$ for values of A greater than 4 or values of a_1 less than 0 or greater than 1

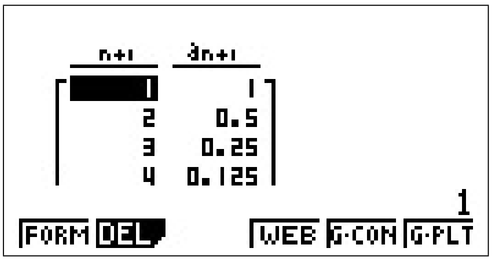
Try both TIME and WEB plots.

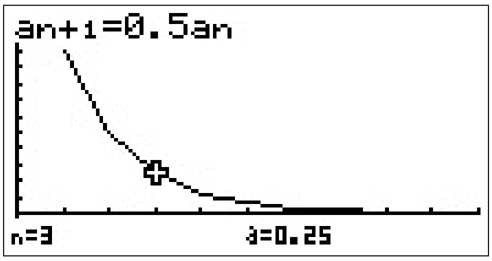
16.6 Solutions to exercises and questions

Exercises

Exercise page 29

Display a table and graph the first 10 terms of the geometric sequence $a_{n+1} = 0.5a_n$, with $a_1 = 1$.





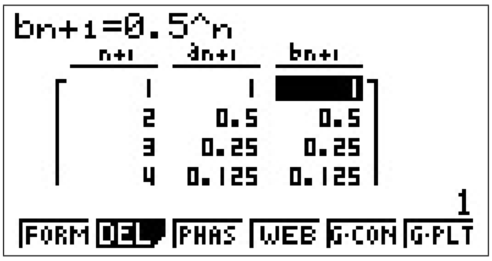
Here, I've used G-Con to plot the graph because the sequence points are widely spaced.

Exercise page 30

For the sequence in the previous exercise, the nth term is given by

$$a_{n+1} = 0.5^n.$$

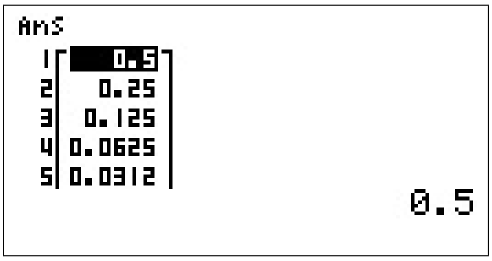
Put this explicit definition in sequence b_n and compare with the recursive definition. Make sure the value for b_1 in SET is correct.



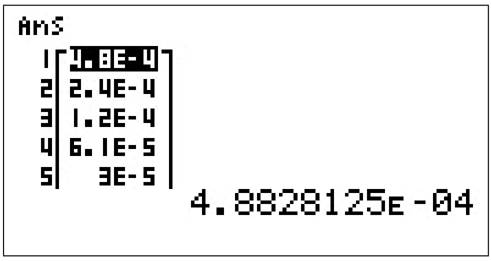
PTO

Exercise page 32

On the RUN screen, store 1 in A, 0.5 in R, type in seq (AR∧(N−1), N, 1, 10, 1) and press $\boxed{\text{EXE}}$.

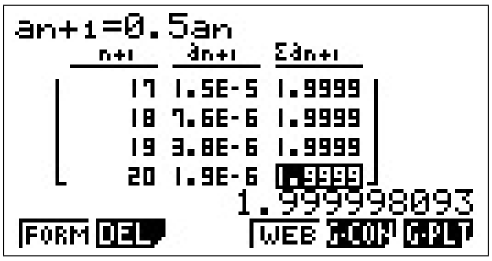


Then use the left or right arrow to recall the previous command and edit it to produce terms 11 to 20 in the sequence.



Exercise page 32

Set up a table of partial sums for our GP with $a_{n+1} = 0.5a_n$, $a_1 = 1$, or $a_{n+1} = 0.5^n$. To what value does the (infinite) series appear to converge. Confirm your answer algebraically.



From the first 20 terms, it appears that the series converges to 2.

Algebraically, the sum of this GP is given by

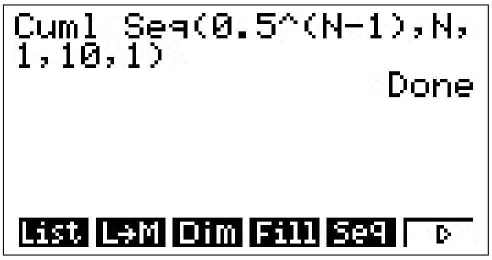
$$\frac{a}{1-r} = \frac{1}{1-0.5} = \frac{1}{0.5} = 2,$$

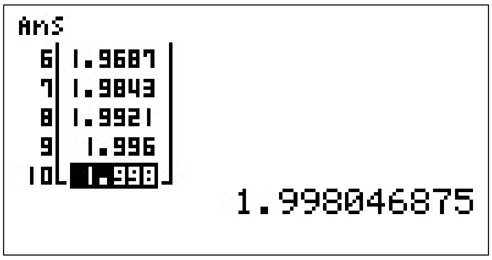
confirming our conjecture above.

Exercise page 36

Use List commands to make a conjecture on the sum of the GP

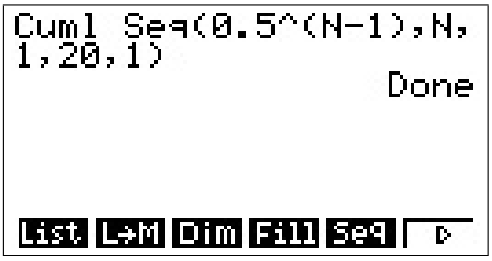
$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots.$$

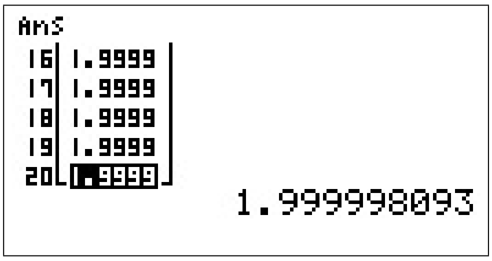




It looks like the answer is 2.

You might want to recall the command you entered (left or right arrow) and increase the end value of N to be (almost) sure of the answer. We can't *prove* this is the answer numerically, but we can be reasonably sure it is the answer.





Almost sure the answer is 2.

Questions page 9

1. The command is Seq (X∧3,X,1,10) → List 1. You can use any letter instead of X: in the text we used N to conform to the standard notation for sequences. X is easiest to use because it requires only one key press.

   To calculate the sum, use Sum List 1 or Sum Seq (X∧3,X,1,10,1) to produce the answer 3025.

2. The command is Sum Seq (1/X∧3,X,1,10,1), giving 1.197531986.

3. (a) The sum of the first 100 positive integers is Sum Seq (X,X,1,100,1) = 5050.

   (b) The sum of the *cubes* of the first 100 positive integers is Sum Seq (X∧3,X,1,100,1) = 25,502,500 = 2.55×10^7 to 3 significant digits.

   (c) The cube of the sum of the first 100 positive integers is $5050^3\approx1.29\times10^{11}$, larger than the sum of the cubes of the first 100 positive integers.

4. **(a)** The amount of radioactive substance in grams at the beginning of successive weeks, starting at Week 1, is (to two decimal places)

$$1000,\ 500,\ 250,\ 125,\ 62.5,\ 31.25,\ 15.63,\ 7.81,\ 3.91,\ 1.95.$$

(b) Explicitly, the amount at the beginning of Week n is $a_n = 1000\big(0.5\big)^{n-1}$. Recursively, $a_{n+1} = 0.5a_n$, with $a_1 = 1000$. The common ratio here is 0.5.

(c) Using a calculator table with $a_n = 1000\big(0.5\big)^{n-1}$, there is 0.005 g remaining sometime in Week 18, i.e. between $n = 18$ and $n = 19$.

(d) A week ago (beginning of Week 0), there was twice as much as there is now (beginning of Week 1), that is 2000 g.

(e) According to this model, there will always be some of the substance left, although the amount becomes small very rapidly. You can't reduce any number to 0 by dividing it by 2 or raising it to a power.

5. **(a)** The height n months from now is $a_n = 30\left(1+\frac{2.5}{100}\right)^n = 30\big(1.025\big)^n$, $n = 1, 2, \ldots, 12$. The table below shows months (top row) and corresponding heights, rounded to 1 decimal place.

0	1	2	3	4	5	6	7	8	9	10	11	12
30	30.8	31.5	32.3	33.1	33.9	34.8	35.7	36.6	37.5	38.4	39.4	40.4

The common ratio is 1.025.

(b) Using the calculator table, the plant would double in height in the 29th month.

6. Let S_n be the amount of money in Sue's account at the start of year n, with $n = 1$ corresponding to 3 years ago. Then,

$$S_n = 1250\left(1+\frac{6.5}{100}\right)^{n-1} = 1250\big(1.065\big)^{n-1}.$$

In 2 year's time, $n = 1+5 = 6$, so the amount of money in her account will be

$$S_5 = 1250\big(1.065\big)^5 = \$1712.61 \quad \text{to the nearest cent.}$$

You could also use the calculator table to reach this answer.

7. The amount in Frank's account is given by

$$S_n = 12876\left(1+\frac{5.75}{12\times 100}\right)^{12(n-1)} = 12876\left(1+\frac{0.0575}{12}\right)^{12(n-1)},$$

where $n = 1$ corresponds to now.

Six years ago, $n = -5$, and the amount in Frank's account was

$$S_{-5} = 12876\left(1+\frac{0.0575}{12}\right)^{-72} = \$9126.56 \quad \text{to the nearest cent.}$$

You could also use the calculator table to reach this answer. Start the table at -5. Do you get the correct value at $n = 1$?

8. **(a)** Set $a_{n+1} = a_n^2$, set a_1 to an appropriate starting value and use the calculator table to see what happens as n becomes large.

Initial value	**Limit of sequence**
$a_1 < -1$	∞
$a_1 = -1$	1
$-1 < a_1 < 1$	0
$a_1 = 1$	1
$a_1 > 1$	∞

(b) Set $a_{n+1} = a_n^2 - 1$.

Initial value	**Limit of sequence**
$a_1 < -1.618\ldots$	∞
$a_1 = -1.618\ldots$	$-1.618\ldots$ (constant)
$-1.618\ldots < a_1 < -0.618\ldots$	oscillates between 0 and -1
$a_1 = -0.618\ldots$	$-0.618\ldots$ (constant)
$-0.618\ldots < a_1 < 0.618\ldots$	oscillates between 0 and -1
$a_1 = 0.618\ldots$	$-0.618\ldots$ (constant)
$0.618\ldots < a_1 < 1.618\ldots$	oscillates between 0 and -1
$a_1 = 1.618\ldots$	$1.618\ldots$ (constant)
$a_1 > 1.618\ldots$	∞

(c) $a_{n+1} = \sqrt{a_n}$ gives 0 if $a_1 = 0$, and tends to 1 otherwise.

(d) $a_{n+1} = \cos\left(a_n\right)$ tends to 0.7391... in Radian mode and 0.99985... in Degree mode.

(e) $a_{n+1} = \tan\left(a_n\right)$ looks several times as though it is going to settle down to a limit, but never does.

9. The sequence $\left(1 + \dfrac{1}{n}\right)^n$ approaches the value $e = 2.71828\ldots$ as $n \to \infty$.

10. $$\sum_{n=0}^{\infty} \frac{1}{n!} = e \qquad \sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2 \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

11. **(a)** approaches $\sqrt{2}$; **(b)** approaches $\sqrt{3}$; **(c)** approaches 2; (d) approaches 3.

PTO

12. **(a)** The following table gives the rebound height after bounce n.

$\boldsymbol{n}$	1	2	3	4	...
Height (m)	6	$\frac{9}{2}$	$\frac{27}{8}$	$\frac{81}{32}$	...

The rebound height after the third bounce is therefore 27/8 m.
The rebound height after bounce n is $6\left(0.75\right)^{n-1}$ m.
Therefore, after the 10th bounce, the rebound height is $6\left(0.75\right)^{9} \approx 0.45$ m.
Alternatively, use a table.

(b) The total distance travelled is 8 + 2 × each rebound height, i.e.

$$8 + 2\left(6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \dots\right).$$

When the ball hits the floor for the 4th time, it has travelled

$$8 + 2\left(6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32}\right) = 35.75\,\text{m}.$$

The distance travelled when it hits the floor for the bth time ($b \geqslant 2$) is therefore

$$8 + 2\sum_{n=1}^{b-1} 6\left(0.75\right)^{n-1} = 8 + 12\sum_{n=1}^{b-1} 0.75^{n-1}.$$

Therefore, when the ball hits the floor for the 20th time, it has travelled

$$8 + 12\sum_{n=1}^{19} 0.75^{n-1} \approx 55.8\,\text{m}.$$

Use $8 + 12\,$Sum Seq (0.75∧(B−1),B,1,19,1) to do the calculation.

13. Number of grains $= 1 + 2 + 4 + 8 + 16 + \dots = \sum_{n=1}^{64} 2^{n-1} \approx 1.84 \times 10^{19}$.

Use Sum Seq (2∧(N−1),N,1,64,1) to calculate this.

To put this number into context, 10^{18} grains would be about the same volume as the Great Wall of China and the five Great Pyramids combined.[10]

PTO

[10] From *Large Numbers* by Victor Scharaschkin, Australian Senior Mathematics Journal 4 (2), 111–125 (1990).

14. We need to evaluate $\sum_{n=a}^{b} 6n$, where

(a) $6a$ is the smallest multiple of 6 greater than 10 and $6b$ is the largest multiple of 6 less than 100. Clearly, $a=2$ and $b=16$, so that the required sum is

$$\sum_{n=2}^{16} 6n = 810,$$

where we use the command Sum Seq (6N, N, 2, 16, 1).

(b) $6a$ is the smallest multiple of 6 greater than 1 and $6b$ is the largest multiple of 6 less than 10,000. Clearly, $a=1$ and b is the integer part of 10,000/6, that is 1666. The required sum is

$$\sum_{n=1}^{1666} 6n = 8,331,666.$$

We can't use the single command Sum Seq (6N,N,1,1666,1) here because there are more than 999 terms in the series[11]. We must break the sum up into two parts: Sum Seq (6N,N,1,999,1); and Sum Seq (6N,N,1000,1666,1).

PTO

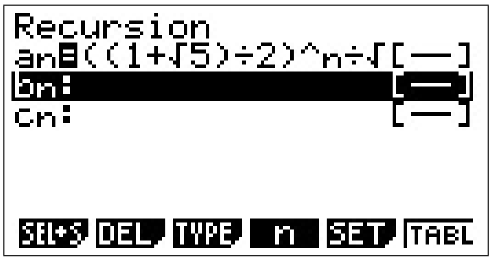
[11] The maximum number of entries in a list is 999.

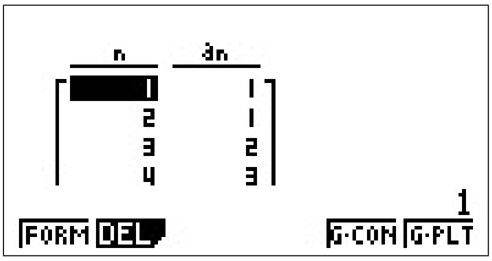
Further investigations page 12

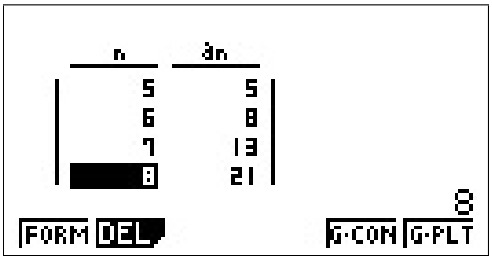
The Fibonacci sequence and the Golden Ratio

1. Generate a table of values of the sequence (be very careful with brackets)

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$







Do you recognise this sequence? It's the Fibonacci sequence.

2. One of the many interesting things about the Fibonacci sequence becomes apparent when we look at ratios of successive terms. In $\boxed{\text{TYPE}}$, press $\boxed{\text{F3}}$ and set

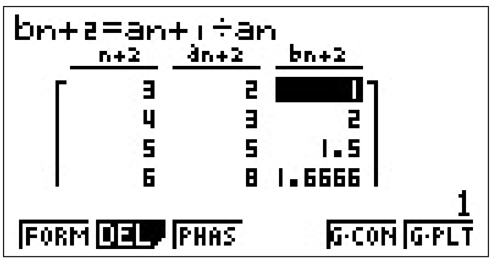
$$a_{n+2} = a_{n+1} + a_n$$

to generate the Fibonacci sequence, and

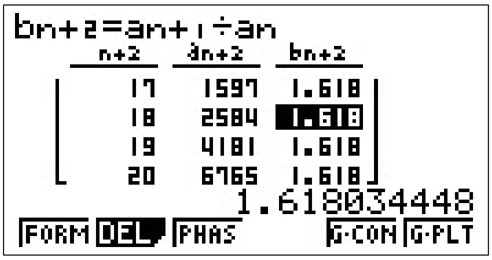
$$b_{n+2} = a_{n+1} \div a_n$$

to calculate the ratios of successive terms.

In $\boxed{\text{SET}}$, put Start = 3, End = 20, $a_1 = 1$, $a_2 = 1$ and generate a table of values.



Scroll down and look at the b_{n+2} column containing the sequence of the ratios of successive terms of a_n.



Does this sequence appear to be approaching a particular value? What value? Find this value accurate to six decimal places.

From the table, the ratio is approaching 1.618034 (to six decimal places); all values after b_{18} round to this value.

Now find a value for the Golden Ratio or Golden Section, $\dfrac{1+\sqrt{5}}{2}$.

$\dfrac{1+\sqrt{5}}{2} = 1.618034$ to six decimal places.

What conclusion do you reach?

The ratio of successive terms in the Fibonacci sequence tends to the Golden Ratio as $n \to \infty$.

17 Probability and Statistics 2 Probability Distributions and Hypothesis Testing

17.1 Introduction

Probability and Statistics 1 in the first volume of this book dealt with descriptive statistics, that is using the calculator to store, process and display statistical data. Here, we deal with discrete and continuous probability distributions, and hypothesis testing.

The most comprehensivel book I have seen on this topic is *Probability and Statistics with the TI-83 Plus: For A-Level Mathematics* by Peter Jones and Chris Barling.[12] Sadly, this is no longer readily available; if you have a copy, treasure it. They explain everything in great detail, with large numbers of screen shots. The exposition here is somewhat briefer but I'll follow their general outline, and even borrow some of their examples (labelled JB). Most of the exercises are also from there. Some of the examples and exercises are from the course notes for a first-year introductory Statistics course by Dr Leesa Sidhu of UNSW Canberra.

Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS[13] by Barry Kissane also has considerable material relevant to this topic. A shorter version is in *Learning Mathematics with Graphics Calculators* by Barry Kissane and Marian Kemp, now on the CMA website.[14]

What follows is not a textbook on Probability and Statistics (you'll need one) but how to do the various operations in such a course on a 9860 graphics calculator. This replaces the various tables usually used in such courses, and makes many of the calculations quite simple. As always, students should do all these operations by hand first, so that they understand the process but, ultimately, always having to do the calculations manually stands in the way of doing any meaningful modelling and interesting applied problems.

17.1.1 Australian Curriculum

References are given here to the texts *Nelson Senior Maths Methods 11 for the Australian Curriculum* (NSM11) and *Nelson Senior Maths Methods 12 for the Australian Curriculum* (NSM12) used in the ACT.

The material here is relevant to the topics *Discrete random variables* (Chapter 2 in NSM12), *Binomial distributions* (Chapter 5 in NSM12), *Continuous random samples and the normal distribution* (Chapter 8 in NSM12), *Random samples and proportions* (Chapter 9 in NSM12), and *Confidence intervals* (Chapter 10 in NSM12).

[12] Cambridge University Press, 2002, ISBN 0 521 52531 4

[13] The Mathematical Association of Western Australia, 2003, ISBN 1-876583-24-X; I have a copy.

[14] *canberramaths.org.au*

17.2 Discrete probability distributions

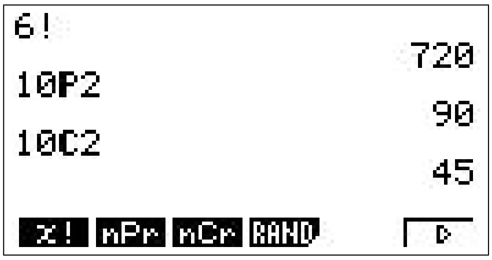
17.2.1 Counting methods

From the RUN screen, the relevant commands here are all in the $\boxed{\text{OPTN}}$ $\boxed{\text{PROB}}$ menu.

To evaluate 6! $\boxed{6}$ $\boxed{\text{F1}}$

To evaluate ${}^{10}P_2$ $\boxed{10}$ $\boxed{\text{F2}}$ $\boxed{2}$

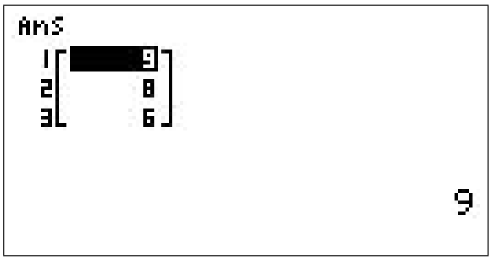
To evaluate ${}^{10}C_2$ $\boxed{10}$ $\boxed{\text{F3}}$ $\boxed{2}$



17.2.2 Random numbers

A random number between 0 and 1 is generated by the *Ran#* command in the $\boxed{\text{RAND}}$ menu.

Random integers are generated by the RanInt# command $\left(\boxed{\text{Int}}\right)$: to generate three random integers between 1 and 17, use the command RanInt#$(1, 17, 3)$. If you leave off the 3, you get a single random integer.



Note the commands in the $\boxed{\text{RAND}}$ menu generating random numbers that are normally or binomially distributed.

Exercises

1. Evaluate each of the following by hand, then check with your calculator.

   (a) 6! **(a)** ${}^{6}P_4$ **(a)** ${}^{8}C_5$.

2. Find: **(a)** 13! **(b)** ${}^{15}P_9$ **(c)** ${}^{35}C_2$.

3. Generate 100 random integers between 1 and 6, simulating throwing a die. Store them in an appropriately named list.

   Find: **(a)** the mean; **(b)** the standard deviation of the mean; **(c)** the median.

17.2.3 Expected value and variance

Theory

The **probability distribution function** (pdf) of a discrete random variable X is

$$p(x) = P(X = x),$$

for each possible value of x; P stands for (*the*) *probability* (*that*).[15]

The **cumulative distribution function** (cdf) is

$$F(x) = P(X \leqslant x) = \sum_{\text{all}\, x} p(x).$$

The **mean** (μ or μ_X) or **expected value** $\big(E[X]\big)$ of a discrete random variable X is

$$\mu = \mu_X = E[X] = \sum_{\text{all}\, x} x\, p(x).$$

The **variance** of a discrete random variable X with mean μ is

$$V[X] = \sum_{\text{all}\, x} (x - \mu)^2 p(x) = \sum_{\text{all}\, x} x^2 p(x) - \mu^2.$$

The **standard deviation** of a random variable is the square root of the variance.

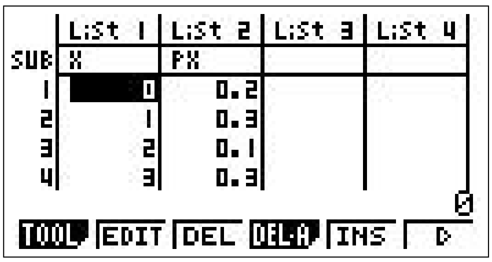
Using LIST operations

To calculate the expected value and variance of a discrete distribution, we must first enter the distribution values into lists.

Example: Consider the following distribution of a discrete random variable X.

x	0	1	2	3	4
$p(x) = P(X = x)$	0.2	0.3	0.1	0.3	0.1

Press $\boxed{\text{MENU}}$ $\boxed{2}$ and put the two sets of numbers in List 1 X and List 2 PX.



Calculations using lists are accessed from the RUN screen in the $\boxed{\text{OPTN}}$ $\boxed{\text{LIST}}$ menu.

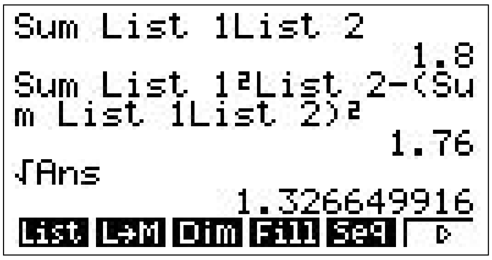
[15]Pr in JB

Mean: On the calculator, the product List 1 List 2 (X PX) is a new list in which each element is the product of the corresponding elements of the individual lists, i.e. a list containing the values $xp(x)$.

If we now sum this list, Sum List 1 List 2, we obtain the mean value μ or the expected value $E[X]$ of the distribution, 1.8 in this case.

Variance: Similarly, the variance $V[X]$ is Sum List 1^2 List 2 − (Sum List 1 List 2)^2, 1.76 here.

Standard deviation: The standard deviation, the square root of the variance, is 1.33 here.



PTO

17.2.4 The Binomial Distribution

A. Calculating binomial probabilities

If a discrete random variable X has a binomial distribution, $X \sim \text{bin}(n, p)$, the **probability distribution function** (pdf)

$$P(X = x) \;=\; {}^nC_x\, p^x\,(1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

The Binomial Distribution is defined by two parameters, the number of independent trials n and the probability p of success at one trial.

If $X \sim \text{bin}(n, p)$, the calculator command to calculate the pdf is BinomialPD (x, n, p).

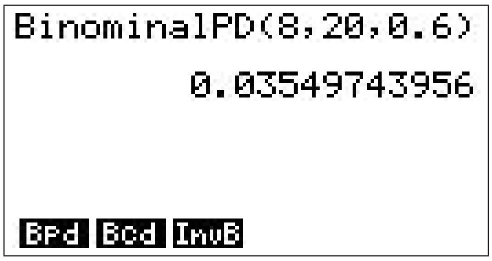
Example: If $X \sim \text{bin}(20, 0.6)$, calculate $P(X = 8)$.

There are two ways to do this.

From the RUN screen

Press $\boxed{\text{OPTN}}$ $\boxed{\text{STAT}}$ $\boxed{\text{DIST}}$ $\boxed{\text{BINM}}$ $\boxed{\text{Bpd}}$.
Put in the three arguments.
Press $\boxed{\text{EXE}}$ to execute the command.

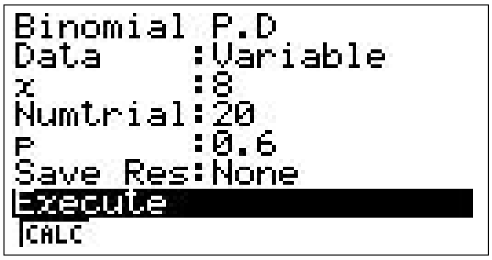
The advantage of this method is that you can then easily do further calculations with the result: see the Examples below.

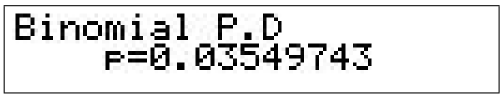


From the STAT screen

Press $\boxed{\text{DIST}}$ $\boxed{\text{BINM}}$ $\boxed{\text{Bpd}}$ to give the input menu.
Select *Variable* for *Data*, put in the three arguments, scroll down to *Execute* and press $\boxed{\text{F1}}$ (CALC) or $\boxed{\text{EXE}}$ to do the calculation.

The advantage of this method is that you are prompted for the inputs; you do not have to remember in which order they go into a command. You can also save the result to a list for further use if you wish.



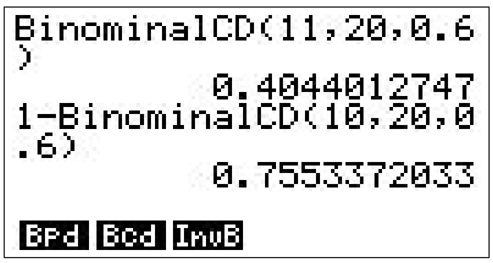


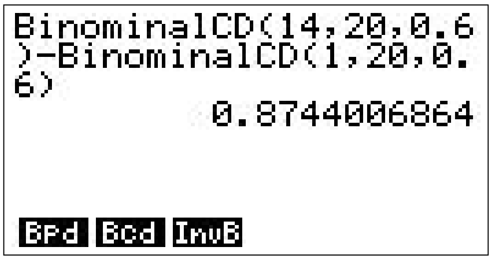
The binomial **cumulative distribution function** (cdf), denoted $F(x)$, is $F(x) = P(X \leqslant x)$.

To calculate the cdf, we use the **binomial cdf** command BinomialCD (x, n, p), accessed in the same way as BinomialPD; here, we use the RUN screen.

Example: If $X \sim \text{bin}(20, 0.6)$, find $P(X \leqslant 11)$.

Example: If $X \sim \text{bin}(20, 0.6)$, find $P(X \geqslant 11)$:
$P(X \geqslant 11) = 1 - P(X \leqslant 10)$.





Example: If $X \sim \text{bin}(20, 0.6)$, find $P(2 \leqslant X \leqslant 14)$:
$P(2 \leqslant X \leqslant 14) = P(X \leqslant 14) - P(X \leqslant 1)$.

Exercises

1. If $X \sim \text{bin}(8, 0.4)$, find
- **(a)** $P(X=4)$
- **(b)** $P(X \leqslant 1)$
- **(c)** $P(X \geqslant 7)$
- **(d)** $P(1 \leqslant X \leqslant 3)$
- **(e)** $P(X<6) = P(X \leqslant 5)$.

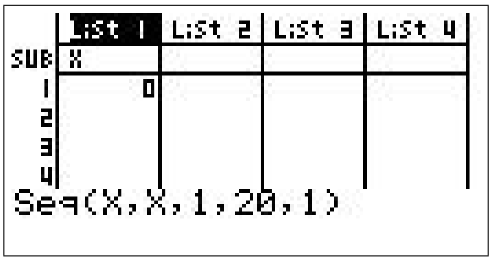
2. If $X \sim \text{bin}(25, 0.2)$, find
- **(a)** $P(X=4)$
- **(b)** $P(X \leqslant 10)$
- **(c)** $P(X \geqslant 7)$
- **(d)** $P(5 \leqslant X \leqslant 8)$
- **(e)** $P(X>4) = 1-P(X \leqslant 4)$.

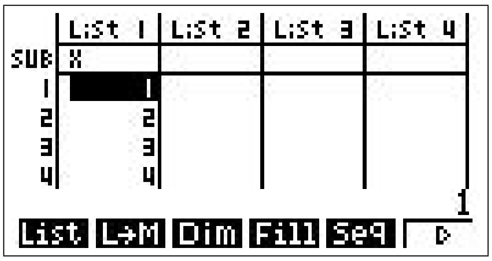
B. Graphing a binomial probability distribution

Example: If $X \sim \text{bin}(20, 0.6)$, graph the distribution of X for $0 \leqslant X \leqslant 20$.

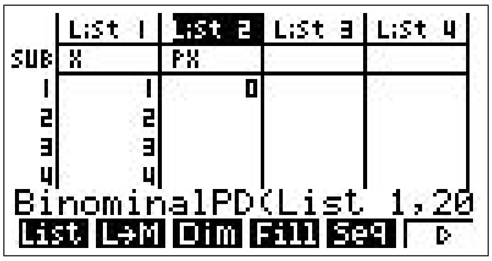
To do this, we need to store values of x in the X list (List 1) and the corresponding values of $P(x=X)$ in the PX list (List 2), like we did in Section 17.2.3.

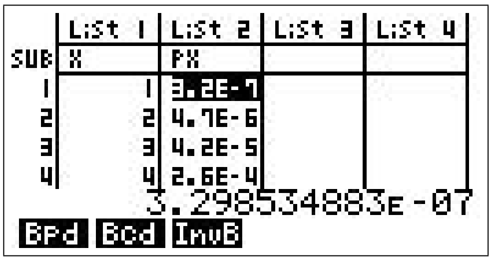
A quick way to fill the X list is to move the cursor to the list name, type in the command Seq(X, X, 0, 20,1); *Seq* is in the $\boxed{\text{OPTN}}$ $\boxed{\text{LIST}}$ menu. Press $\boxed{\text{EXE}}$ to fill the list.





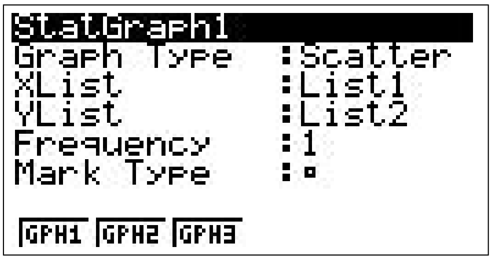
Similarly, move the cursor to the header of the PX list, go to the $\boxed{\text{OPTN}}$ (F6 twice) $\boxed{\text{STAT}}$ $\boxed{\text{DIST}}$ $\boxed{\text{BINM}}$ menu and select $\boxed{\text{Bpd}}$. Complete the command BinomialPD(List 1, 20, 0.6) (below left) and press $\boxed{\text{EXE}}$ to fill the list (below right).

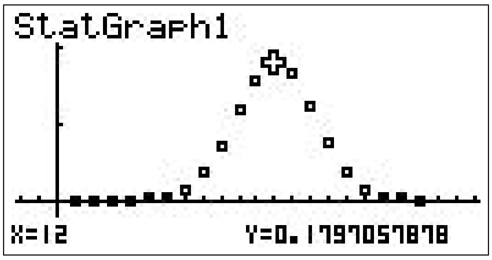




Press EXIT several times to return to the top level of the STAT menu and press GRPH (F1). Press SET and set up StatGraph1 as shown below left. Press EXIT, then SEL. Check that StatGraph1 is On, the other two off and press DRAW. The View Window is set automatically (in SET UP).

In the plot below right, Trace has been pressed and the cursor moved to X = 12.





Exercises

1. If $X \sim \text{bin}(10, 0.4)$, display the distribution of X graphically.

2. On the one graph, plot the distribution of:[16]

$X_1 \sim \text{bin}(10, 0.1)$ use StatGraph1 with the large-square marker;

$X_2 \sim \text{bin}(10, 0.5)$ use StatGraph2 with the $\times$ marker;

$X_3 \sim \text{bin}(10, 0.9)$ use StatGraph3 with the small-square marker.

Comment on how changing the value of p changes the shape of the distribution.

[16] You might try line graphs ($\boxed{xy}$ on the StatGraph screens) rather than scatterplots to show the three distributions better.

17.2.5 The Geometric Distribution

A. Calculating geometric probabilities

If a discrete random variable X has a geometric distribution, $X \sim \text{geom}(p)$, the **probability distribution function** (pdf)

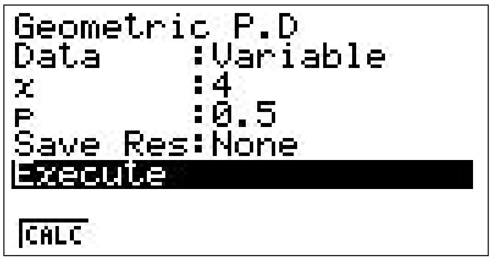
$$P(X = x) = p\,(1 - p)^{x-1} \quad \text{for } x = 1, 2, \ldots.$$

The Geometric Distribution is defined by a single parameter p.

Example: If $X \sim \text{geom}(0.5)$, calculate $P(X = 4)$.

From the STAT screen: press $\boxed{\text{DIST}}$ $\boxed{\text{GEO}}$ $\boxed{\text{Gpd}}$ to give the input menu (top figure).

Select *Variable* for *Data*, put in the values for x and p, scroll down to *Execute* and press $\boxed{\text{F1}}$ (CALC) or $\boxed{\text{EXE}}$ to do the calculation.

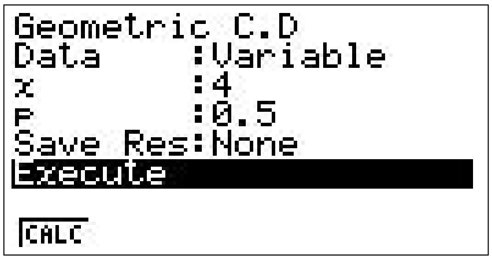


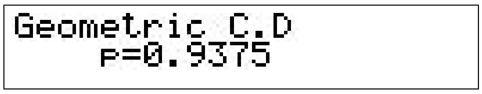
To calculate $P(X \leqslant x)$, we use the **geometric cdf**.

Example: If $X \sim \text{geom}(0.5)$, calculate $P(X \leqslant 4)$.

From the STAT screen: press $\boxed{\text{DIST}}$ $\boxed{\text{GEO}}$ $\boxed{\text{Gcd}}$ to give the input menu (top figure).

Select *Variable* for *Data*, put in the values for x and p, scroll down to *Execute* and press $\boxed{\text{F1}}$ (CALC) or $\boxed{\text{EXE}}$ to do the calculation.



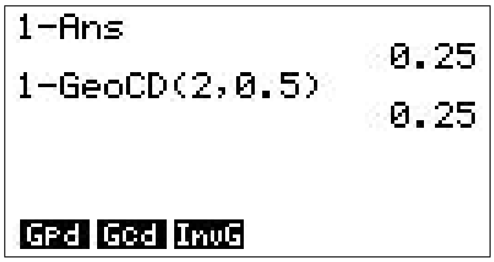


Example: If $X \sim \text{geom}(0.5)$, calculate $P(X \geqslant 3)$:

$P(X \geqslant 3) = 1 - P(X \leqslant 2)$.

Using the method of the previous example, we can only calculate $P(X \leqslant 2)$ (top figure). To do calculations with this value, which is stored in *Ans*, we have to go to the RUN screen (next figure, first calculation).

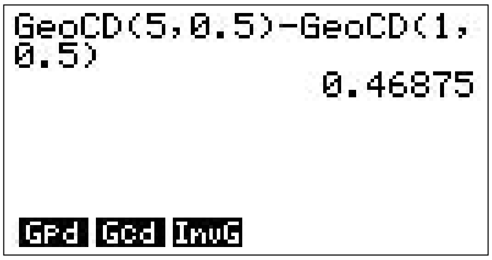
Alternatively, use the method in Section 17.2.4 to do all the calculations on the RUN screen (bottom figure, second calculation): $\boxed{\text{OPTN}}$ $\boxed{\text{STAT}}$ $\boxed{\text{DIST}}$ $\boxed{\text{GEO}}$ $\boxed{\text{Gcd}}$; GeoCD$(2, 0.5)$.



Example: If $X \sim \text{geom}(0.5)$, find $P(2 \leqslant X \leqslant 5)$:

$P(2 \leqslant X \leqslant 5) = P(X \leqslant 5) - P(X \leqslant 1)$.

Here it makes sense to do all the calculations on the RUN screen.



Exercises

If $X \sim \text{geom}(0.2)$, find: **(a)** $P(X=5)$; **(b)** $P(X \leqslant 8)$; **(c)** $P(X \geqslant 7)$; **(d)** $P(3 \leqslant X \leqslant 9)$.

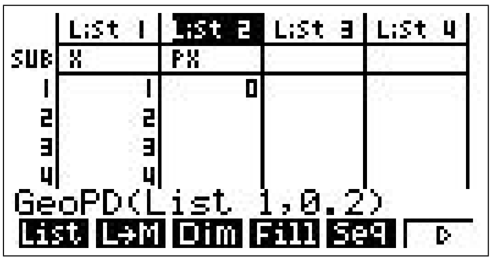
B. Graphing a geometric probability distribution

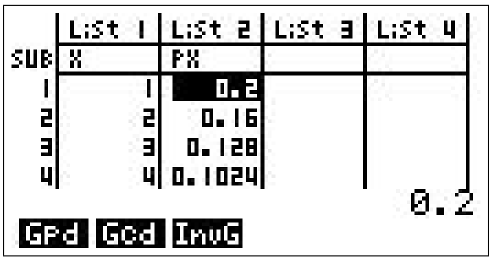
Example: If $X \sim \text{geom}(0.2)$, graph the distribution of X for $x = 1, 2, \ldots, 10$.

To do this, we need to store values of x in List 1 (X) and the corresponding values of $P(x = X)$ in List 2 (PX), as we did in Section 17.2.4.

Here, to fill the X list, execute the command Seq(X, X, 1, 10, 1).

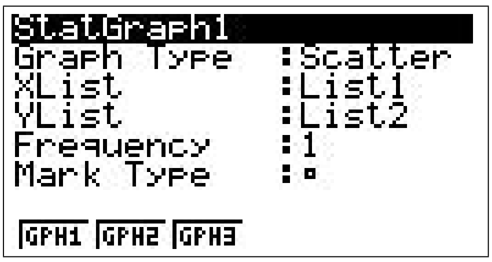
Move the cursor to the header of the PX list, go to the OPTN (F6 twice) STAT DIST GEO menu and select Gpd. Complete the command GeoPD(List 1, 0.2) (below left) and press EXE to fill the list (below right).

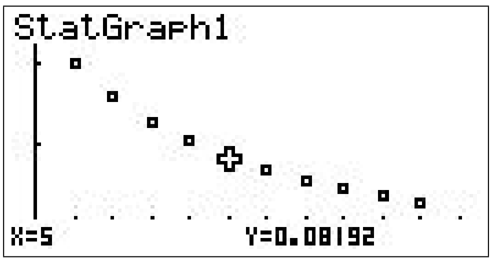




Press EXIT several times to return to the top level of the STAT menu and press GRPH (F1). Press SET and set up StatGraph1 as shown below left. Press EXIT, then SEL. Check that StatGraph1 is On, the other two off and press DRAW. The View Window is set automatically (in SET UP).

In the plot below right, Trace has been pressed and the cursor moved to X = 5.





Exercises

1. If $X \sim \text{geom}(0.4)$, display the distribution of X graphically for $1 \leqslant x \leqslant 8$.
2. On the one graph, plot the distribution of:

   $X_1 \sim \text{geom}(0.1)$ use StatGraph1 with a large-square marker;

   $X_2 \sim \text{geom}(0.5)$ use StatGraph2 with a $\times$ marker;

   Comment on how changing the value of p changes the shape of the distribution.

17.2.6 The Hypergeometric Distribution

A. Calculating hypergeometric probabilities

If a discrete random variable X has a hypergeometric distribution, $X \sim \text{hypg}(n, M, N)$, the **probability distribution function** (pdf)

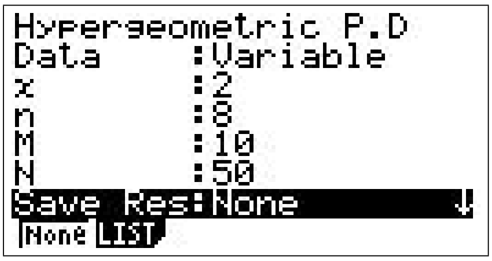
$$P(X=x) \quad = \quad \frac{{}^{M}C_x \, {}^{N-M}C_{n-x}}{{}^{N}C_n} \quad \text{with } 0 \leqslant x \leqslant n \text{ and } x \leqslant M.$$

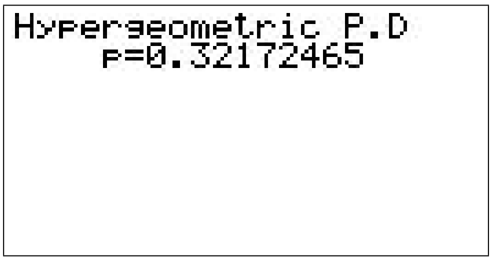
The Hypergeometric Distribution is defined by three parameters, the sample size n, the number of successes in the population M and the population size N.

Example: If $X \sim \text{hypg}(8, 10, 50)$, calculate $P(X=2)$.

From the STAT screen: press DIST H-GEO Hpd to give the input menu (below left).

Select *Variable* for *Data*, put in the values for x, n, M and N, scroll down to *Execute* and press F1 (CALC) or EXE to do the calculation.



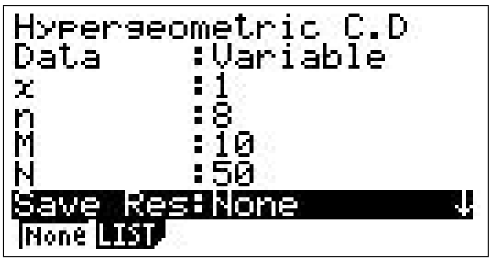


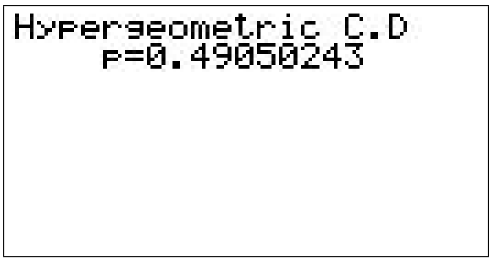
To calculate $P(X \leqslant x)$, we use the **hypergeometric cdf**.

Example: If $X \sim \text{hypg}(8, 10, 50)$, calculate $P(X \leqslant 1)$.

From the STAT screen: press DIST H-GEO Hcd to give the input menu (below left).

Select *Variable* for *Data*, put in the values for x, n, M and N, scroll down to *Execute* and press F1 (CALC) or EXE to do the calculation.





Example: If $X \sim \text{hypg}(50, 10, 8)$, calculate $P(X \geqslant 2)$.

$P(X \geqslant 2) = 1 - P(X \leqslant 1) = 1 - 0.4905 = 0.509$ from the previous example.

Exercises

If $X \sim \text{hypg}(100, 15, 5)$, find: **(a)** $P(X=3)$; **(b)** $P(X \leqslant 3)$; **(c)** $P(X>3)$; **(d)** $P(1 \leqslant X \leqslant 2)$.

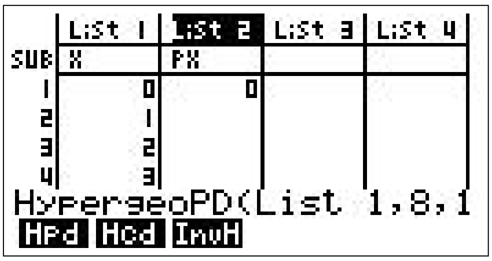
B. Graphing a hypergeometric probability distribution

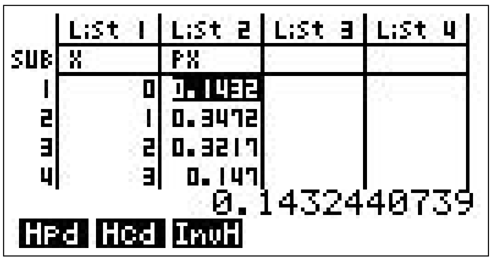
Example: If $X \sim \text{hypg}(8, 10, 50)$, graph the distribution of X for $x = 0, 2, \ldots, 8$.

To do this, we need to store values of x in List 1 (X) and the corresponding values of $P(x = X)$ in List 2 (PX), as we did in Section 17.2.4.

Here, to fill the X list, execute the command seq(X, X, 0, 8).

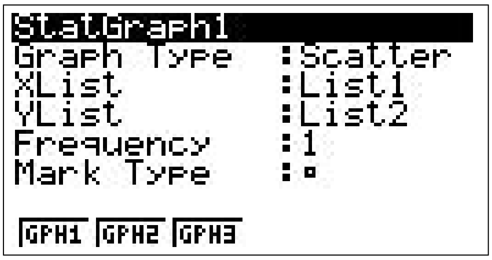
Move the cursor to the header of the PX list, go to the OPTN (F6 twice) STAT DIST H-GEO menu and select Hpd. Complete the command HypergeoPD(List 1, 8, 10, 50) (below left) and press EXE to fill the list (below right).

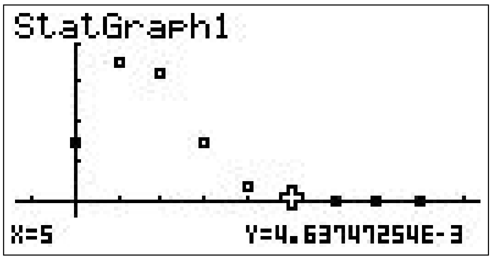




Press EXIT several times to return to the top level of the STAT menu and press GRPH (F1). Press SET and set up StatGraph1 as shown below left. Press EXIT, then SEL. Check that StatGraph1 is On, the other two off and press DRAW. The View Window is set automatically (in SET UP).

In the plot below right, Trace has been pressed and the cursor moved to X = 5.





Exercises

1. If $X \sim \text{hypg}(8, 15, 100)$, display the distribution of X graphically for $0 \leqslant x \leqslant 8$.

2. On the one graph, plot the distribution of:

$X \sim \text{hypg}(8, 15, 100)$ use StatGraph1 with a large-square marker;

$X \sim \text{hypg}(8, 30, 100)$ use StatGraph2 with a $\times$ marker;

Comment on how changing the value of M changes the shape of the distribution.

17.2.7 The Poisson Distribution

A. Calculating Poisson probabilities

If a discrete random variable X has a Poisson distribution, $X \sim \text{pois}(\lambda)$, the **probability distribution function** (pdf)

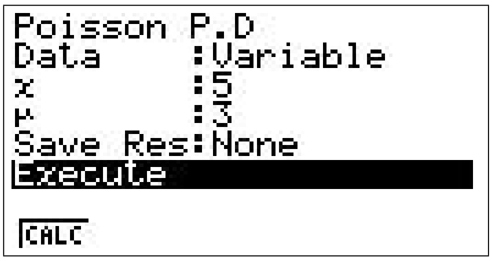
$$P(X = x) = \frac{\mu^x}{x!}\, e^{-x} \quad \text{for } x = 0, 1, 2, \ldots.$$

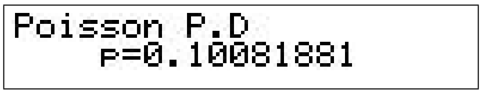
The Poisson Distribution is defined by a single parameter μ.

Example: If $X \sim \text{pois}(3)$, calculate $P(X = 5)$.

From the STAT screen: press DIST POISN Ppd to give the input menu (below left).

Select *Variable* for *Data*, put in the values for x and μ, scroll down to *Execute* and press F1 (CALC) or EXE to do the calculation.



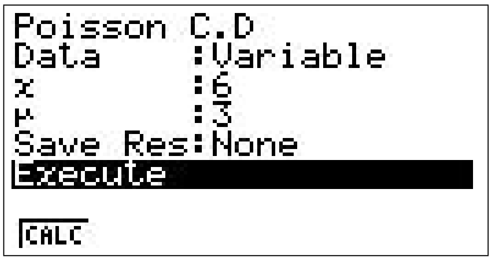


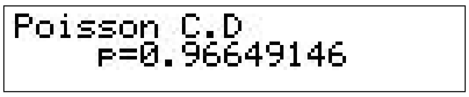
To calculate $P(X \leqslant x)$, we use the **Poisson cdf**.

Example: If $X \sim \text{pois}(3)$, calculate $P(X \leqslant 6)$.

From the STAT screen: press DIST POISN Pcd to give the input menu (below left).

Select *Variable* for *Data*, put in the values for x and μ, scroll down to *Execute* and press F1 (CALC) or EXE to do the calculation.

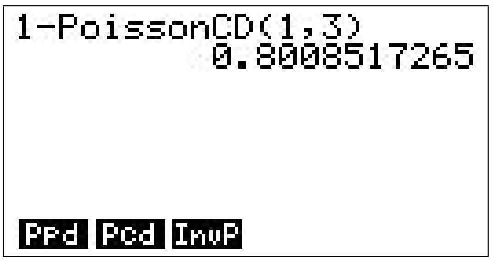




Example: If $X \sim \text{pois}(3)$, calculate $P(X \geqslant 2)$:

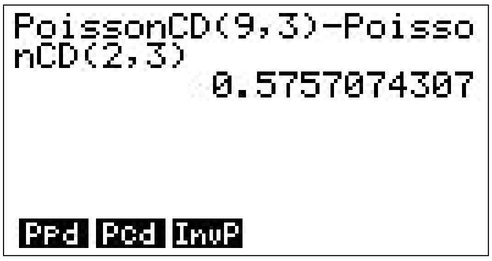
$P(X \geqslant 2) = 1 - P(X \leqslant 1)$.

Here we use the RUN screen: OPTN STAT DIST POISN Pcd.



Example: If $X \sim \text{pois}(3)$, calculate $P(3 \leqslant X \leqslant 9)$:
$P(3 \leqslant X \leqslant 9) = P(X \leqslant 9) - P(X \leqslant 2)$.

Again we use the RUN screen.



Exercises: If $X \sim \text{pois}(0.5)$, find: **(a)** $P(X=0)$; **(b)** $P(X \leqslant 3)$; **(c)** $P(X \geqslant 1)$; **(d)** $P(1 \leqslant X \leqslant 4)$.

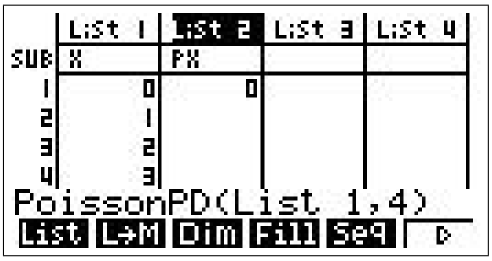
B. Graphing a Poisson probability distribution

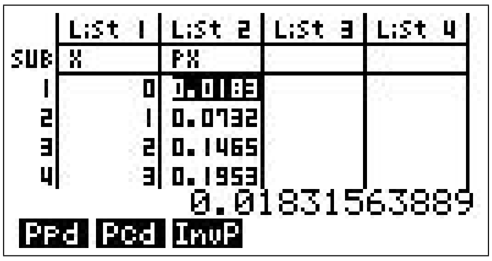
Example: If $X \sim \text{pois}(4)$, graph the distribution of X for $x = 0, 1, \ldots, 10$.

To do this, we need to store values of x in List 1 (X) and the corresponding values of $P(x=X)$ in List 2 (PX) (Section 17.2.4).

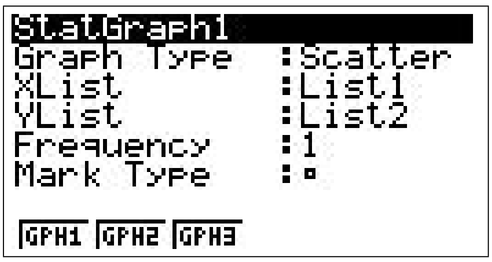
Here, to fill the X list, execute the command seq(X, X, 0, 10).

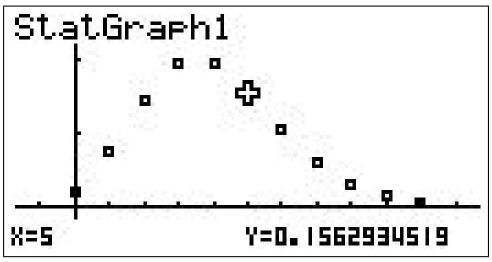
Move the cursor to the header of the PX list, go to the $\boxed{\text{OPTN}}$ (F6 twice) $\boxed{\text{STAT}}$ $\boxed{\text{DIST}}$ $\boxed{\text{POISN}}$ menu and select $\boxed{\text{Ppd}}$. Complete the command PoissonPD(List 1, 4) (below left) and press $\boxed{\text{EXE}}$ to fill the list (below right).





Press $\boxed{\text{EXIT}}$ several times to return to the top level of the STAT menu and press $\boxed{\text{GRPH}}$ (F1). Press $\boxed{\text{SET}}$ and set up StatGraph1 as shown below left. Press $\boxed{\text{EXIT}}$, then $\boxed{\text{SEL}}$. Check that StatGraph1 is On, the other two off and press $\boxed{\text{DRAW}}$. The View Window is set automatically (in SET UP). In the plot below right, $\boxed{\text{Trace}}$ has been pressed and the cursor moved to X = 5.





Exercises

1. If $X \sim \text{pois}(1.5)$, display the distribution of X graphically for $0 \leqslant x \leqslant 6$.

2. On the one graph, for $0 \leqslant x \leqslant 10$, plot the distribution of:
$X_1 \sim \text{pois}(2)$ use StatGraph1 with a large-square marker;
$X_2 \sim \text{pois}(5)$ use StatGraph2 with a $\times$ marker;
$X_2 \sim \text{pois}(9)$ use StatGraph3 with a small-square marker.

Comment on how changing the value of λ changes the shape of the distribution.

17.3 Continuous probability distributions

For continuous random variables we do not specify the probability for every value in the range. In fact, $P(X=a)=0$ for any value of a but we can work out the probability that X lies in a range of values. For example, the probability of finding a person whose height is exactly 2 metres is zero, but the probability of finding someone with a height between 2 and 2.01 metres might be 0.05.

For a continous random variable X, we define a **probability density function** (pdf) $f(x)$, and we find the probability that X lies in an interval by finding the area under the pdf curve for that interval.

The **cumulative distribution function** (cdf), $F(x)=P(X\leqslant x)$ still exists for continous random variables. The continuous analogue of a summation is an integral, and $F(x)$ is naturally defined as

$$F(x)=\int_{-\infty}^{x} f(u)\,du.$$

17.3.1 The Normal Distribution

A continuous random variable X with a Normal Distribution has pdf

$$f(z)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right) \qquad -\infty<z<\infty.$$

The Normal Distribution is defined by two parameters, the mean μ and variance σ^2; we write $X\sim\mathrm{N}(\mu,\sigma^2)$.

A. Calculations with the Normal Distribution

The basic (cdf) calculation is

$$P(a\leqslant X\leqslant b)=\int_{a}^{b} f(z)\,dz.$$

From the RUN screen, this is achieved by the calculator command (with appropriate numbers inserted)

NormCD (a,b,σ,μ).   *note the opposite order of* μ *and* σ

The *NormCD* command is found in the $\boxed{\text{OPTN}}$ $\boxed{\text{STAT}}$ $\boxed{\text{DIST}}$ $\boxed{\text{NORM}}$ menu.

If μ and σ are omitted, the standard values $\sigma=1$ and $\mu=0$ are assumed. *It's best to always put them in.*

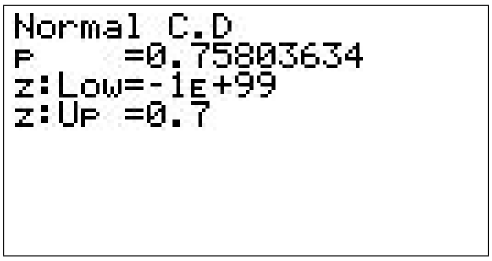
If the lower limit a is $-\infty$, set $a=-\mathrm{E}99=-1\times10^{99}$, the smallest number on the calculator.[17] Similarly, if the upper limit b is ∞, set $b=\mathrm{E}99=1\times10^{99}$, the largest number on the calculator.

Always think geometrically here: What area under the normal-distribution curve (pdf) am I calculating? The graphical representation in Section 17.3.1 below should help. You need to do a certain number of examples by hand first to understand this.

[17] The symbol E is $\boxed{\text{EXP}}$.

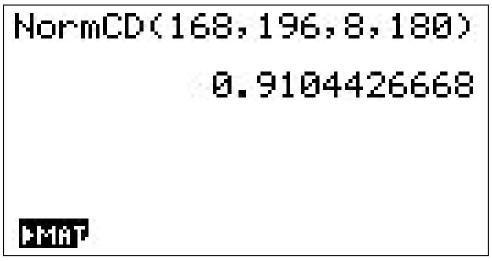
Example: Find $P(Z \leqslant 0.7) = P(-\infty < Z \leqslant 0.7)$, where $Z \sim \mathrm{N}(0,1)$ is the Standard Normal Distribution.

You can also do this calculation from the STAT screen: $\boxed{\text{DIST}}$ $\boxed{\text{NORM}}$ $\boxed{\text{Ncd}}$. This gives the screen below left. Put in the relevant values, scroll down to *Execute* and press $\boxed{\text{F1}}$ (CALC) or $\boxed{\text{EXE}}$ to do the calculation.



Example: Find $P(168 \leqslant X \leqslant 196)$, where $X \sim \mathrm{N}(180, 8^2)$.

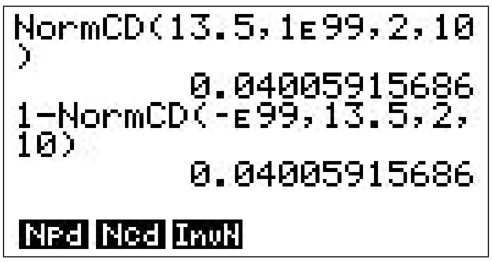
Normally you would rewrite this problem in terms of the Standard Normal Distribution, then use tables to find the answer. However, because we can specify the mean and standard deviation in the relevant calculator command, we can do this calculation in one go. Learn how to do both.



Example: Find $P(X \geqslant 13.5)$, where $X \sim \mathrm{N}(10, 2^2)$.

Two ways to do this.

(a) $P(X \geqslant 13.5) = P(13.5 \leqslant X < \infty)$ (first command below).

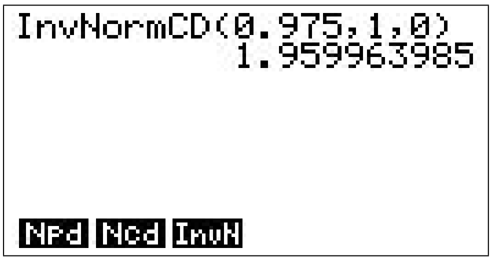


(b) $P(X \geqslant 13.5) = 1 - P(X \leqslant 13.5)$ (second command above).

Example: Find c such that $P(Z \leqslant c) = 0.975$, where $Z \sim \mathrm{N}(0,1)$ *an inverse-normal problem*

Here we could generate a table of normal-distribution cdf values using STAT on the calculator and search for the appropriate value.

However, there is an easier way. The command InvNormCD(M, σ, μ) (also in the $\boxed{\text{NORM}}$ menu) finds c such that $P(Z \leqslant c) = M$, the inverse of the normal-distribution function.



Exercises

1. If $Z \sim \mathrm{N}(0,1)$, find
- **(a)** $P(-1.5 \leqslant Z \leqslant 2)$
- **(b)** $P(Z \leqslant -0.8)$
- **(c)** $P(Z \geqslant 1.6)$
- **(d)** $P(-1 \leqslant Z \leqslant 1)$.

2. If $X \sim \mathrm{N}(100, 20)$, find
- **(a)** $P(88 \leqslant X \leqslant 112)$
- **(b)** $P(100 \leqslant X \leqslant 105)$
- **(c)** $P(X \leqslant 107)$
- **(d)** $P(X \geqslant 97)$.

3. If $Z \sim \mathrm{N}(0,1)$, find the value of z to 2 decimal places if
- **(a)** $P(Z \leqslant z) = 0.8413$
- **(b)** $P(Z \leqslant z) = 0.95$
- **(c)** $P(Z \geqslant z) = 0.9772$
- **(d)** $P(|Z| \leqslant 0.95)$.

You'll need to think geometrically here.

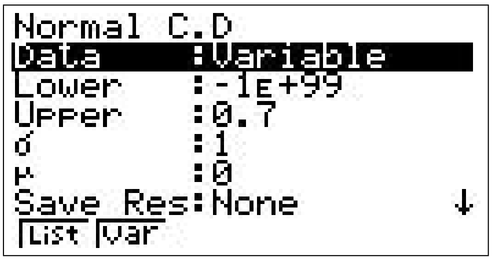
4. If $X \sim \mathrm{N}(10, 4)$, find the value of x to 1 decimal place if
- **(a)** $P(X \leqslant x) = 0.05$
- **(b)** $P(X \geqslant x) = 0.90$
- **(c)** $P(X \geqslant x) = 0.025$

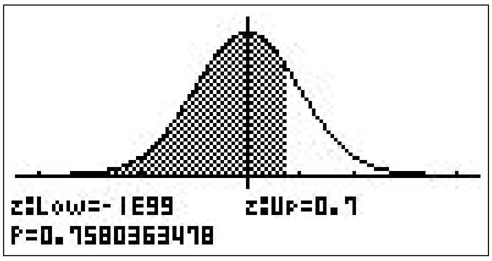
Here too.

B. Graphical representation

Example: Graph $P(Z \leqslant 0.7) = P(-\infty < Z \leqslant 0.7)$, where $Z \sim \mathrm{N}(0,1)$.

From the STAT screen, press $\boxed{\text{DIST}}$ $\boxed{\text{NORM}}$ $\boxed{\text{Ncd}}$ for the Normal cdf. This gives the screen below left. Fill in the relevant values, scroll down to *Execute* and press $\boxed{\text{DRAW}}$ to give the graph below right.





Exercises

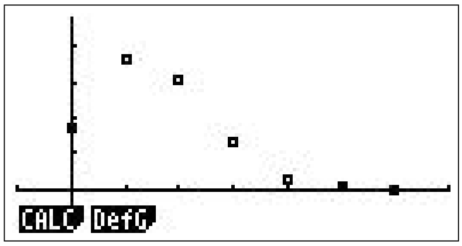
If $X \sim \mathrm{N}(60, 25)$, display each of the following probabilities as areas under the Normal pdf curve:
(a) $P(X \leqslant 55)$; **(b)** $P(X \geqslant 62.5)$; **(c)** $P(54 \leqslant X \leqslant 58)$.

C. Using the Normal Distribution to approximate a binomial distribution

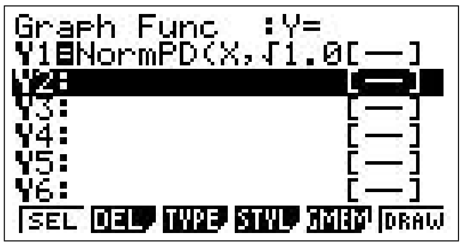
When n is large, the binomial distribution $\mathrm{bin}(n,p)$ can be approximated by the Normal distribution $\mathrm{N}\big(np,\ np(1-p)\big)$. This can be seen by plotting the two distributions on the one graph.

Example: Compare $\mathrm{bin}(5, 0.3)$ with $\mathrm{N}(1.5, 1.05)$.

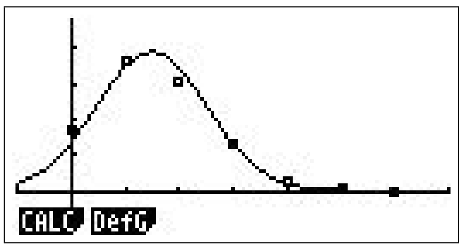
Construct a scatterplot of $\mathrm{bin}(5, 0.3)$.



Put $\mathrm{N}(1.5, 1.05)$ into Y1 as shown in the figure. Note that $\sigma = \sqrt{1.05}$.

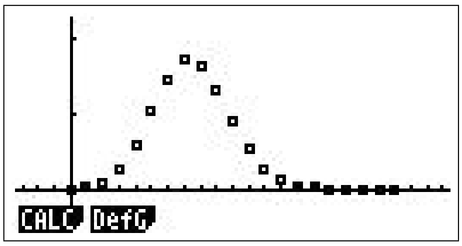


Graph the binomial data again, press $\boxed{\text{DefG}}$ (F2) to display the Graph Function screen, $\boxed{\text{SEL}}$ to turn on Y1, then $\boxed{\text{DRAW}}$ (F6) to plot the function over the data. A reasonable but not good approximation with $n = 5$.

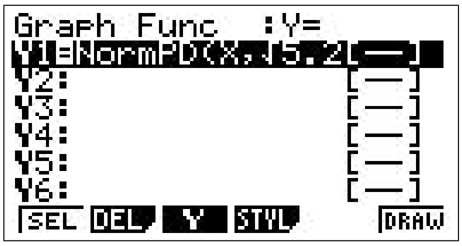


Example: Compare bin(25, 0.3) with N(7.5, 5.25).

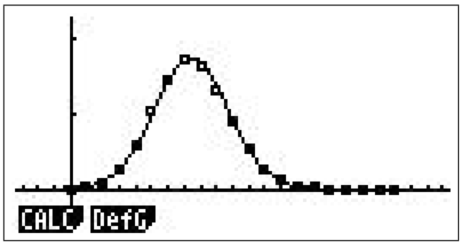
Construct a scatterplot of bin(25, 0.3).



Put N(7.5, 5.25) into Y1 as shown in the figure.
Note that $\sigma = \sqrt{5.25}$.



Plot both graphs.
A much better approximation with $n = 25$.



Exercise: Graphically compare the normal approximation to the binomial ($p = 0.5$) for $n = 5$ and $n = 20$.

17.3.2 The Exponential Distribution

A continuous random variable X with an Exponential Distribution has pdf

$$p(x) = \lambda e^{-\lambda x} \qquad 0 < x < \infty.$$

The Exponential Distribution is defined by a one parameter, λ. We write $X \sim \text{Exp}(\lambda)$.

The cdf $P(a \leqslant X \leqslant b)$ is given by

$$P(a \leqslant X \leqslant b) \;=\; \int_a^b \lambda e^{-\lambda x}\, dx \;=\; e^{-\lambda a} \;-\; e^{-\lambda b}.$$

Example: If $X \sim \text{Exp}(0.5)$, find $P(1 \leqslant X \leqslant 4)$.
$P(1 \leqslant X \leqslant 4) \;=\; e^{-0.5\times 1} - e^{-0.5\times 4} \;=\; e^{-0.5} - e^{-2} \;=\; 0.471$ to 3 significant digits.

Exercises

If $X \sim \text{Exp}(0.2)$, find to 4 decimal places: **(a)** $P(3 \leqslant X \leqslant 5)$; **(b)** $P(X \leqslant 5)$; **(c)** $P(X \geqslant 2)$.

17.4 Statistical inference

17.4.1 Hypothesis testing and confidence intervals for means

1. Single-sample mean drawn from a normal population: σ known

Testing the significance of the difference of a sample mean from a hypothesised population mean: Z test

Example: (JB) A sample of 25 workers in a factory spent, on average, 2.2 minutes to assemble the co-processor of a personal computer. Assuming that the assembly times are normally distributed with a standard deviation $\sigma = 0.4$ seconds, can we conclude that the average assembly time exceeds 2 minutes? Test at the 5% level of significance.

Steps

H_0: $\mu = 2$.

H_1: $\mu > 2$.

Significance level: 5%.

Test statistic: $Z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$.

Given: $\mu = 2$; $\sigma = 0.4$; $\bar{x} = 2.2$; and $n = 25$.

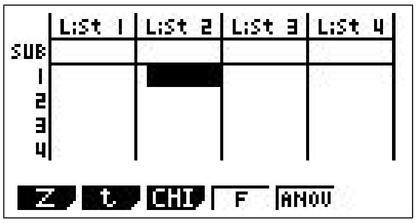
Then: $z = 2.5$ the observed value of Z, from the calculator (see below); and

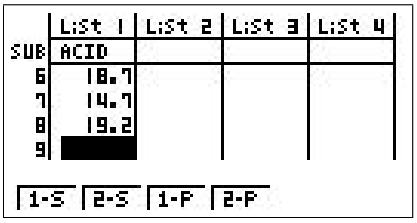
$p = P(Z \geqslant 2.5) = 0.0062$ from the calculator.

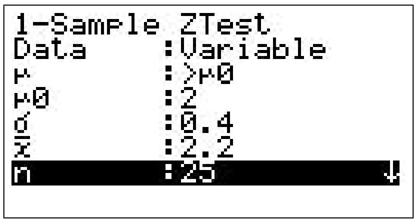
As $p < 0.05$, reject H_0 and conclude that, on average, assembly times exceed 2 minutes.

Calculator operations

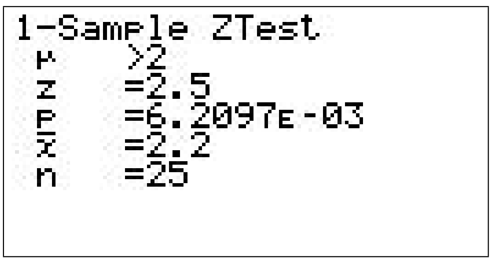
From the STAT screen, go to the TEST menu and select Z (below left), then 1-S for a single-sample Z test (below centre).

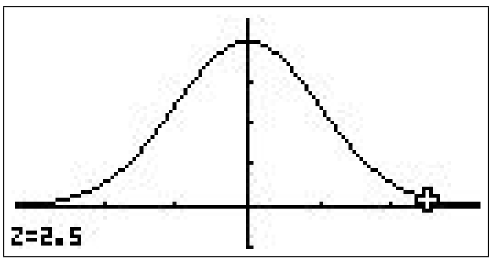






Complete the menu items as shown (above right): *Variable* because we are given the sample mean rather than the raw data; $>\mu_0$ because H_1: $\mu > 2$. Scroll down to *Execute* and press CALC (F1) or EXE to obtain the screen below left. If you select DRAW instead of CALC, the result appears graphically (below right): shading at the right-hand end of the curve. Pressing Z (F1) shows where the calculated Z is located.





Estimate the population mean from the sample mean

Use a 95% confidence interval to form an interval estimate for the population mean μ.

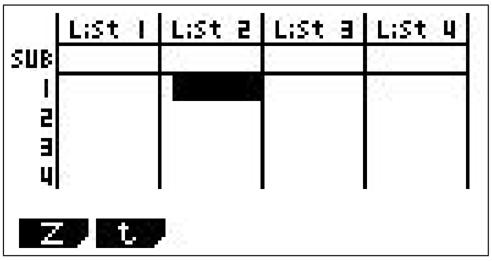
Steps

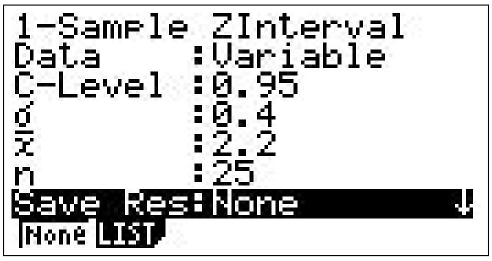
Given: $\sigma = 0.4$; $\bar{x} = 2.2$; $n = 25$; and C(onfidence)-Level 0.95.

From the calculator (see below), we are 95% confident that the interval $2.04 < \mu < 2.36$ captures the population mean.

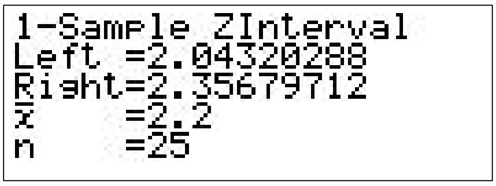
Calculator operations

After running the Z test above, from the top level of the STAT screen go to the $\boxed{\text{INTR}}$ menu and select $\boxed{\text{Z}}$ (below left), then $\boxed{\text{1-S}}$ for a single-sample Z test.





Complete the menu items as shown (above right) if necessary. Scroll down to *Execute* and press $\boxed{\text{CALC}}$ or $\boxed{\text{EXE}}$ to obtain the screen below giving the confidence interval.



2. Single-sample mean drawn from a normal population: σ unknown

Testing the significance of the difference of a sample mean from a hypothesised population mean: T test

Example: (JB) A study was conducted to see whether the ascorbic-acid (Vitamin C) content (mg/100g) of a certain brand of frozen vegetables was less than the recommended level of 18.6 mg/100g. Eight packets of frozen vegetables were randomly selected from the production line and were found to have the following ascorbic-acid levels:

14.3 19.1 16.3 15.8 16.2 18.7 14.7 19.2.

Assuming that assembly times are normally distributed, can we conclude that the average ascorbic-acid level of this brand of vegetables is less than 18.6 mg/100g?

Steps

H_0: $\mu = 18.6$.

H_1: $\mu < 18.6$.

Significance level: 5%.

Test statistic: $T = \dfrac{\bar{x}-\mu}{s/\sqrt{n}}$, with $n-1$ degrees of freedom.

Then, from the calculator (see below), $\bar{x} = 16.79$; $s = 1.961$; and $n = 8$, giving

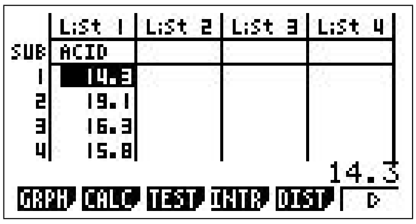
$t = -2.614$, the observed value of T with df $= n-1 = 7$, and

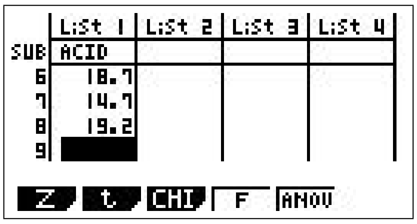
$p = P(T \leqslant 2.614) = 0.017$.

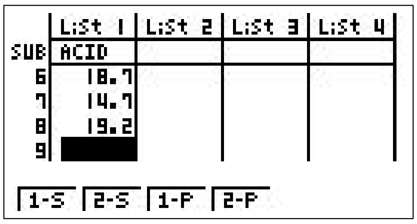
As $p < 0.05$, reject H_0 and conclude that the mean ascorbic-acid level of this brand of vegetables is less than 18.6 mg/100g.

Calculator operations

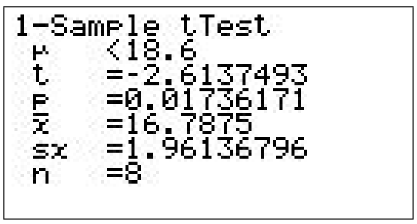
Enter the data into List 1 called ACID (below left). Go to the $\boxed{\text{TEST}}$ menu and select $\boxed{\text{t}}$ (below centre), then $\boxed{\text{1-S}}$ for a single-sample t test (below right).

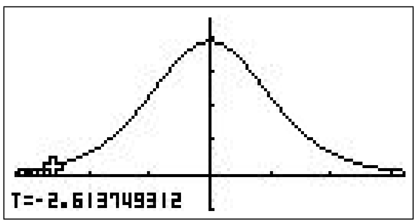






Complete the menu items as shown (below left). Scroll down to *Execute* and press $\boxed{\text{CALC}}$ or $\boxed{\text{EXE}}$ to obtain the screen below centre If you select $\boxed{\text{DRAW}}$ instead of $\boxed{\text{CALC}}$, the result appears graphically (below right): shading at the left-hand end of the curve. Pressing $\boxed{\text{T}}$ (F1) shows where the calculated T is located.





Estimate the population mean from the sample mean

Use a 95% confidence interval to form an interval estimate for the mean ascorbic-acid level μ of the brand of vegetables.

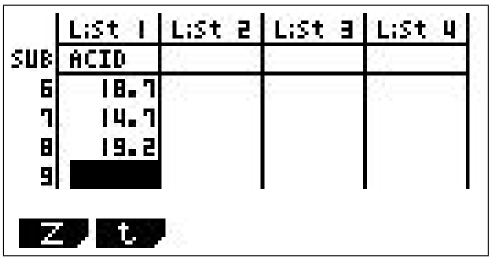
Steps

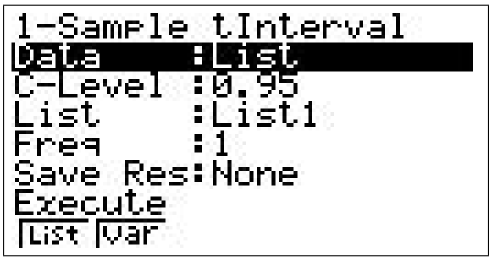
Given: C(onfidence)-Level 0.95.

From the calculator (see below), we are 95% confident that the interval $15.1 < \mu < 18.4$ captures the population mean.

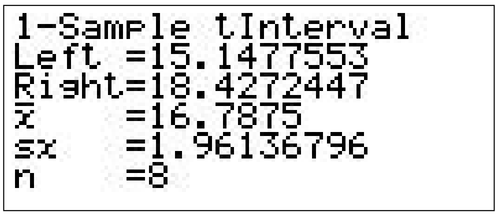
Calculator operations

After running the t test above, from the top level of the STAT screen go to the INTR menu and select t (below left), then 1-S for a single-sample t test.





Complete the menu items as shown (above right) if necessary. Scroll down to *Execute* and press CALC or EXE to obtain the screen below giving the confidence interval.



17.4.2 Hypothesis testing and confidence intervals for population proportion

1. Single-sample proportion

Significance test using the binomial distribution (exact test)

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 15 components contains 4 faulty ones. Does this challenge the manufacturer's claim? Test at the 5% level of significance.

Steps

H_0: $\pi = 0.10$.

H_1: $\pi > 0.10$.

Significance level: 5%.

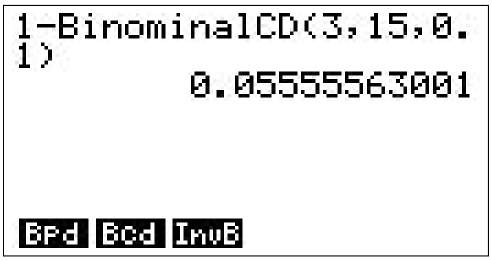
Let X be the number of faulty components.

Then, under the null hypothesis, $X \sim \text{bin}(15, 0.10)$.

Then, from the calculator (see below), $p = P(X \geqslant 4) = 1 - P(X \leqslant 3) = 0.056$.

As $p > 0.05$, do not reject H_0. There is not sufficient evidence that the percentage of faulty components exceeds 10%.

Calculator operation



Significance test using the normal approximation to the binomial distribution:
Z test with no continuity correction

For n large, $n\pi > 10$ and $n(1-\pi) > 10$, the binomial distribution $\text{bin}(n, \pi)$ can be approximated by a normal distribution with $\mu = n\pi$ and $\sigma^2 = n\pi(1-\pi)$ (Section 17.3.1).

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 120 components contains 23 faulty ones. Does this challenge the manufacturer's claim? Test at the 5% level of significance.

Steps

H_0: $\pi = 0.10$.

H_1: $\pi > 0.10$.

Significance level: 5%.

Given: $x = 23$; $n = 120$.

Test statistic: $Z = \dfrac{x - n\pi}{s\sqrt{n\pi(1-\pi)}}$.

Then, from the calculator (see below), the observed value of Z, $z = 3.347$, and

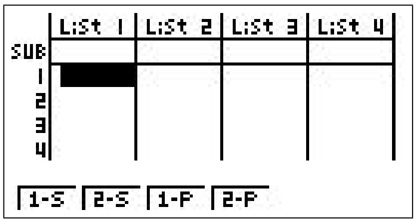
$p = P(Z > 3.347) = 0.0004$ to 4 decimal places.

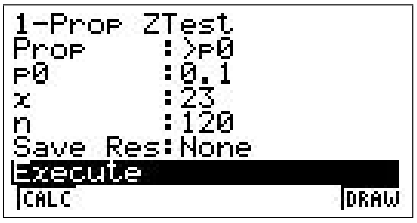
As $p < 0.05$, reject H_0, and conclude that the percentage of faulty components exceeds 10%.

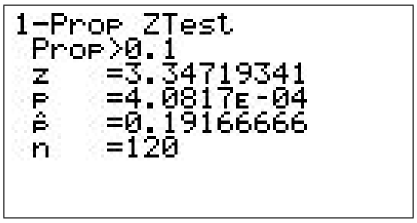
Calculator operation

From the STAT screen, go to the $\boxed{\text{TEST}}$ menu and select $\boxed{\text{Z}}$, then $\boxed{\text{1-P}}$ for a single-sample proportion Z test (below left).

Complete the menu items as shown (below centre), scroll down to *Execute* and press $\boxed{\text{CALC}}$ or $\boxed{\text{EXE}}$ to obtain the screen below right.







Significance test using the normal approximation to the binomial distribution: Z test with continuity correction

For n large, $n\pi > 10$ and $n(1-\pi) > 10$, the binomial distribution can be approximated by a normal distribution with $\mu = n\pi$ and $\sigma^2 = n\pi(1-\pi)$ (Section 17.3.1). The approximation can be improved by using the continuity correction.

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 120 components contains 23 faulty ones. Does this challenge the manufacturer's claim? Test at the 5% level of significance.

Steps

H_0: $\pi = 0.10$.

H_1: $\pi > 0.10$.

Significance level: 5%.

Given: $x = 23$; $n = 120$.

Test statistic: $Z = \dfrac{x - n\pi}{\sqrt{n\pi(1-\pi)}}$.

Given $x = 22.5$ (taking into account the continuity correction), $\pi = 0.10$ and $n = 120$, we find from the calculator (see below) that the observed value of Z,

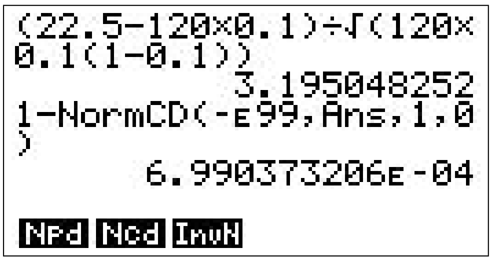
$$z = \frac{22.5 - 120\times 0.1}{\sqrt{120\times 0.1(1-0.1)}} = 3.195,$$

and

$p = P(Z > 3.195) = 1 - P(Z < 3.195) = 0.0007$ to 4 decimal places.

As $p < 0.05$, reject H_0, and conclude that the percentage of faulty components exceeds 10%.

Calculator operation



Estimating the population proportion from a sample proportion: normal approximation

Example: (JB) An electronic-component manufacturer claims that fewer than 10% of components have manufacturing faults. A random sample of 120 components contains 23 faulty ones. Use a 95% confidence interval to form an interval estimate for the true proportion of components that have manufacturing faults.

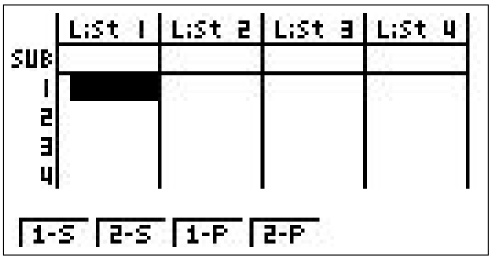
Steps

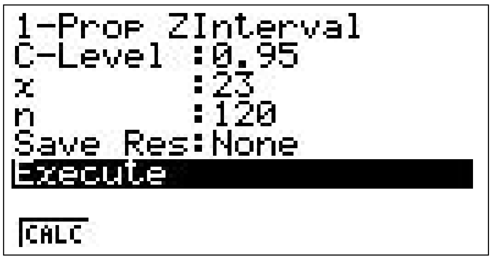
Given: C(onfidence)-Level 0.95.

From the calculator (see below), we are 95% confident that the interval $0.12 < \pi < 0.26$ captures the true population proportion.

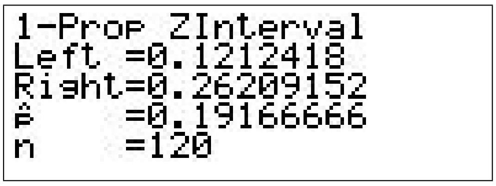
Calculator operations

From the top level of the STAT screen, go to the $\boxed{\text{INTR}}$ menu and select $\boxed{\text{Z}}$, then $\boxed{\text{1-P}}$ (below left).





Complete the menu items as shown (above right) if necessary. scroll down to *Execute* and press $\boxed{\text{CALC}}$ or $\boxed{\text{EXE}}$ to obtain the the confidence interval below.



17.5 Solutions to exercises

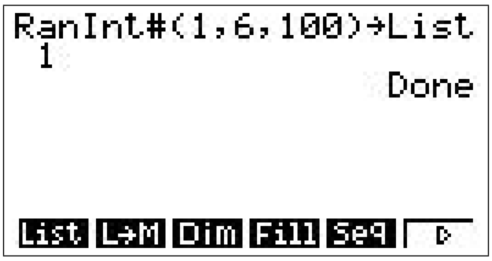
Exercises page 25

1. Evaluate each of the following by hand, then check with your calculator.

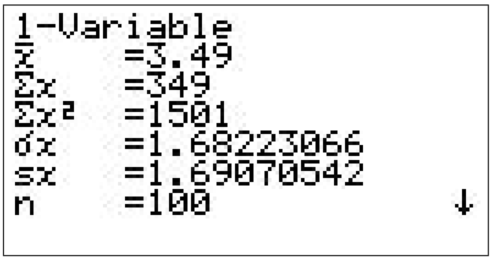
   (a) $6! = 720$; **(a)** ${}^{6}P_{4} = 360$; **(a)** ${}^{8}C_{5} = 56$.

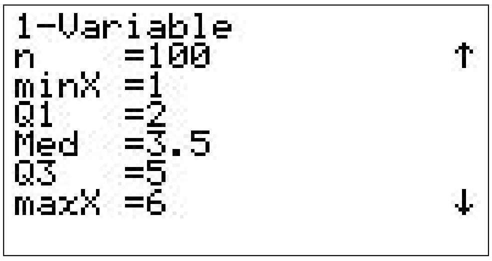
2. **(a)** $13! = 6,227,020,800$; **(b)** ${}^{15}P_{9} = 1,816,214,400$; **(c)** ${}^{35}C_{2} = 1,476,337,800$.

3. Generate 100 random integers between 1 and 6, simulating throwing a die. Store them in an appropriately named list.



   From 1-Var Stats, **(a)** the mean is 3.49; **(b)** the standard deviation is 1.68; and **(c)** the median is 3.5. Answers will vary here.





Exercises page 29

1. If $X \sim \text{bin}(8, 0.4)$, to 4 decimal places,

   (a) $P(X=4) = 0.2322$

   (b) $P(X \leqslant 1) = 0.1064$

   (c) $P(X \geqslant 7) = 0.0085$

   (d) $P(1 \leqslant X \leqslant 3) = 0.5773$

   (e) $P(X<6) = P(X \leqslant 5) = 0.9502$.

2. If $X \sim \text{bin}(25, 0.2)$, to 4 decimal places,

   (a) $P(X=4) = 0.1867$

   (b) $P(X \leqslant 10) = 0.9944$

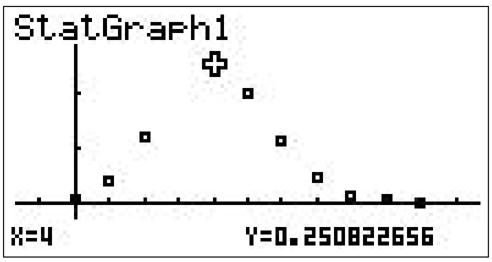
   (c) $P(X \geqslant 7) = 0.2200$

   (d) $P(5 \leqslant X \leqslant 8) = 0.5326$

   (e) $P(X>4) = 1 - P(X \leqslant 4) = 0.5793$.

Exercises page 30

1. If $X \sim \text{bin}(10, 0.4)$, display the distribution of X graphically.



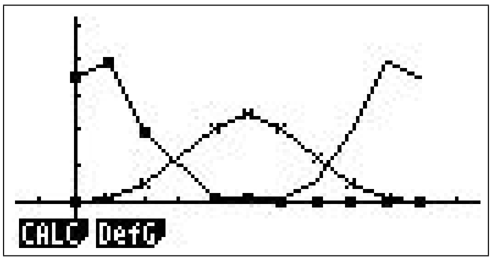
2. On the one graph, plot the distribution of:

$X_1 \sim \text{bin}(10, 0.1)$ use StatGraph1 with a large-square marker;

$X_2 \sim \text{bin}(10, 0.5)$ use StatGraph2 with a $\times$ marker;

$X_3 \sim \text{bin}(10, 0.9)$ use StatGraph3 with a small-square marker.

Here, I've done line graphs rather than scatterplots to show the three distributions better.



Comment on how changing the value of p changes the shape of the distribution.

As p increases, the peak of the distribution moves to the right.

Exercises page 32

If $X \sim \text{geom}(0.2)$, to 4 decimal places,

(a) $P(X=5)=0.0819$

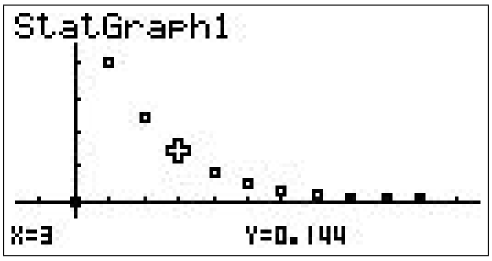
(b) $P(X \leqslant 8)=0.8322$

(c) $P(X \geqslant 7)=0.2621$

(d) $P(3 \leqslant X \leqslant 9)=0.5058$.

Exercises page 32

1. If $X \sim \text{geom}(0.4)$, display the distribution of X graphically for $1 \leqslant x \leqslant 10$.

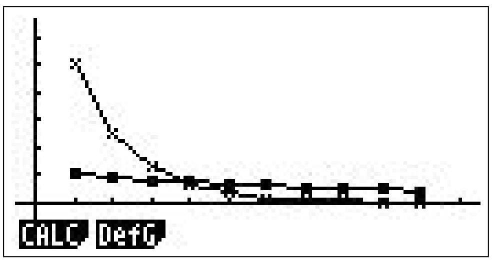


2. On the one graph, plot the distribution of:

$X_1 \sim \text{geom}(0.1)$ use StatGraph1 with a large-square marker;

$X_2 \sim \text{geom}(0.5)$ use StatGraph2 with a $\times$ marker;

Here, I've done line graphs rather than scatterplots to show the two distributions better.



Comment on how changing the value of p changes the shape of the distribution.

As p increases, the distribution has a larger initial value but decays more rapidly.

Exercises page 33

If $X \sim \text{hypg}(100, 15, 5)$, find

(a) $P(X = 3) = 0.0216$

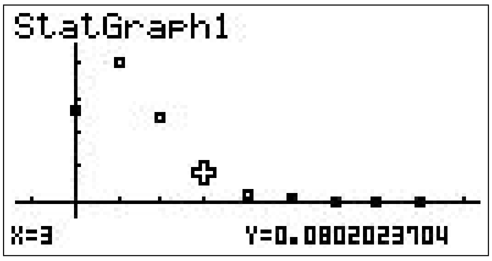
(b) $P(X \leqslant 3) = 0.9984$

(c) $P(X > 3) = 0.0016$

(d) $P(1 \leqslant X \leqslant 2) = 0.5411$.

Exercises page 34

1. If $X \sim \text{hypg}(8, 15, 100)$, display the distribution of X graphically for $0 \leqslant x \leqslant 8$.

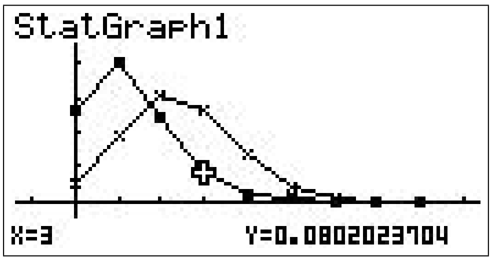


2. On the one graph, plot the distribution of:

$X \sim \text{hypg}(8, 15, 100)$ use StatGraph1 with a large-square marker;

$X \sim \text{hypg}(8, 30, 100)$ use StatGraph2 with a $\times$ marker;

Here, I've done line graphs rather than scatterplots to show the two distributions better.



V-Window $[0, 9, 1] \times [0, 0.5, 0.1]$

Comment on how changing the value of M changes the shape of the distribution.

As M increases, the distribution has a smaller initial value, and a smaller peak, shifted to the right.

Exercises page 36

If $X \sim \text{pois}(0.5)$, to 4 decimal places,

(a) $P(X = 0) = 0.6065$

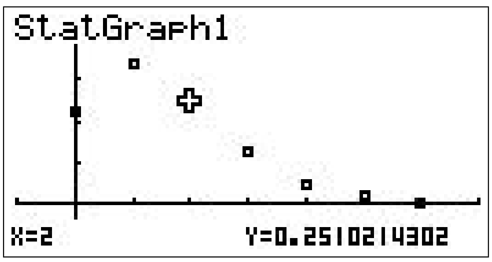
(b) $P(X \leqslant 3) = 0.9982$

(c) $P(X \geqslant 1) = 0.3935$

(d) $P(1 \leqslant X \leqslant 4) = 0.3933$.

Exercises page 36

1. If $X \sim \text{pois}(1.5)$, display the distribution of X graphically for $0 \leqslant x \leqslant 6$.



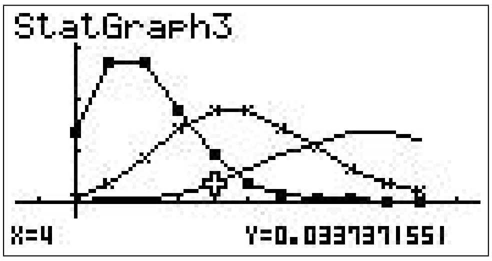
2. On the one graph, for $0 \leqslant x \leqslant 10$, plot the distribution of:

$X_1 \sim \text{pois}(2)$ use StatGraph1 with a large-square marker;

$X_2 \sim \text{pois}(5)$ use StatGraph2 with a $\times$ marker;

$X_2 \sim \text{pois}(9)$ use StatGraph2 with a small-square marker.

Here, I've done line graphs rather than scatterplots to show the three distributions better.



Comment on how changing the value of λ changes the shape of the distribution.

The distribution changes from positively skewed to symmetric, while the peak moves to the right.

Exercises page 39

1. If $Z \sim \text{N}(0,1)$, to 4 decimal places,

(a) $P(-1.5 \leqslant Z \leqslant 2) = 0.9104$

(b) $P(Z \leqslant -0.8) = 0.2119$

(c) $P(Z \geqslant 1.6) = 0.0548$

(d) $P(-1 \leqslant Z \leqslant 1) = 0.6827$.

2. If $X \sim \text{N}(100, 20)$, to 4 decimal places,

(a) $P(88 \leqslant X \leqslant 112) = 0.9927$

(b) $P(100 \leqslant X \leqslant 105) = 0.3682$

(c) $P(X \leqslant 107) = 0.9412$

(d) $P(X \geqslant 97) = 0.7488$.

3. If $Z \sim \text{N}(0,1)$, find the value of z to 2 decimal places if

(a) $P(Z \leqslant z) = 0.8413$: $x = 1.00$

(b) $P(Z \leqslant z) = 0.95$: $x = 1.64$

(c) $P(Z \geqslant z) = 0.9772$: $x = -2.00$

(d) $P(|Z| \leqslant 0.95)$: $x = 1.96$.

4. If $X \sim \text{N}(10, 4)$, find the value of x to 1 decimal place if

(a) $P(X \leqslant x) = 0.05$: $x = 6.7$

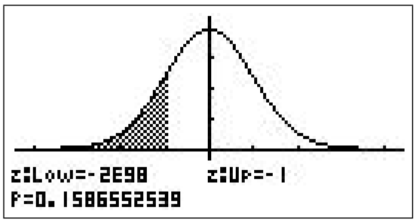
(b) $P(X \geqslant x) = 0.90$: $x = 7.4$

(c) $P(X \geqslant x) = 0.025$: $x = 13.9$

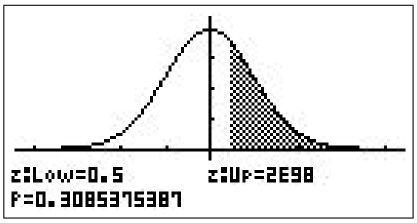
Exercises page 40

If $X \sim \mathrm{N}(60, 25)$, display each of the following probabilities as an area under the normal pdf curve.

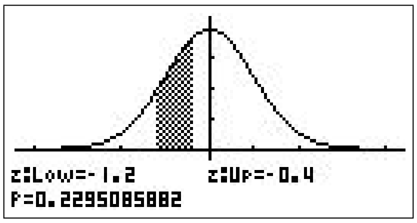
(a) $P(X \leqslant 55)$



(b) $P(X \geqslant 62.5)$

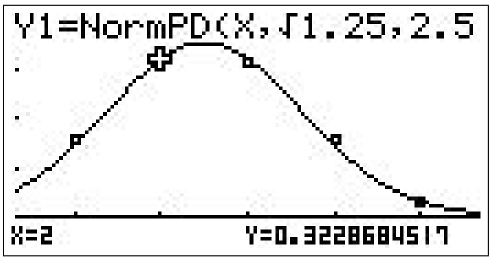


(c) $P(54 \leqslant X \leqslant 58)$

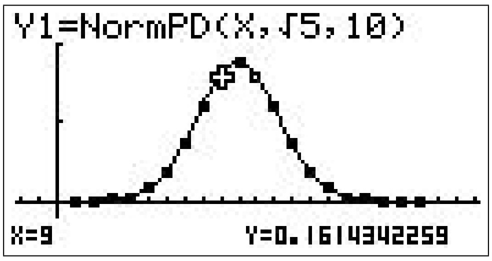


Graphically compare the normal approximation to the binomial pdf for $p = 0.5$ when $n = 5$ and $n = 20$.

$\boldsymbol{n = 5}$



$\boldsymbol{n = 20}$



Exercises page 41

If $X \sim \mathrm{Exp}(0.2)$, to 4 decimal places,

(a) $P(3 \leqslant X \leqslant 5) = 0.1809$ **(b)** $P(X \leqslant 5) = 0.6321$ **(c)** $P(X \geqslant 2) = 0.6703$

18 Matrix and Vector Operations

The material here is directly relevant to the topics *Vectors in the plane*, *Vectors in three dimensions* and *Matrices* in the Australian Curriculum for Specialist Mathematics.

As with all technology, you should do the first few calculations by hand to make sure you understand the method. Use the technology firstly to check your hand calculations and then when the numbers make hand calculation difficult or prone to error (often), or when the calculations are a relatively minor part of a larger process.

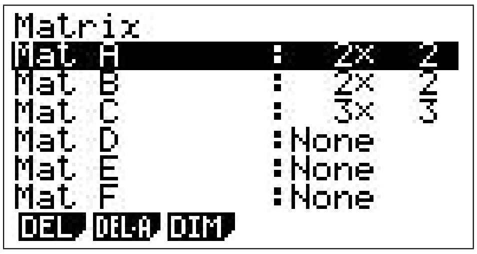
18.1 The basics

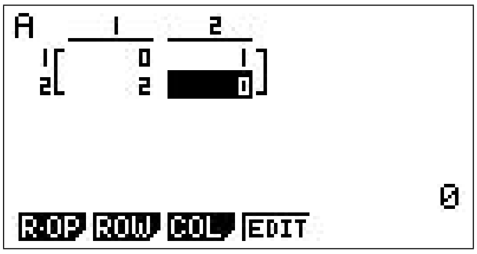
The basic matrix operations on the calculator work just like the number operations with one or two minor exceptions.

Let's start out with matrices $\underline{\underline{\boldsymbol{A}}} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, $\underline{\underline{\boldsymbol{B}}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $\underline{\underline{\boldsymbol{C}}} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$.

18.1.1 Putting matrices in your calculator

Press $\boxed{\text{MENU}}$ $\boxed{\text{1}}$ $\boxed{\text{F1}}$ (►MAT) to reach the matrix input menu. You are now ready to edit one of the twenty-six matrices Mat A – Mat Z.





With Mat A highlighted, select $\boxed{\text{F3}}$ (DIM), input the order (rows × columns), then the elements of Mat A, with $\boxed{\text{EXE}}$ after each.

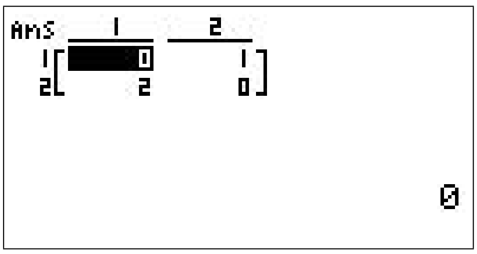
Press $\boxed{\text{EXIT}}$ and input Mat B the same way.

Press $\boxed{\text{EXIT}}$ to return to the RUN-MAT screen.

Press $\boxed{\text{OPTN}}$ $\boxed{\text{F2}}$ to select the matrix operations menu.

18.1.2 Displaying matrices

You call up matrices by pressing $\boxed{\text{F1}}$ (Mat) and the appropriate letter. For example, pressing $\boxed{\text{F1}}$ $\boxed{\text{ALPHA}}$ $\boxed{\text{A}}$ will display 'Mat A' on the screen. Pressing $\boxed{\text{EXE}}$ will show you the numbers in Mat A.

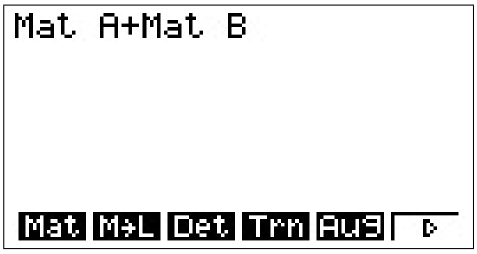


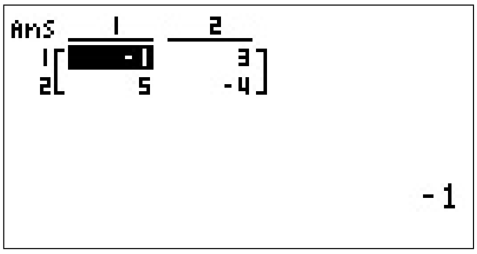
Press $\boxed{\text{EXIT}}$ to return to the RUN-MAT screen.

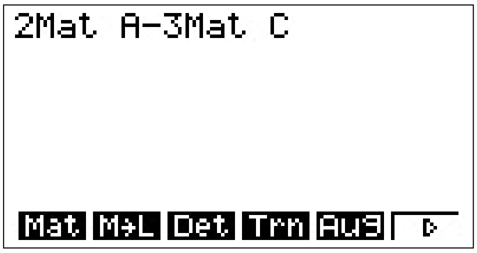
From now on, we omit the $\boxed{\text{ALPHA}}$ before letters.

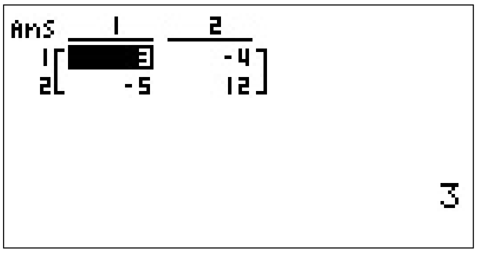
18.1.3 Adding and subtracting matrices

Matrices can only be added or subtracted if they have the same dimensions, that is the same number of rows and the same number of columns. We can add/subtract multiples of Mat A and Mat C but not Mat B with either Mat A or Mat C.





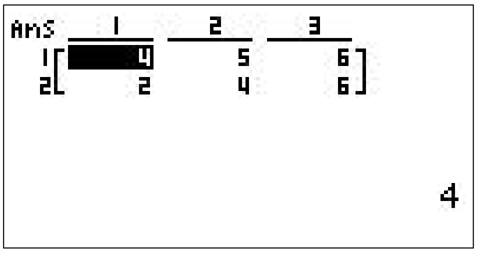




18.1.4 Multiplying matrices

Press $\boxed{\text{F1}}$ $\boxed{\text{A}}$ $\boxed{\text{F1}}$ $\boxed{\text{B}}$ to display 'Mat A Mat B'; press $\boxed{\text{EXE}}$ to display the numerical result

$$\underline{\underline{\boldsymbol{AB}}} = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}.$$



The result is stored in Mat Ans. Press $\boxed{\text{EXIT}}$ to return to the RUN-MAT screen. If you now want to pre-multiply by $\underline{\underline{\boldsymbol{A}}}$) again, i.e. find $\underline{\underline{\boldsymbol{A}}}\big(\underline{\underline{\boldsymbol{AB}}}\big)$, press $\boxed{\text{F1}}$ $\boxed{\text{A}}$ $\boxed{\text{F1}}$ $\boxed{\text{Ans}}$ $\big($on the $\boxed{(-)}$ key$\big)$ to display 'Mat A Mat Ans'; press $\boxed{\text{EXE}}$ to display the answer

$$\underline{\underline{\boldsymbol{A}}}\big(\underline{\underline{\boldsymbol{AB}}}\big) = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

18.1.5 Squaring matrices

To evaluate $\underline{\underline{\boldsymbol{A}}}\big(\underline{\underline{\boldsymbol{AB}}}\big)$, we can also evaluate $\underline{\underline{\boldsymbol{A}}}^2\underline{\underline{\boldsymbol{B}}}$.

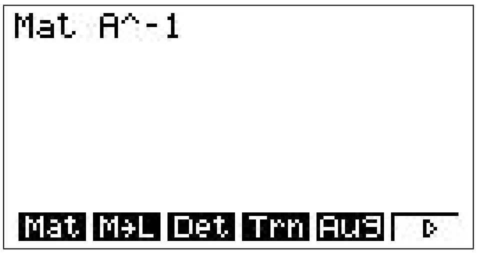
Press $\boxed{\text{F1}}$ $\boxed{\text{A}}$ $\boxed{x^2}$ $\boxed{\text{F1}}$ $\boxed{\text{B}}$ to display 'Mat A^2 Mat B' and $\boxed{\text{EXE}}$ to give the same result

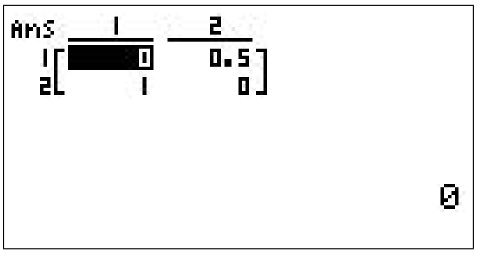
$$\underline{\underline{\boldsymbol{A}}}^2\underline{\underline{\boldsymbol{B}}} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Integer powers of a matrix are produced the same way as with numbers (for positive integers). Only square matrices can be raised to a power.

$$\underline{\underline{\boldsymbol{A}}}^4 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

The only negative integer 'power' that works is -1: however, Mat A^-1 produces the **inverse** of (square) matrix $\underline{\underline{\boldsymbol{A}}}$ (see Section 18.3). The reciprocal of a matrix is not defined.



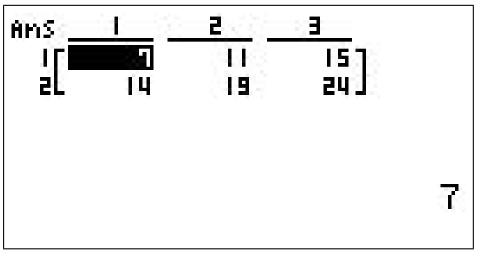


18.1.6 More-involved expressions

Extensions work just as you would expect. For example, to work out $\underline{\underline{\boldsymbol{A}}}\underline{\underline{\boldsymbol{B}}}+3\underline{\underline{\boldsymbol{B}}}$, press

F1 A F1 B + 3 F1 B to display 'Mat A Mat B + 3 Mat B', and EXE to give

$$\underline{\underline{\boldsymbol{A}}}\underline{\underline{\boldsymbol{B}}} + 3\underline{\underline{\boldsymbol{B}}} = \begin{bmatrix} 7 & 11 & 15 \\ 14 & 19 & 24 \end{bmatrix}.$$



18.1.7 Storing matrices

If you wanted to keep the previous answer for later use, you might store it in Mat C by pressing → F1 C EXE instead of the final EXE above.

If you have already done the calculation, 'Mat Ans → Mat C' will do the same thing.

Note that the calculator automatically makes Mat C have the correct order.

18.1.8 Illegal operations

The calculator will not let you do invalid operations. For example, if you try to calculate $\underline{\underline{\boldsymbol{A}}}+\underline{\underline{\boldsymbol{B}}}$ by entering F1 A + F1 B, you will see 'Mat A + Mat B' on the screen, but pressing EXE produces an error message. *Why?*

18.1.9 Other matrix operations

A number of operations are contained in the $\boxed{\text{OPTN}}$ $\boxed{\text{MAT}}$ menu, accessed from the RUN-MAT screen.

- *determinant*: $\boxed{\text{F3}}$ $\boxed{\text{F1}}$ $\boxed{\text{A}}$ gives $\text{Det Mat A} = -2$.
- *transpose*: $\boxed{\text{F4}}$ $\boxed{\text{F1}}$ $\boxed{\text{B}}$ gives $\text{Trn Mat B} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.
- *dimension*: $\boxed{\text{F6}}$ $\boxed{\text{F2}}$ $\boxed{\text{F6}}$ $\boxed{\text{F1}}$ $\boxed{\text{B}}$ gives $\text{Dim Mat B} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
- *'filling' a matrix*: $\boxed{\text{F6}}$ $\boxed{\text{F3}}$ $\boxed{1}$ $\boxed{,}$ $\boxed{\text{F6}}$ $\boxed{\text{F1}}$ $\boxed{\text{A}}$ gives $\text{Fill}(1, \text{Mat A})$, which produces a matrix of '1's.

  Press $\boxed{\text{F1}}$ $\boxed{\text{A}}$ $\boxed{\text{EXE}}$ to see the result $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

  Note that the dimension of Mat A must be specified first.
- *identity matrix*: $\boxed{\text{F6}}$ $\boxed{\text{F1}}$ n produces the $n \times n$ identity matrix.
- *row-echelon form*: $\boxed{\text{F6}}$ $\boxed{\text{F4}}$ $\boxed{\text{F6}}$ $\boxed{\text{F1}}$ $\boxed{\text{A}}$ gives Ref Mat A, the row-echelon form of $\underline{\underline{\boldsymbol{A}}}$ (Gaussian reduction — see Section 18.2)
- *reduced-row-echelon form*: $\boxed{\text{F6}}$ $\boxed{\text{F5}}$ $\boxed{\text{F6}}$ $\boxed{\text{F1}}$ $\boxed{\text{A}}$ gives Rref Mat A, the reduced row-echelon form of $\underline{\underline{\boldsymbol{A}}}$ (Gauss-Jordan reduction — see Section 18.2).

See the calculator manual for more details on the other $\boxed{\text{OPTN}}$ $\boxed{\text{MAT}}$ menu items.

18.2 Gauss Elimination

18.2.1 The method

The endpoint of Gauss elimination or Gauss reduction is a matrix in **row-echelon** form, characterised by:

(a) the first non-zero element in each row is a 1 (called a **pivot**);

(b) all elements in the column below a pivot are 0.

To reduce a matrix to row-echelon form, we use **elementary row operations**.[18] The three elementary row operations are

(A) exchange two rows ($R_i \rightleftharpoons R_j$)

(B) multiply a row by a constant ($R_i \to cR_i,\ c \neq 0$)

(C) add a constant multiple of one row to another ($R_i \to R_i + cR_j,\ c \neq 0$).

Procedure

1. If the first column of the matrix is all zeros, "cross" out this column to leave a smaller matrix.
2. If the element in the first row and first column (top left element) of the matrix is 0, exchange Row 1 with another row (Operation A).

   Sometimes it is convenient to do this even if the top left element is non-zero, so as to avoid fractions in Step 3.
3. Multiply the top row by a constant to make the first element 1 (Operation B).
4. Use Operation C to obtain '0's **below** the 1 (pivot) by adding multiples of **Row 1** to each successive row ($R_i \to R_i + cR_1$, $i = 2, 3, \dots$).
5. "Cross out" the first row and first column to leave a smaller matrix.
6. Go back and start at Step 1 on this smaller matrix.

Notes

- The only choice in this version of Gauss elimination is which rows to interchange in Step 2. All the other operations are prescribed.
- The solutions to the simultaneous equations described by the row-echelon matrix are the same as those of the original matrix, hence the usefulness of the method. Write down the equations corresponding to the row-echelon matrix and use back substitution to find the solutions.
- The matrix changes wherever we perform an elementary row operation, so that we cannot use equal signs between the steps. Use a $\sim$ instead. The matrices are said to be row-equivalent.
- We can proceed in a similar manner to obtain '0' above the pivots too, the reduced row-echelon matrix. This is called *Gauss-Jordan elimination.* The solution can be read directly from the final matrix, without the need for back substitution.

[18] Each row operation corresponds to the multiplication of the matrix by a corresponding elementary matrix.

18.2.2 Using the calculator

Casio calculators have two built-in commands in the $\boxed{\text{OPTN}}$ $\boxed{\text{MAT}}$ menu: *Ref* ($\boxed{\text{F6}}$ $\boxed{\text{F4}}$) and *Rref* ($\boxed{\text{F6}}$ $\boxed{\text{F5}}$).

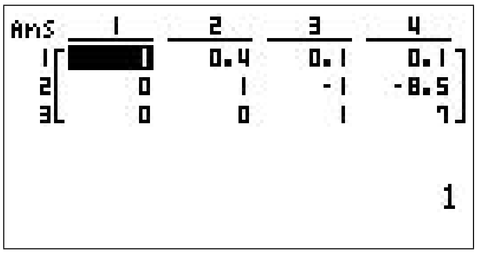
Ref *matrix*, where *matrix* is one of the twenty-six matrices used by the calculator, produces the row-echelon form of *matrix* (Gauss elimination).

Rref *matrix* produces the reduced row-echelon form of *matrix* (Gauss-Jordan elimination).

The GAUSS program[19] just automates these proceedures. Enter the matrix into Mat A, then run the program from the PRGM menu ($\boxed{\text{MENU}}$ $\boxed{\text{B}}$). Press $\boxed{\text{EXE}}$ to generate successively the row-echelon matrix and the reduced row-echelon matrix.

Examples: Gauss elimination

$$\begin{bmatrix} 10 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.4 & 0.1 & 0.1 \\ 0 & 1 & -1 & -8.5 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2/5 & 1/0 & 1/10 \\ 0 & 1 & -1 & -17/2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$



Using the *Ref* command in the $\boxed{\text{OPTN}}$ $\boxed{\text{MAT}}$ menu

$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.\dot{3} & 0.1\dot{6} & 0.\dot{6} \\ 0 & 1 & 0.25 & 0.25 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 1/6 & 2/3 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

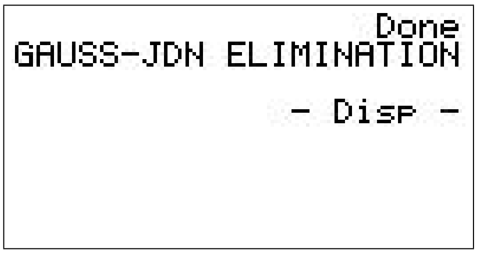
$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0.25 & 0.25 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

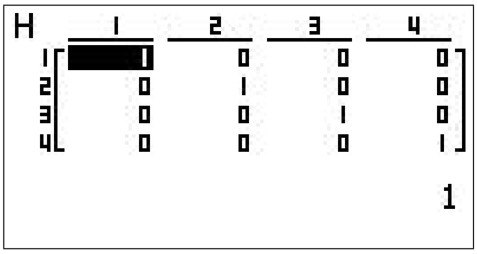
PTO

[19] available on the CMA website *canberramaths.org.au* under *Resources*

Example: Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 5 & 1 & 3 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$





Using the GAUSS program

18.3 Exercises

Work through the matrix operations in the first section of these notes, if you haven't already done so.

1. Enter the matrix below carefully into Mat A using $\boxed{\text{F1}}$ (MAT) on the RUN-MAT screen.

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

Mat A is the augmented matrix $\left[\underline{\underline{A}} \,|\, \underset{\sim}{b}\right]$ for the following system of equations: in matrix form $\underline{\underline{A}}\underset{\sim}{x} = \underset{\sim}{b}$.

$$\begin{aligned} x_2 + \ x_3 - 2x_4 &= \ -3 \\ x_1 + 2x_2 - \ ,x_3 \qquad &= \quad 2 \\ 2x_1 + 4x_2 + \ x_3 - 3x_4 &= \ , -2 \\ x_1 - 4x_2 - 7x_3 - \ x_4 &= -19. \end{aligned}$$

Solve this system

A. using the calculator to find the row-echelon form (Gaussian reduction), and then back substitution (you have to do this). In the result, you can convert individual entries to fractions using the $\boxed{\text{F}\leftrightarrow\text{D}}$ key.

B. using the calculator to find the reduced row-echelon form (Gauss-Jordan reduction), from which you can read off the answer.

C. using $\underset{\sim}{\boldsymbol{x}} = \underline{\underline{\boldsymbol{A}}}^{-1}\underset{\sim}{\boldsymbol{b}}$. Do you understand why this works?

A little manipulation on the calculator allows us to evaluate $\underline{\underline{\boldsymbol{A}}}^{-1}\underset{\sim}{\boldsymbol{b}}$.

Change Mat A to the matrix $\underline{\underline{\boldsymbol{A}}}$ using $\boxed{\text{F1}}$ (MAT). Highlight Mat A and press $\boxed{\text{EXE}}$.
Press $\boxed{\text{F3}}$ (COL), move the cursor to Column 5 and press $\boxed{\text{F1}}$ (DEL).
Store the column matrix $\underset{\sim}{\boldsymbol{b}}$ in Mat B, then evaluate Mat A^(−1) Mat B.

2. Solve the systems of equations below using each of the three methods. Check your answers by substituting them back into the equations.

Can you use Method 2(c) to solve these? What happens in (b) and (c) below? Can you explain?

What is $\det\left(\underline{\underline{\boldsymbol{A}}}\right)$? What does this tell you about the inverse matrix?

(a)
$$\begin{aligned} 2x - 5y + 5z &= 17 \\ x - 2y + 3z &= 9 \\ -x + 3y \quad &= -4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 \quad - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 \quad + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

18.4 Eigenvalues and eigenvectors

18.4.1 Theory

The eigenvalues of an $n \times n$ matrix $\underline{\underline{\boldsymbol{A}}}$ are the constants λ, and the eigenvectors of $\underline{\underline{\boldsymbol{A}}}$ are the corresponding n-dimensional vectors $\underset{\sim}{\boldsymbol{v}}$ ($n \times 1$ matrices) satisfying the equation

$$\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}} \;=\; \lambda \underset{\sim}{\boldsymbol{v}}.$$

The eigenvalues λ are the zeros or roots of the nth-degree characteristic polynomial

$$\det\left(\underline{\underline{\boldsymbol{A}}} - \lambda \underline{\underline{\boldsymbol{I}}}\right), \tag{1}$$

where $\underline{\underline{\boldsymbol{I}}}$ is the $n \times n$ identity matrix.

A nice way to find the eigenvalues on the 9860 is graphically: for 3×3 matrices, use the EIGENVAL program. With the 3×3 matrix in Mat A, the program graphs the characteristic (cubic) polynomial. Use $\boxed{\text{ROOT}}$ in the $\boxed{\text{G-Solv}}$ menu to find the zeros of the polynomial: these are the eigenvalues λ_i, $i = 1, 2, 3$.

To find the eigenvectors, we have to solve the homogeneous matrix equation

$$\left(\underline{\underline{\boldsymbol{A}}} - \lambda_i \underline{\underline{\boldsymbol{I}}}\right) \underset{\sim}{\boldsymbol{v}} \;=\; \underset{\sim}{\boldsymbol{0}} \tag{2}$$

for each eigenvalue λ_i found above. We do this using Gauss or Gauss-Jordan elimination, most simply done using the commands *Ref* or *Rref* on the matrix $\underline{\underline{\boldsymbol{A}}} - \lambda_i \underline{\underline{\boldsymbol{I}}}$ for each eigenvalue λ_i or using the GAUSS program with $\underline{\underline{\boldsymbol{A}}} - \lambda_i \underline{\underline{\boldsymbol{I}}}$ stored in Mat A.

18.4.2 Example

Find the eigenvalues and corresponding eigenvectors of

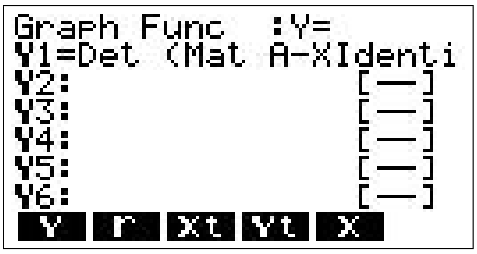
$$\underline{\underline{\boldsymbol{A}}} \;=\; \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}.$$

This is a simple problem that can be done easily by hand. We use it to illustrate the calculator method, which can be used for more complicated problems.

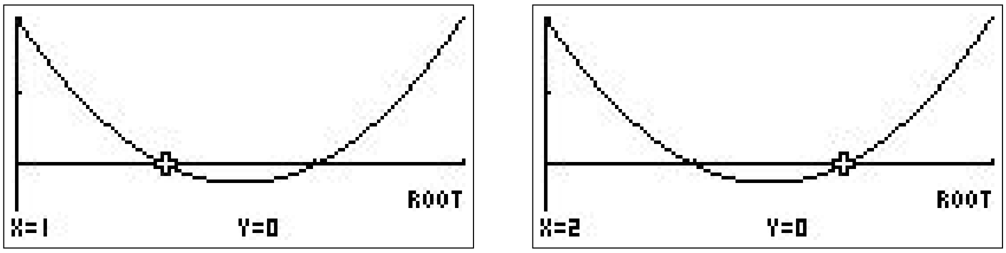
Eigenvalues

Put matrix $\underline{\underline{\boldsymbol{A}}}$ into Mat A in your calculator and set Y1 = Det (Mat A−X Identity 2) (Eq. (1).

Identity is produced by $\boxed{\text{Iden}}$ in the $\boxed{\text{OPTN}}$ $\boxed{\text{MAT}}$ menu (see Section 18.1.9).



Start with Xmin $=-5$ and Xmax $=5$ in V-Window. Press $\boxed{\text{EXIT}}$ and $\boxed{\text{DRAW}}$ to graph the characteristic polynomial, here a quadratic (because Mat A is 2×2). It is clear that the zeros lie in the range $0<x<3$. Change the V-Window to magnify this region.



V-Window $[0,3,1]\times[-1,2,1]$

Using $\boxed{\text{ROOT}}$[20] in the $\boxed{\text{G-Solv}}$ menu gives the eigenvalues as $\lambda_1=1$ and $\lambda_2=2$.

In this simple case, it is easy to show by hand that the characteristic polynomial is $p(\lambda)=\lambda^2-3\lambda+2$, with zeros 1 and 2.

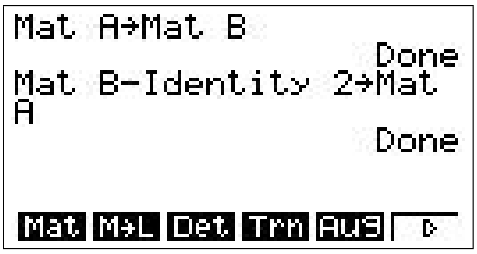
Eigenvectors

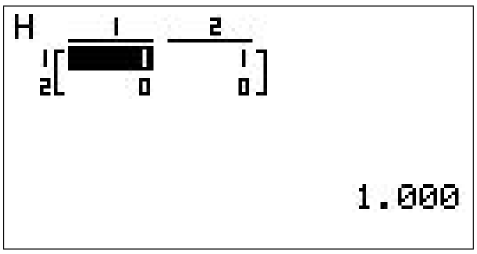
$\boldsymbol{\lambda_1=1}$: We have to find $\underset{\sim}{\boldsymbol{v}}_1=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, such that $\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}}_1=\underset{\sim}{\boldsymbol{v}}_1$ or $\big(\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{I}}}\big)\underset{\sim}{\boldsymbol{v}}_1=\underset{\sim}{\boldsymbol{0}}$ (Eq. (2), i.e. solve

$$\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

To use the GAUSS program, we need to put $\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{I}}}$ in Mat A. As we want to use $\underline{\underline{\boldsymbol{A}}}$, now in Mat A, for the second eigenvector, store Mat A in Mat B. Then store Mat B$-$Identity 2 in Mat A and run GAUSS to give the reduced row-echelon form. Alternatively, execute the command Rref $\big($Mat A$-$Identity 2$\big)$.

We should really use the augmented matrix here, with a third column of 0s, but these remain 0 in the Gauss elimination, so we just remember to put them back in afterwards.





This gives the row-echelon form of $\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{I}}}$, with the third column of 0s back in, as

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the bottom row is all zeros. There must be at least one row of zeros (the bottom row) in all eigenvector problems.

[20]Press the right arrow to move to successive roots.

The corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The bottom row tells us that x_2 is arbitrary, so we set $x_2 = t$, where t is any non-zero number.[21] The top row tells us that $x_1 + x_2 = 0$, so that $x_1 = -x_2 = -t$. The eigenvector $\underset{\sim}{\boldsymbol{v}}_1$ corresponding to $\lambda_1 = 1$ is therefore

$$\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero) multiples of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We usually say that this vector is the eigenvector, with the understanding that all non-zero multiples of it are also eigenvectors.

Check:

$$\underset{=}{\boldsymbol{A}}\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underset{\sim}{\boldsymbol{v}}_1,$$

as required.

$\boldsymbol{\lambda_2 = 2}$: We have to solve $\underset{=}{\boldsymbol{A}}\underset{\sim}{\boldsymbol{v}}_2 = 2\underset{\sim}{\boldsymbol{v}}_2$ or $\left(\underset{=}{\boldsymbol{A}} - 2\,\underset{=}{\boldsymbol{I}}\right)\underset{\sim}{\boldsymbol{v}}_2 = \underset{\sim}{\boldsymbol{0}}$. Store Mat B−2Identity 2 in Mat A and run GAUSS to give the reduced row-echelon form.[22] With the third column of 0s back in, this is

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $x_2 = t$, where t is any non-zero number, and the top row gives $x_1 + 2x_2 = 0$, so that $x_1 = -2x_2 = -2t$.

The eigenvector $\underset{\sim}{\boldsymbol{v}}_2$ corresponding to $\lambda_2 = 2$ is therefore

$$\underset{\sim}{\boldsymbol{v}}_2 = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The eigenvector is an arbitrary (non-zero) multiple of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Check:

$$\underset{=}{\boldsymbol{A}}\underset{\sim}{\boldsymbol{v}}_2 = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2\underset{\sim}{\boldsymbol{v}}_2,$$

as required.

[21] If $t = 0$, we obtain a zero vector, which, by definition, cannot be an eigenvector.

[22] Alternatively, execute the command Rref (Mat A−2 Identity 2).

18.4.3 EIGENV5 program

This program semi-automates the whole process. With matrix $\underline{\underline{\boldsymbol{A}}}$ in Mat A, run the program. It plots the characteristic polynomial, finds the zeros (eigenvalues) on the screen (change the X scale if necessary) and displays the eigenvectors (specified by their index/place in List 4). If it cannot find an eigenvector, it displays the reduced-row echelon form of the matrix $\underline{\underline{\boldsymbol{A}}} - \lambda_i \underline{\underline{\boldsymbol{I}}}$ for you to analyse.

Note: If an eigenvalue is a repeated root of the eigenvalue equation (the graph just touches the X axis), the search routine will not pick it up. You will need to add it to the list of eigenvalues using the option in the FIND EIGENVECTORS menu. The program will then find the corresponding eigenvector or display the reduced-row echelon form of the matrix $\underline{\underline{\boldsymbol{A}}} - \lambda_i \underline{\underline{\boldsymbol{I}}}$ if there are multiple eigenvectors.

18.4.4 Exercises

Solutions are on page 75.

1. Find, using the calculator (but not EIGENV5), the eigenvectors and eigenvalues of the matrix

$$\underline{\underline{\boldsymbol{A}}} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}.$$

Check with EIGENV5 and by verifying that $\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}} = \lambda \underset{\sim}{\boldsymbol{v}}$ for each eigenvalue.

2. Show, using the calculator (but not EIGENV5), that the eigenvalues of

$$\underline{\underline{\boldsymbol{A}}} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

are -1, 1 and 2, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

3. Find, using the calculator (but not EIGENV5), the eigenvectors and eigenvalues of the matrix

$$\underline{\underline{\boldsymbol{A}}} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Check with EIGENV5 and by verifying that $\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}} = \lambda \underset{\sim}{\boldsymbol{v}}$ for each eigenvalue.

18.5 Vector operations

18.5.1 Introduction

All the operations here, except one, are based on the dot or scalar product. This can be evaluated with the vectors stored as (column) matrices or as lists. If $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ are two vectors of the same dimension n, they can be represented on the calculator by the $n \times 1$ column matrices Mat A and Mat B, say, or the components stored in lists, say, List 1 and List 2. Then the dot or scalar product is given by the matrix multiplication

$$\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} = \text{Trn Mat A Mat B},$$

where Trn denotes the transpose, or by the list operation

$$\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} = \text{sum List 1 List 2}.$$

The list product List 1 List 2 produces a list whose components are the products of the corresponding components of the two lists; summing these then gives the scalar product.

Note that the scalar product is defined for vectors of any length, i.e. $\underset{\sim}{\boldsymbol{a}}, \underset{\sim}{\boldsymbol{b}} \in \mathbb{R}^n$; this is handled by both methods on the calculator.

The one exception mentioned above is the cross or vector product, which is only defined for three-dimensional vectors.

The list method is a little easier to use, so we continue with it. The operations described below are contained in the VECTOR program,[23] which prompts for the vectors, then does the calculations.

Note: The CG50 (and presumably the CG20) has built-in vector operations (accessed from the Run-Matrix screen ($\boxed{\text{MENU}}$ $\boxed{1}$) by pressing $\boxed{\text{OPTN}}$ $\boxed{\text{F2}}$, then $\boxed{\text{F6}}$ twice), so that the list commands below are not necessary. However, the VECTOR program still makes these operations easier.

Vectors are entered on the CG50 like matrices but after first pressing $\boxed{\text{F6}}$ (M⇔V).

18.5.2 Operations

Use the STAT menu ($\boxed{\text{MENU}}$ $\boxed{2}$) to input the vectors.

Assume vector $\underset{\sim}{\boldsymbol{a}}$ is in List 1, vector $\underset{\sim}{\boldsymbol{b}}$ in List 2.

sum is in the $\boxed{\text{OPTN}}$ LIST menu (press $\boxed{\text{F6}}$ twice).

Scalar multiplication ($\mathbb{R}^n$)

If c is a constant, the scalar multiple $c\underset{\sim}{\boldsymbol{a}} = c\,\text{List 1}$.

Scalar or dot product ($\mathbb{R}^n$)

$\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} = \text{sum List 1 List 2}$.

Vector or cross product ($\mathbb{R}^3$ only)

$\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}}$ is a vector, given in terms of the entries in the two lists by

$$\begin{aligned}\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}} = \big(&\text{List 1(2) List 2(3)} - \text{List 1(3) List 2(2)},\ \text{List 1(3) List 2(1)} \\ &- \text{List 1(1) List 2(3)},\ \text{List 1(1) List 2(2)} - \text{List 1(2) List 2(1)}\big).\end{aligned}$$

[23] available at *canberramaths.org.au* under *Resources*

Norm ($\mathbb{R}^n$)

norm $\underset{\sim}{\boldsymbol{a}} \;=\; \sqrt{(\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{a}})} \;=\; \sqrt{}\text{sum List 1 List 1}$

Projection ($\mathbb{R}^n$)

The (vector) projection of $\underset{\sim}{\boldsymbol{b}}$ onto $\underset{\sim}{\boldsymbol{a}}$ is given by

$$\text{proj}_{\underset{\sim}{\boldsymbol{a}}} \underset{\sim}{\boldsymbol{b}} \;=\; \frac{\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}}}{\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{a}}} \underset{\sim}{\boldsymbol{a}} \;=\; \text{sum List 1 List 2} \div (\text{sum List 1 List 1})\,\text{List 1.}$$

Angle between vectors ($\mathbb{R}^n$)

The angle θ between vectors $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ is given by

$$\cos(\theta) \;=\; \frac{\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}}}{\sqrt{(\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{a}})}\sqrt{(\underset{\sim}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{b}})}} \;=\; \text{sum List 1 List 2} \div \sqrt{}\text{sum List 1 List 1} \div \sqrt{}\text{sum List 2 List 2.}$$

18.5.3 Exercises

1. Given vectors $\underset{\sim}{\boldsymbol{a}} = (1, 2, 3)$ and $\underset{\sim}{\boldsymbol{b}} = (-1, 0, 3)$, find

- **(a)** $\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}}$ — *Answer*: 8.
- **(b)** $\underset{\sim}{\boldsymbol{a}} \times \underset{\sim}{\boldsymbol{b}}$ — *Answer*: $(6, 6, –2)$.
- **(c)** norm of $\underset{\sim}{\boldsymbol{b}}$ — *Answer*: $\sqrt{10}$.
- **(d)** $\text{proj}_{\underset{\sim}{\boldsymbol{a}}} \underset{\sim}{\boldsymbol{b}}$ — *Answer*: $\left(\frac{4}{7}, \frac{8}{7}, \frac{12}{7}\right)$.
- **(e)** the angle between $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ in radians and degrees. — *Answer*: 0.828 rad; 47.5°.

2. Given vectors $\underset{\sim}{\boldsymbol{a}} = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\right)$ and $\underset{\sim}{\boldsymbol{b}} = \left(\frac{2}{3}, \frac{3}{5}, \frac{2}{7}, -1\right)$, find

- **(a)** $\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}}$ — *Answer*: $\frac{47}{105}$.
- **(b)** norm of $\underset{\sim}{\boldsymbol{b}}$ — *Answer*: 1.373 (to 3DP).
- **(c)** $\text{proj}_{\underset{\sim}{\boldsymbol{a}}} \underset{\sim}{\boldsymbol{b}}$ — *Answer*: $\big(0.531,\ 0.354,\ 0.212,\ 0.152\big)$ (to 3DP).
- **(d)** the angle between $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ in radians and degrees. — *Answer*: 1.05 rad; 59.9°.

18.6 Solutions

18.6.1 Gauss elimination

1. Enter the matrix below carefully into Mat A using RUN-MAT $\boxed{\text{F1}}$.

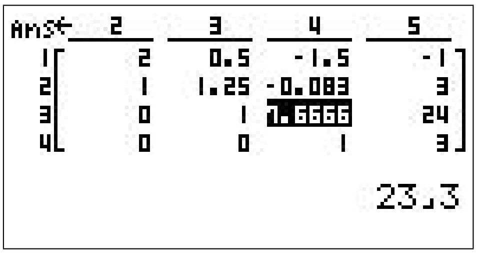
$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

Mat A is the augmented matrix $\left[\underline{\underline{\boldsymbol{A}}} \,|\, \underset{\sim}{\boldsymbol{b}}\right]$ for the following system of equations: in matrix form $\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{x}} = \underset{\sim}{\boldsymbol{b}}$.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 \qquad &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

A. Solve this system using the calculator to find the row-echelon form (Gaussian elimination), then back substitution (you have to do this);

Either run the GAUSS program or the *Ref* command to give the row-echelon form of the matrix. In the result, you can convert individual entries to fractions using the $\boxed{\text{F}\leftrightarrow\text{D}}$ key. The figure below omits the first column, which is 1 0 0 0.



The corresponding (equivalent) equations are

$$\begin{aligned} x_1 + 2x_2 + \frac{1}{2}x_3 - \frac{3}{2}x_4 &= -1 \\ x_2 + \frac{5}{4}x_3 - \frac{1}{12}x_4 &= 3 \\ x_3 + \frac{23}{3}x_4 &= 24 \\ x_4 &= 3 \end{aligned}$$

The last equation gives $x_4 = 3$.
Substituting this into the third equation gives $x_3 + 23 = 24$, giving $x_3 = 1$.
Substituting these into the second equation gives

$$x_2 = 3 - \frac{5}{4}x_3 + \frac{1}{12}x_4 = 3 - \frac{5}{4} + \frac{3}{12} = 2.$$

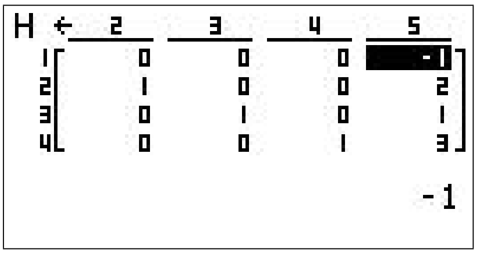
Substituting these into the first equation gives

$$x_1 = -1 - 2x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 = -1 - 4 - \frac{1}{2} + \frac{9}{2} = -1.$$

Therefore, $x_1 = –1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$.

B. Solve this system using the calculator to find the reduced row-echelon form (Gauss-Jordan elimination), from which you can read off the answer;

Either run the GAUSS program again or use the *Rref* command to give the reduced-row-echelon form of the matrix. The figure below omits the first column: 1 0 0 0.



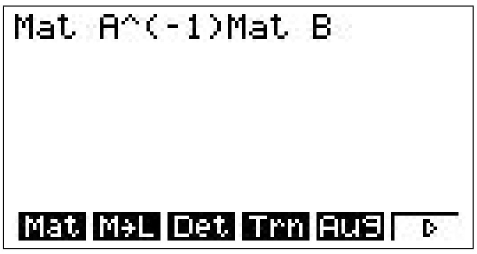
The corresponding (equivalent) equations are

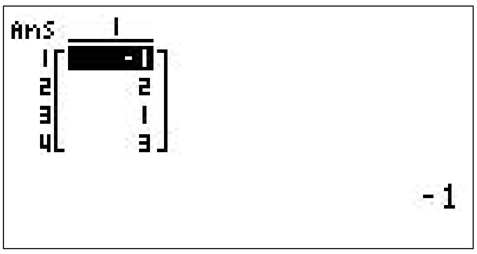
$$\begin{aligned} x_1 &= -1 \\ x_2 &= 2 \\ x_3 &= 1 \\ x_4 &= 3, \end{aligned}$$

the answer directly.

C. Solve this system using using $\underset{\sim}{\boldsymbol{x}} = \underline{\underline{\boldsymbol{A}}}^{-1}\underset{\sim}{\boldsymbol{b}}$. Do you understand why this works?

Change Mat A back to the matrix $\underline{\underline{\boldsymbol{A}}}$ using $\boxed{\text{F1}}$ (MAT). Highlight Mat A and press $\boxed{\text{EXE}}$. Press $\boxed{\text{F3}}$ (COL), move the cursor to Column 5 and press $\boxed{\text{F1}}$ (DEL).
Store the column matrix $\underset{\sim}{\boldsymbol{b}}$ in Mat B, then evaluate Mat A^(−1) Mat B.





This method works here because the matrix $\underline{\underline{\boldsymbol{A}}}$ is invertible, corresponding to the case in which there is a unique solution.
Again the answer is $x_1 = –1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$.

2. Solve the following systems of equations using each of the three methods using the calculator to do the Gauss or Gauss-Jordan elimination.

Check your answers by substituting them back into the equations.

(a)

$$\begin{aligned} 2x - 5y + 5z &= 17 \\ x - 2y + 3z &= 9 \\ -x + 3y \quad &= -4 \end{aligned}$$

In matrix form,

$$\begin{bmatrix} 2 & -5 & 5 \\ 1 & -2 & 3 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 9 \\ -4 \end{bmatrix},$$

with augmented matrix

$$\begin{bmatrix} 2 & -5 & 5 & 17 \\ 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \end{bmatrix}.$$

The row-echelon form from GAUSS (Method A) is

$$\begin{bmatrix} 1 & -2.5 & 2.5 & 8.5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

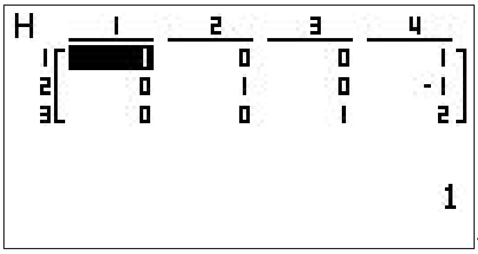
The corresponding equations are

$$\begin{aligned} x - 2.5y + 2.5z &= 8.5 \\ y + z &= 1 \\ z &= 2. \end{aligned}$$

The last equation gives $z=2$ directly. Substituting this into the second equation gives $y=-1$, and these into the first equation $x=1$.

The unique solution from Method A is $x=1$, $y=-1$ and $z=2$.

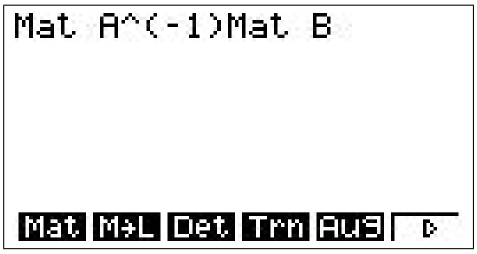
Using GAUSS for Method B, Gauss-Jordan elimination:

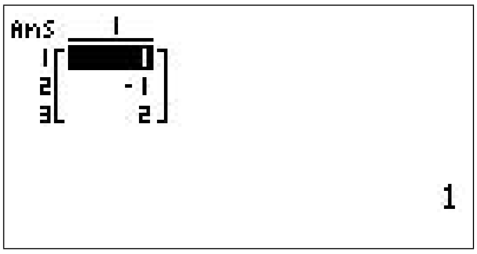


.

The right-hand column of the reduced row-echelon form gives the solution; each equation now only contains one variable.

Following the hint in Question 1 for putting $\underline{\underline{\boldsymbol{A}}}$ back in Mat A, the RHS of the original system in Mat B and evaluating Mat A^(−1) Mat B. gives the Method C result





which gives the solution directly.

Substituting the solution $x = 1$, $y = -1$ and $z = 2$ back into the original equations:

$$\text{LHS} = 2\times 1 - 5\times -1 + 5\times 2 = 17 = \text{RHS}.$$

$$\text{LHS} = 1 - 2\times -1 + 3\times 2 = 9 = \text{RHS}.$$

$$\text{LHS} = -1 + 3\times -1 = -4 = \text{RHS}.$$

The solution is verified.

(b)

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 \qquad - 2x_3 &= 1 \\ 2x_1 - x_2 - \ x_3 &= 0 \end{aligned}$$

In matrix form,

$$\begin{bmatrix} 1 & 1 & -5 \\ 1 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$

with augmented matrix

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix}.$$

GAUSS gives the row-echelon form (Method A)

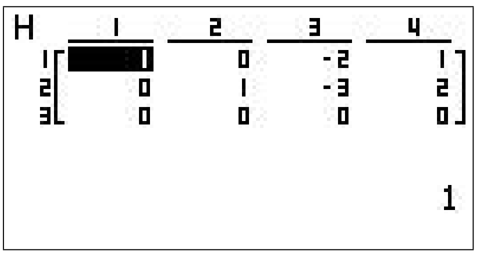
$$\begin{bmatrix} 1 & -0.5 & -0.5 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The bottom row of all zeros tells us there is an infinite number of solutions.
The corresponding equations are

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_2 - 3x_3 &= 2 \end{aligned}$$

Here, x_3 is a free variable: let $x_3 = t$, $t \in \mathbb{R}$.
The second equation gives $x_2 = 2 + 3t$, and the first $x_1 = 1 + 2t$.
The solution is $x_1 = 1 + 2t$, $x_2 = 2 + 3t$ and $x_3 = t$, $t \in \mathbb{R}$

Using GAUSS for Method B, Gauss-Jordan elimination:



The right-hand column of the reduced row-echelon form does not gives the solution here. However, back substitution is easier with the reduced row-echelon form than with the row-echelon form.

Putting $\underline{\underline{\boldsymbol{A}}}$ back in matrix Mat A, the RHS of the original system in Mat B and evaluating Mat A^(−1) Mat B gives an error message: *No inverse exists*. Method C cannot be used, consistent with the fact that there is not a unique solution.

Substituting the solution $x_1 = 1+2t$, $x_2 = 2+3t$ and $x_3 = t$ back into the original equations:

$$\text{LHS} = 1+2t + 2+3t - 5t = 3 = \text{RHS}.$$

$$\text{LHS} = 1+2t - 2t = 1 = \text{RHS}.$$

$$\text{LHS} = 2(1+2t) - (2+3t) - t = 0 = \text{RHS}.$$

The solution is verified.

(c)

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 \qquad + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

In matrix form,

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & -3 & 5 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \\ 1 \end{bmatrix},$$

with augmented matrix

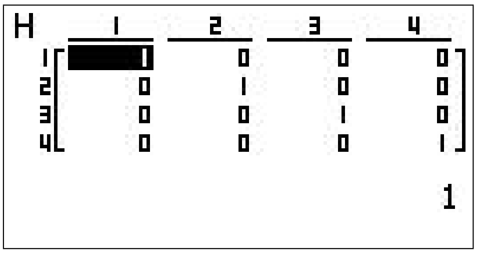
$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}.$$

GAUSS gives the row-echelon form (Method A)

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{17}{3} & -\frac{10}{3} \\ 0 & 0 & 1 & \frac{31}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The bottom row tells us there is no solution: the corresponding equation is $0x_1 + 0x_2 + 0x_2 = 1$.

Using GAUSS for Method B, Gauss-Jordan elimination:



Again no solution.

Putting $\underline{\underline{A}}$ back in Mat A, the RHS of the original system in Mat B and evaluating Mat A^(−1) Mat B gives an error message: *No inverse exists.* Method C cannot be used, consistent with the fact that there is not a unique solution.

What is det($\underline{\underline{A}}$)? What does this tell you about the inverse matrix?

In (b) and (c), det($\underline{\underline{A}}$) = 0, confirming that an inverse matrix does not exist.

18.6.2 Eigenvalues and eigenvectors

1. Find, using the calculator (but not EIGENV5), the eigenvectors and eigenvalues of the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}.$$

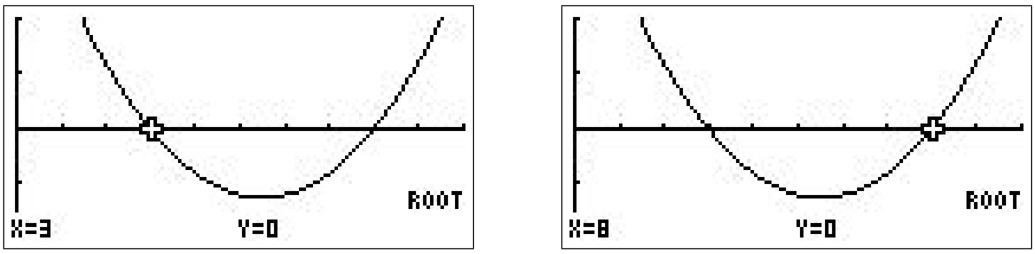
Check with EIGENV5 and by verifying that $\underline{\underline{A}}\underset{\sim}{v} = \lambda \underset{\sim}{v}$ for each eigenvalue.

Eigenvalues

Put the 2×2 matrix $\underline{\underline{A}}$ into Mat A in your calculator.

Set Y1 = Det (Mat A−X Identity 2).

Start with V-Window STD. Press EXIT and DRAW to plot the characteristic polynomial, here a quadratic. It is clear that the zeros lie in the range $0 < x < 10$.



V-Window $[0, 10, 1] \times [-10, 10, 5]$

Using ROOT in the G-Solv menu gives the eigenvalues as $\lambda_1 = 3$ and $\lambda_2 = 8$.

Eigenvectors

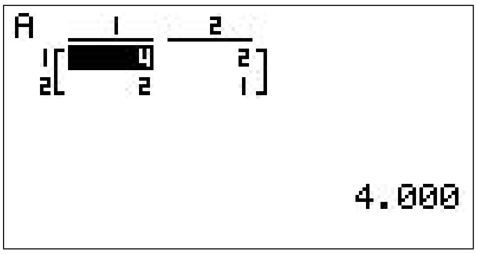
$\boldsymbol{\lambda_1 = 3}$

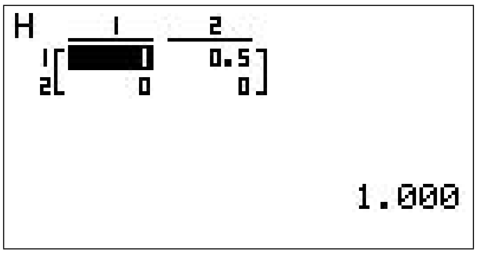
Let $\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$.

Then $\left(\underline{\underline{\boldsymbol{A}}} - \lambda \underline{\underline{\boldsymbol{I}}}\right) \underset{\sim}{\boldsymbol{v}}_1 \;=\; \left(\underline{\underline{\boldsymbol{A}}} - 3\,\underline{\underline{\boldsymbol{I}}}\right) \underset{\sim}{\boldsymbol{v}}_1 \;=\; \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

To use the GAUSS program, we need to put $\underline{\underline{\boldsymbol{A}}} - 3\,\underline{\underline{\boldsymbol{I}}}$ in Mat A. As we want to use $\underline{\underline{\boldsymbol{A}}}$), now in Mat A, for the second eigenvector, store Mat A in Mat B. Then store Mat B−3Identity 2 in Mat A (below left) and run GAUSS to give the reduced row-echelon form (below right).

We should really use the augmented matrix here, with a third column of 0s, but these remain 0 in the Gauss elimination, so we just remember to put them back in afterwards.





The reduced row-echelon form of $\underline{\underline{\boldsymbol{A}}} - 3\,\underline{\underline{\boldsymbol{I}}}$, with the third column of 0s back in, is

$$\begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the bottom row is all zeros. There must be at least one row of zeros (the bottom row) in all eigenvector problems.

Therefore, y_1 is a free variable. Set $y_1 = t \in \mathbb{R}$, $t \neq 0$.

The top row gives the equation $x_1 + 0.5y_1 = 0$, so that $x_1 = -0.5y_1 = -0.5t$.

The solution is therefore $y_1 = t$, $x_1 = -0.5t$, where t is any **non-zero** real number.

Therefore, $\underline{\underline{\boldsymbol{A}}}$ has an eigenvalue $\lambda_1 = 3$, with corresponding eigenvector

$$\underset{\sim}{\boldsymbol{v}}_1 \;=\; \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \;=\; \begin{bmatrix} -0.5t \\ t \end{bmatrix} \;=\; t \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad t \neq 0 \in \mathbb{R}.$$

Given that t is arbitrary, set $s = -\frac{1}{2}t$, $s \neq 0 \in \mathbb{R}$, so that

$$\underset{\sim}{\boldsymbol{v}}_1 \;=\; s \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Check your answer by verifying that $\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}}_1 = 3\underset{\sim}{\boldsymbol{v}}_1$.

$$\begin{aligned}
\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}}_1 &= \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -6 \end{bmatrix} \\
&= 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= 3\underset{\sim}{\boldsymbol{v}}_1.
\end{aligned}$$

$\boldsymbol{\lambda_2 = 8}$

Let $\underset{\sim}{\boldsymbol{v}}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$.

Then $\left(\underline{\underline{\boldsymbol{A}}} - \lambda \underline{\underline{\boldsymbol{I}}}\right) \underset{\sim}{\boldsymbol{v}}_1 = \left(\underline{\underline{\boldsymbol{A}}} - 8\,\underline{\underline{\boldsymbol{I}}}\right) \underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Gauss elimination on the augmented matrix:

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, y_2 is a free variable. Set $y_2 = t \neq 0 \in \mathbb{R}$.

The top row gives the equation $x_2 - 2y_2 = 0$, so that $x_2 = 2y_2 = 2t$.

The solution is therefore $y_2 = t$, $x_2 = 2t$, where t is any **non-zero** real number.

Therefore, $\underline{\underline{\boldsymbol{A}}}$ has an eigenvalue $\lambda_2 = 8$, with corresponding eigenvector

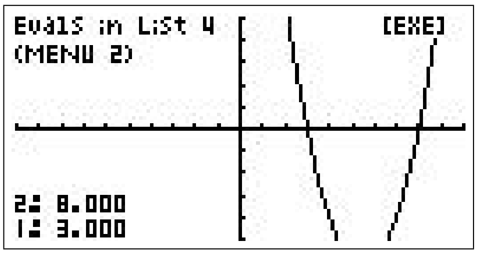
$$\underset{\sim}{\boldsymbol{v}}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad t \neq 0 \in \mathbb{R}.$$

Check your answer by verifying that $\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}}_2 = 8\underset{\sim}{\boldsymbol{v}}_2$.

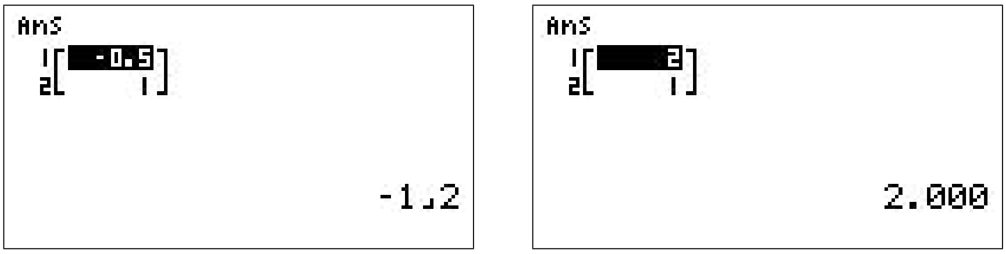
$$\begin{aligned}
\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}}_2 &= \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 16 \\ 8 \end{bmatrix} \\
&= 8 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= 8\underset{\sim}{\boldsymbol{v}}_2.
\end{aligned}$$

Note that we usually take $t = 1$ when writing down the eigenvectors, knowing that any non-zero multiple is also an eigenvector.

Using EIGENV5, with the original matrix $\underline{\underline{A}}$ in Mat A,



eigenvalues



eigenvectors

2. Show, using the calculator (but not EIGENV5), that the eigenvalues of

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

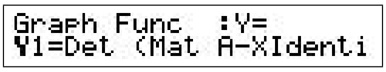
are -1, 1 and 2, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Eigenvalues

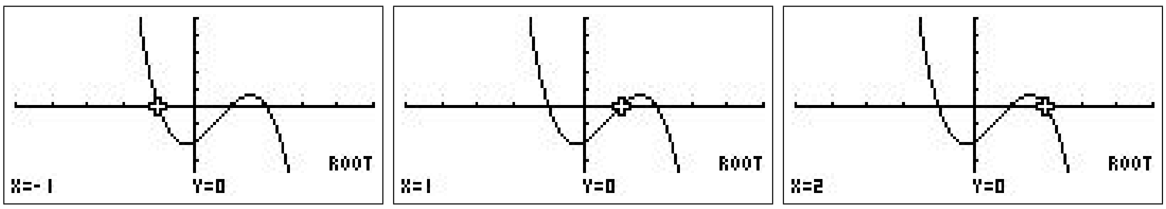
Put the 3×3 matrix $\underline{\underline{A}}$ into Mat A in your calculator.

Set Y1 = Det (Mat A−X Identity 3).



Start with Xmin $= -5$ and Xmax $= 5$. Press DRAW to plot the characteristic polynomial, here a cubic. It is clear that the zeros lie in the range $-2 < x < 3$.

Using ROOT in the G-Solv menu gives the eigenvalues as $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 3$.



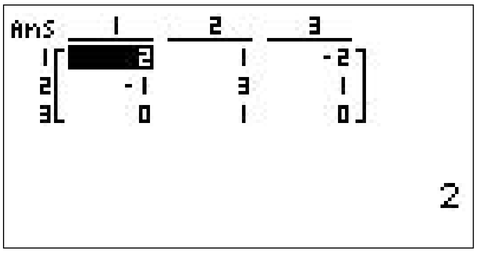
V-Window $[-5, 5, 1] \times [-5, 5, -1]$

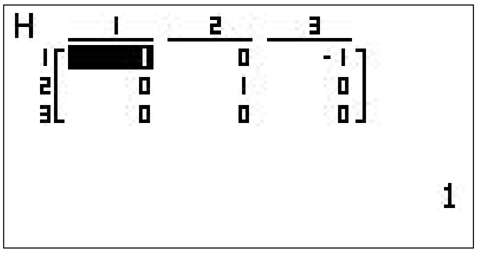
Eigenvectors

$\boldsymbol{\lambda_1 = -1}$: We have to find $\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, such that $\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}}_1 = -\underset{\sim}{\boldsymbol{v}}_1$ or $\left(\underline{\underline{\boldsymbol{A}}} + \underline{\underline{\boldsymbol{I}}}\right)\underset{\sim}{\boldsymbol{v}}_1 = \underset{\sim}{\boldsymbol{0}}$,
i.e.

$$\left(\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To use the GAUSS program, we need to put $\underline{\underline{\boldsymbol{A}}} + \underline{\underline{\boldsymbol{I}}}$ in Mat A. As we want to use $\underline{\underline{\boldsymbol{A}}}$, now in Mat A, for the second eigenvector, store Mat A in Mat B. Then store Mat B+Identity 3 in Mat A (below left) and run GAUSS to give the reduced row-echelon form of $\underline{\underline{\boldsymbol{A}}} + \underline{\underline{\boldsymbol{I}}}$ (below right).





This matrix, with the fourth column of 0s put back in, is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the bottom row is all zeroes. There must be at least one row of zeroes (the bottom row) in all eigenvector problems.

The bottom row tells us that x_3 is arbitrary, so we set $x_3 = t$, where t is any number. The second row gives $x_2 = 0$. The top row tells us that $x_1 - x_3 = 0$, so that $x_1 = x_3 = t$.

The eigenvector $\underset{\sim}{\boldsymbol{v}}_1$ corresponding to $\lambda_1 = -1$ is therefore

$$\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero[24]) multiples of the vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

We usually say that this vector is the eigenvector, with the understanding that all non-zero multiples of it are also eigenvectors.

[24]If $t = 0$, we obtain a zero vector, which, by definition, cannot be an eigenvector.

$\boldsymbol{\lambda_2 = 1}$: We have to solve $\underline{\underline{A}}\underset{\sim}{v}_2 = \underset{\sim}{v}_2$ or $\left(\underline{\underline{A}} - \underline{\underline{I}}\right)\underset{\sim}{v}_2 = \underset{\sim}{0}$ for $\underset{\sim}{v}_2$.

Store Mat B− Identity 3 in Mat A and run GAUSS. The reduced row-echelon form of $\underline{\underline{A}} - \underline{\underline{I}}$, with the fourth column of 0s put back in, is

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $x_3 = t$, where t is any number. The second row gives the equation $x_2 - 2x_3 = 0$, so that $x_2 = 2x_2 = 2t$. The top row gives the equation $x_1 - 3x_3 = 0$, so that $x_1 = 3x_3 = 3t$.

The eigenvector $\underset{\sim}{v}_2$ corresponding to $\lambda_2 = 2$ is therefore

$$\underset{\sim}{v}_2 = \begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

The eigenvector is an arbitrary (non-zero) multiple of the vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

$\boldsymbol{\lambda_3 = 2}$: We have to solve $\underline{\underline{A}}\underset{\sim}{v}_3 = 2\underset{\sim}{v}_3$ or $\left(\underline{\underline{A}} - 2\underline{\underline{I}}\right)\underset{\sim}{v}_3 = \underset{\sim}{0}$ for $\underset{\sim}{v}_3$.

Store Mat B−2 Identity 3 in Mat A and run GAUSS. The row-equivalent reduced row-echelon form of $\underline{\underline{A}} - 2\underline{\underline{I}}$, with the fourth column of 0s put back in, is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $x_3 = t$, where t is any number. The second row gives the equation $x_2 - 3x_3 = 0$, so that $x_2 = 3x_2 = 3t$. The top row gives the equation $x_1 - x_3 = 0$, so that $x_1 = x_3 = t$.

The eigenvector $\underset{\sim}{v}_3$ corresponding to $\lambda_3 = 2$ is therefore

$$\underset{\sim}{v}_3 = \begin{bmatrix} t \\ 3t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

The eigenvector is an arbitrary (non-zero) multiple of the vector $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

3. Find, using the calculator (but not EIGENV5), the eigenvectors and eigenvalues of the matrix

$$\underline{\underline{\boldsymbol{A}}} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

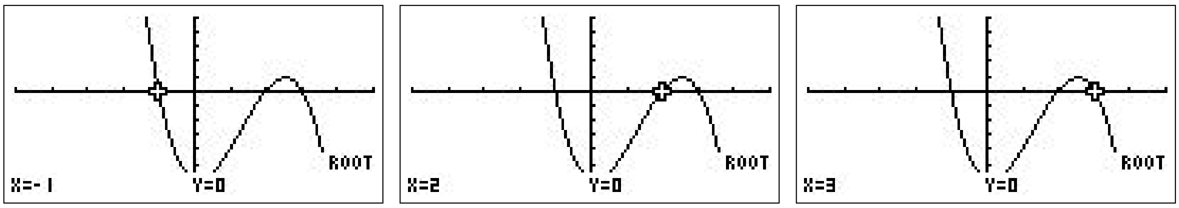
Check with EIGENV5 and by verifying that $\underline{\underline{\boldsymbol{A}}}\underset{\sim}{\boldsymbol{v}} = \lambda \underset{\sim}{\boldsymbol{v}}$ for all the eigenvalues.

Eigenvalues

Put the 3×3 matrix $\underline{\underline{\boldsymbol{A}}}$ into Mat A in your calculator.

Set Y1 = Det $\big($Mat A−X Identity 3$\big)$.

Start with Xmin = −5 and Xmax = 5. Press $\boxed{\text{DRAW}}$ to plot the characteristic polynomial, here a cubic.



V-Window $[-5, 5, 1] \times [-7, 5, -1]$

Using $\boxed{\text{ROOT}}$ in the $\boxed{\text{G-Solv}}$ menu gives the eigenvalues as $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

Eigenvectors

$\boldsymbol{\lambda_1 = -1}$: We have to solve $\big(\underline{\underline{\boldsymbol{A}}} + \underline{\underline{\boldsymbol{I}}}\big)\underset{\sim}{\boldsymbol{v}}_1 = \underset{\sim}{\boldsymbol{0}}$ for $\underset{\sim}{\boldsymbol{v}}_1$.

Let $\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. We need to find the components x, y and z.

Then,

$$\big(\underline{\underline{\boldsymbol{A}}} + \underline{\underline{\boldsymbol{I}}}\big)\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The augmented matrix for this problem is 3×4, with the augmented fourth column a column of zeros.

To use the GAUSS program, we need to put $\underline{\underline{\boldsymbol{A}}} + \underline{\underline{\boldsymbol{I}}}$ in Mat A. Store Mat A in Mat B, then store Mat B+Identity 3 in Mat A and run GAUSS.

This gives the reduced row-echelon form of $\underline{\underline{\boldsymbol{A}}} + \underline{\underline{\boldsymbol{I}}}$, with the fourth column of 0s put back in, as

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

(note the bottom row of zeros), with corresponding equations

$$\begin{aligned} x + y &= 0 \\ y + 3z &= 0. \end{aligned}$$

Back substitution gives $z = t$, $y = -3t$, $x = 3t$.

Thus the required eigenvector is $\underset{\sim}{\boldsymbol{v}}_1 = t \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$, where $t \neq 0 \in \mathbb{R}$.

Check that $\underline{\underline{\boldsymbol{A}}}\,\underset{\sim}{\boldsymbol{v}}_1 = -\underset{\sim}{\boldsymbol{v}}_1$.

$\underline{\boldsymbol{\lambda_2 = 2}}$

Let $\underset{\sim}{\boldsymbol{v}}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Solve $\left(\underline{\underline{\boldsymbol{A}}} - 2\underline{\underline{\boldsymbol{I}}}\right)\underset{\sim}{\boldsymbol{v}}_2 = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Gauss elimination

$$\begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

corresponding to equations

$$\begin{array}{rcl} x - 2y & = & 0 \\ y & = & 0, \end{array}$$

with free variable z and solution (back substitution) $z = t$, $y = 0$, $x = 0$, with $t \neq 0 \in \mathbb{R}$.

Thus the required eigenvector is $\underset{\sim}{\boldsymbol{v}}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Check that $\underline{\underline{\boldsymbol{A}}}\,\underset{\sim}{\boldsymbol{v}}_2 = 2\underset{\sim}{\boldsymbol{v}}_2$.

$\lambda_3 = 3$

Let $\underset{\sim}{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Solve $\left(\underline{\underline{A}} - 3\,\underline{\underline{I}}\right)\underset{\sim}{v}_3 = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Gauss elimination

$$\left[\begin{array}{cccc} -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

corresponding to equations

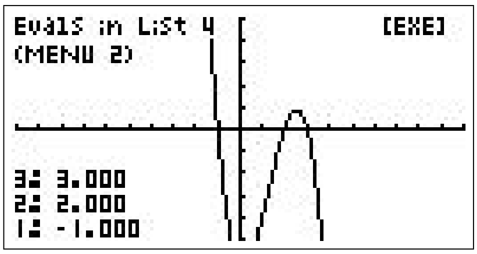
$$\begin{aligned} x - y &= 0 \\ y - z &= 0, \end{aligned}$$

with free variable z and solution (back substitution) $z=t$, $y=t$, $x=t$, with $t \neq 0 \in \mathbb{R}$.

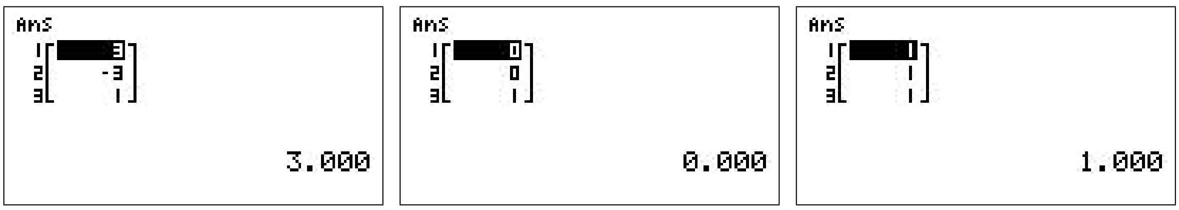
Thus the required eigenvector is $\underset{\sim}{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Check that $\underline{\underline{A}}\underset{\sim}{v}_3 = 3\underset{\sim}{v}_3$.

Using EIGENV5, with the original matrix $\underline{\underline{A}}$ (from Mat B) in Mat A,



eigenvalues



eigenvectors

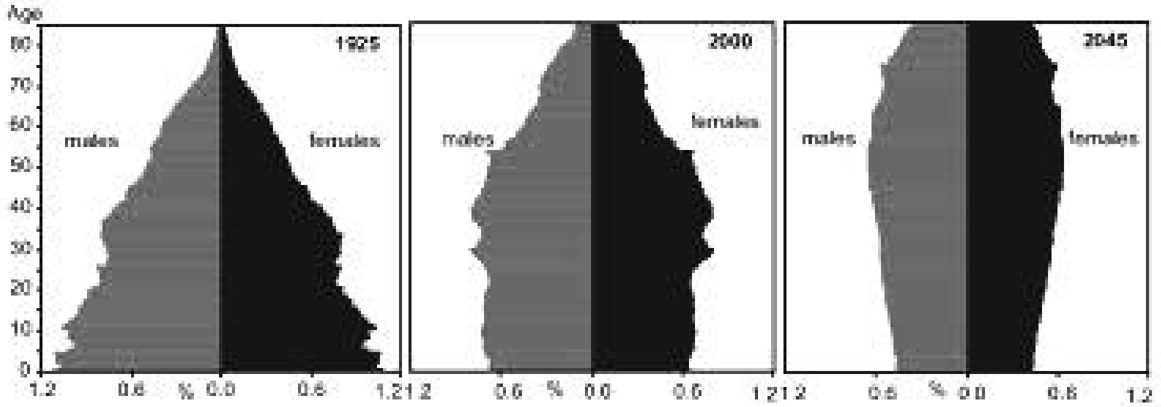
19 Population Modelling 3: Matrix Models

19.1 Population-projection matrices

In 2005, the Australian Government's Productivity Commission released its research report, *Economic Implications of an Aging Australia*[25]. Planners were concerned that the proportion of older people was increasing, which would eventually put pressure on social services like health care. There would also be fewer people in the working population to pay for the required services and pensions. The distribution of the population with respect to age is changing.

To describe the population, we can divide it into age groups, for example into five-year groups, 0–5 years, 6–10 years, etc. It is also useful to split the population into males and females.

Here are some data taken from the Productivity Report, with the Australian population in one-year age groups and split into males and females.



From pyramid to coffin. Changing age structure of the Australian population, 1925–2045.

Some of the results are measured data (1925 and 2000), while those for 2045 are clearly predictions. The trend is very clear, with a predominance of young people in 1925 (after World War 1) changing to a more even distribution in 2000 up to the age of the 'baby boomers' (40–55 years).

We follow this approach in the examples in this section: splitting a population into age classes and seeing how these classes evolve over time.

19.1.1 Leslie matrices

The Leslie matrix is a discrete, age-structured model of population growth that is very popular in population ecology. It was invented by and named after Patrick H. Leslie.

We divide a population into a number of classes — here we shall assume three classes, referring to three age groups, young, adults and seniors, with respective numbers y, a and s. The population is then described by the vector or 3×1 column matrix

$$\underset{\sim}{v} = \begin{bmatrix} y \\ a \\ s \end{bmatrix}.$$

[25]www.pc.gov.au/inquiries/completed/ageing/report

A 3×3 transition matrix $\underline{\underline{T}}$ tells us how the population evolves. For example, if the population to start with is

$$\underset{\sim}{v}_0 = \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix},$$

after one cycle it is

$$\underset{\sim}{v}_1 = \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \underline{\underline{T}} \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix} = \underline{\underline{T}}\,\underset{\sim}{v}_0.$$

In problems leading to a Leslie transition matrix:

- in each cycle, members of the other classes produce a certain number of new young in Class 1;
- a certain fraction of each class survives to move into the next class; the rest die;
- all members of the top class die.

This leads to a Leslie matrix $\underline{\underline{T}}$ that is zero everywhere except possibly:

- along the top row after the first element — the birth rates for each class;
- in the elements along the diagonal parallel to and just below the main diagonal — the survival rates for each class.

$$\begin{bmatrix} 0 & * & * & * & * & \cdots & * & * \\ * & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & * & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & * & 0 \end{bmatrix}$$

Leslie discovered these matrices in the 1940s when he pioneered this way of exploring how populations can develop. He taught himself matrix algebra while he was in hospital with TB.

Exercise: Leslie matrices and beetles

(a) During each cycle, each adult beetle produces on average 2.75 young and each senior beetle produces on average 2.5 young; one quarter of the young beetles survive to become adults; and one half of the adult beetles survive to become seniors. In a Leslie-matrix problem, all the seniors die.

Find the Leslie transition matrix $\underline{\underline{T}}$.

Good strategy: Write out the linear equations for y_1, a_1 and s_1 in terms of y_0, a_0 and s_0, and convert to matrix form.

$$\begin{aligned} y_1 &= ?y_0 + ?a_0 + ?s_0 \\ a_1 &= ?y_0 + ?a_0 + ?s_0 \\ s_1 &= ?y_0 + ?a_0 + ?s_0 \end{aligned}$$

(b) If we start with 40 young and no adults or seniors, show that after one cycle

$$\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}.$$

You should have entered $\underline{\underline{\boldsymbol{T}}}$ into Mat A, $\underset{\sim}{\boldsymbol{v}}_0$ into Mat B and evaluated Mat A Mat B.

(c) Multiply repeatedly by $\underline{\underline{\boldsymbol{T}}}$, and record $\underset{\sim}{\boldsymbol{v}}$ and the total population $P = y + a + s$ after 11, 12 and 13 cycles.[26]

What happens to the total population? to the ratios of the numbers in the different classes?

Solutions are in Section 19.2.

19.1.2 Populations and oscillations

Workers other than Leslie had independently used matrix algebra in population models. The first was Harro Bernardelli, who published a paper in 1941 in the Journal of the Burma Research Society with the title *Population Waves.* Bernardelli's paper was unusual in focussing not on the eventual stability of the population structure, but on intrinsic oscillations in the population structure. He had observed oscillations in the age structure of the Burmese population between 1901 and 1931.

As an abstract model for such oscillations, he proposed a matrix model for the evolution of the population with

$$\underline{\underline{\boldsymbol{T}}} = \begin{bmatrix} 0 & 0 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix},$$

and showed by numerical calculations that this gave rise to apparently permanent oscillations in the age structure.

Exercise

(a) Record the total population P at each cycle for 12 cycles using Bernardelli's matrix $\underline{\underline{\boldsymbol{T}}}$ with $\underset{\sim}{\boldsymbol{v}}_0 = \begin{bmatrix} 1 \\ 0.01 \\ 0.01 \end{bmatrix}$, the initial populations in three age groupings in millions.

Do this by hand or using POP (Section 19.1.5).

Plot P versus cycle number, joining up the points with straight lines.

Discuss your findings.

(b) Repeat (a) using $\underline{\underline{\boldsymbol{T}}} = \begin{bmatrix} 0 & 0 & 5 \\ 0.7 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$.

Describe your results in words. Explain in terms of the entries in $\underline{\underline{\boldsymbol{T}}}$.

Solutions are in Section 19.2.

[26] the POP program (Section 19.1.5) helps here

19.1.3 Exercise: Killer whales

Based on a Year-12 project set by Margaret McLaughlan then at St Francis Xavier College.

Leslie matrices are just a special case of a more general population-projection matrix. In the more general case, animals in a class may remain in that class for more than one cycle. The probability that an animal remains in a particular class for any given cycle is an element on the diagonal of the matrix, immediately above the entry in the Leslie matrix, which gives the probability of moving to the next class in any given cycle.

Therefore we have a matrix whose elements along the top row give the fecundity or birth rate per animal per cycle for each class, whose diagonal elements give the probability of an animal remaining in a particular class in any cycle and whose elements below the diagonal give the probability of an animal moving to the next class in any cycle. The fact that the latter two numbers in any column do not add up to 1 means that some animals in each class die each cycle.

For female killer whales, we have four classes — yearlings (individuals in the first year of life), juveniles (past the first year, but not mature), mature females and post-reproductive females. The mean period in the juvenile stage is 13.4 years and in the mature stage 22.1 years, with an overall lifetime of 80–90 years. Details in Brault and Caswell, *Pod-specific demography of killer whales*, Ecology 74, 1444–1454 (1993).

The population-projection matrix for female killer whales is given below. The time for one cycle is one year.

$$\underline{\underline{\boldsymbol{T}}} = \begin{bmatrix} 0 & 0.0043 & 0.1138 & 0 \\ 0.9775 & 0.9111 & 0 & 0 \\ 0 & 0.0736 & 0.9534 & 0 \\ 0 & 0 & 0.0452 & 0.9804 \end{bmatrix}$$

1. A project you are involved in wants to re-introduce killer whales into an area of ocean from which they have disappeared. The project leader wants to know what is the best combination of juveniles and mature females to re-introduce, assuming an overall total of 50 (plus an appropriate number of adult males). You decide to model three options over 40 cycles (years).

   (a) 50 female juveniles.

   (b) 40 female juveniles and 10 mature females.

   (c) 50 mature females.

   What happens to the different population classes of female killer whales over time in each of these options (according to this model)? What is the best strategy for re-establishing the killer-whale population?

   Hint: (*manual method*) If $\underline{\underline{\boldsymbol{T}}}$ is in the 4×4 matrix Mat A and the initial population $\underset{\sim}{\boldsymbol{v}}_0$ in the 4×1 matrix Mat B, executing the command Mat A Mat B → Mat B, then repeatedly pressing $\boxed{\text{EXE}}$ will generate successive population vectors $\underset{\sim}{\boldsymbol{v}}_1$, $\underset{\sim}{\boldsymbol{v}}_2$, $\underset{\sim}{\boldsymbol{v}}_3$, etc.[27]

2. Does the trend in the total population depend on which option you choose?

3. The value of 0.1138 in the top row of $\underline{\underline{\boldsymbol{T}}}$ gives the number of live births per mature female per cycle (year). To what value could this birth rate fall before the total population starts to decrease? This birth rate is clearly important for the overall survival of killer whales.

4. How sensitive is the population to the survival rates of yearlings (0.9775), juveniles (0.9111), mature females (0.9534) and post-reproductive females (0.9804)?

   You might like to quantify your answers here by determining what percentage decrease in each rate is needed to stop the population growing. *Do your answers make sense?*

[27] You could also use the POP program (Section 19.1.5). Run POP repeatedly by pressing $\boxed{\text{EXE}}$.

19.1.4 Exercise: Age distribution of trees in a forest

The difference-equation version of this problem can be found in *Population Modelling 2* in Volume 2 of this book.

The population of trees in a forest is split into four age groups: b_n is the number of baby trees ($0-15$ years old) at time-point n; y_n the number of young trees ($16-30$ years); m_n middle-aged trees ($31-45$ years old) ; and o_n old trees (more than 45 years old).

The time step for our difference equations is 15 years.

In order to simplify the model we make the following assumptions:

A. a certain percentage of trees in each age group dies in each time interval;

B. surviving trees age into the next age group each time step; old trees remain old trees (or die);

C. dead trees are replaced by an equal number of baby trees.

Define α, β, γ, δ as the fraction of dead trees in the respective age groups in each time interval. Then, the difference-equation model is

$$\begin{aligned} b_{n+1} &= \alpha b_n + \beta y_n + \gamma m_n + \delta o_n && \text{(Assumption C)} && (1) \\ y_{n+1} &= (1-\alpha) b_n && \text{(Assumptions A, B)} && (2) \\ m_{n+1} &= (1-\beta) y_n && \text{(Assumptions A, B)} && (3) \\ o_{n+1} &= (1-\gamma) m_n + (1-\delta) o_n && \text{(Assumptions A, B).} && (4) \end{aligned}$$

1. If the population of trees in time interval n is $N = b_n + y_n + m_n + o_n$, show that the population stays the same size after one more time step, and so by induction the population of trees is a constant N.

2. Equations (1) – (4) are linear in the dependent variables so that matrices are a way of solving the problem. Define $\underset{\sim}{\boldsymbol{v}}_n = (b_n, y_n, m_n, o_n)^{\mathrm{T}}$ (a column vector) and write down the matrix $\underline{\underline{\boldsymbol{T}}}$ that you would use in the matrix model $\underset{\sim}{\boldsymbol{v}}_{n+1} = \underline{\underline{\boldsymbol{T}}}\,\underset{\sim}{\boldsymbol{v}}_n$ of these equations.

3. Take $\alpha = 0.2$, $\beta = 0.5$, $\gamma = 0.3$, $\delta = 0.2$ and $N = 1000$. Write down the matrix $\underline{\underline{\boldsymbol{T}}}$ with these values.

4. Four initial conditions are needed in order to fully solve this matrix equation. Assume all baby trees initially.

Run the matrix model in Problems 2 and 3 through 10 cycles (150 years). Do the individual populations appear to be stabilising?

5. Use the EIGENV5 program to find the eigenvalues of $\underline{\underline{\boldsymbol{T}}}$.

6. Find the eigenvector of $\underline{\underline{\boldsymbol{T}}}$ corresponding to $\lambda = 1$ (*explain why we use this value*) and relate this to the behaviour of the model that you found in Problem 4.

Solutions to these problems are in Section 19.2.

19.1.5 POP program

POP multiplies an $M\times1$ column vector $\underset{\sim}{\boldsymbol{v}}$ (M age classes) by an $M\times M$ matrix $\underline{\underline{\boldsymbol{T}}}$ (transition matrix), displays the new $\underset{\sim}{\boldsymbol{v}}$ and the sum of the components of $\underset{\sim}{\boldsymbol{v}}$ (total population).

Use: Store $\underline{\underline{\boldsymbol{T}}}$ in Mat A and the initial $\underset{\sim}{\boldsymbol{v}}$ in Mat B. Run the program for the first step; press $\boxed{\text{EXE}}$ repeatedly for subsequent steps. Press $\boxed{\text{AC}^{/\text{ON}}}$ to stop the program.

19.2 Solutions

Exercise: Leslie matrices and beetles

(a) Writing out the equations for the three beetle age classes,

$$\begin{aligned} y_1 &= 0y_0 + 2.75a_0 + 2.5s_0 \\ a_1 &= 0.25y_0 + 0a_0 + 0s_0 \\ s_1 &= 0y_0 + 0.5a_0 + 0s_0 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 & 2.75 & 2.5 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix}.$$

Therefore, $\underline{\underline{\boldsymbol{T}}} = \begin{bmatrix} 0 & 2.75 & 2.5 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$.

(b) $\underset{\sim}{\boldsymbol{v}}_1 = \underline{\underline{\boldsymbol{T}}} \begin{bmatrix} 40 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$.

(c)

After cycle	$\underset{\sim}{\boldsymbol{v}}$	Total pop'n
1	$\begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$	10
2	$\begin{bmatrix} 27.5 \\ 0 \\ 5 \end{bmatrix}$	32.5
$\vdots$	$\vdots$	$\vdots$
11	$\begin{bmatrix} 17.320 \\ 4.305 \\ 2.184 \end{bmatrix}$	23.809
12	$\begin{bmatrix} 17.299 \\ 4.330 \\ 2.153 \end{bmatrix}$	23.781
13	$\begin{bmatrix} 17.289 \\ 4.325 \\ 2.165 \end{bmatrix}$	23.779

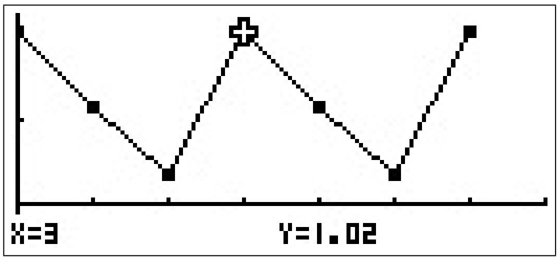
The total population seems to be stabilising at a little under 24, and the ratios of the populations in the 3 classes at about 8 : 2 : 1. Divide the first two numbers by the third (smallest) number to see this.

Populations and oscillations

(a) After 1 cycle: $\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} 0 & 0 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.01 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.08 \\ 0.5 \\ 0.0025 \end{bmatrix}$, so $P = 0.5825$.

From successive cycles, we build up a table:

Cycle	0	1	2	3	4	5	6	⋯
Pop'n	1.02	0.5825	0.185	1.02	0.5825	0.185	1.02	⋯



V-Window $[0, 7, 1] \times [0, 1.3, 0.5]$

The population is oscillating or going in waves, with no overall growth or decline.

A group of young must first become adults (survival rate 0.5), then seniors (survival rate 0.25) before producing a new group of 8 young; the process then repeats itself. The overall survival rate between young and seniors of $0.5 \times 0.25 = 1/8$ is balanced by a birth rate of 8, so that the overall population is not growing or declining.

(b) The populations are now 1.02, 0.755, 0.41, 1.785, 1.321, 0.718, 3.124, 2.312, 1.256, 5.467, 4.046, 2.197, 9.566, 7.081, 3.846, 16,741, ….

The population is oscillating, but growing overall. The birth rate of 5 and the survival rate of $0.7 \times 0.5 = 0.35$ gives an overall growth rate of 1.75 (dashed line).

Exercise: Killer whales

1. A project you are involved in wants to re-introduce killer whales into an area of ocean from which they have disappeared. The project leader wants to know what is the best combination of juveniles and mature females to re-introduce, assuming an overall total of 50 (plus an appropriate number of adult males).

   You model three options, with the numbers in each class after 40 years shown below, rounded to whole whales.

   (a) 50 juveniles.

   $$\begin{bmatrix} 6 \\ 52 \\ 53 \\ 42 \end{bmatrix}$$

   Total population 153

   (b) 40 juveniles and 10 mature females.

   $$\begin{bmatrix} 7 \\ 57 \\ 58 \\ 47 \end{bmatrix}$$

   Total population 169

   (c) 50 adult females.

   $$\begin{bmatrix} 9 \\ 78 \\ 80 \\ 70 \end{bmatrix}$$

   Total population 237

   What happens to the different population classes of female killer whales over time in each of these options (according to this model)? What is the best strategy for re-establishing the killer-whale population?

   After about 15 years, in all three options, there are approximately equal numbers of juvenile and mature females, and this remains the case: both populations are increasing. The greater the inital number of mature females, the greater the numbers of both juvenile and mature females at any given cycle. The number of post-reproductive females follows a similar trend in all three options.

   In terms of the best strategy for re-introducing killer whales, it probably comes down to cost. Clearly the best option, according to the model, is to introduce all mature females but it could take quite a long time (and be expensive) to breed up a sufficient number, given it takes at least 14 years to maturity. Releasing all juveniles, a much cheaper option, runs the risk that not enough of the inexperienced juveniles survive long enough to establish a sustainable population. Some combination of juvenile and mature females may be the best option.

2. Does the trend in the total population depend on which option you choose?

In the three options with different initial populations, the trend is the same: the overall female population increases. However, the greater the inital number of mature females, the greater the total population at any given cycle. The fact of an increase in total population (and the eventual ratios of the populations in the four classes) does not depend on the initial populations in each of the classes.

3. The value of 0.1138 in the top row of $\underline{\underline{\boldsymbol{T}}}$ gives the number of live births per mature female per cycle (year). To what value could this birth rate fall before the total population starts to decrease? This birth rate is clearly important for the overall survival of killer whales.

Experimenting with different values of the birth rate for mature females shows that the population becomes steady when the value is about 0.055, i.e. about 48% of the observed value. For birth rates less than 0.055, the total population will decline.

4. How sensitive is the population to the survival rates of yearlings (0.9775), juveniles (0.9111), mature females (0.9534) and post-reproductive females (0.9804)?

The female population is more sensitive to the survival rates than to the birth rates, especially that of the mature females. Reducing the survival rate of 0.9534 for mature females to 0.905 (a reduction of only 5%) is enough to stop the population growing.

For the other classes, the corresponding values are: yearlings 0.9775 down to 0.49 (50%); juveniles 0.9111 down to 0.82 (10%). The growth or otherwise of the population is unaffected by the survival rate of post-reproductive females in this model.

Exercise: Age distribution of trees in a forest

1. If the population of trees in time interval n is $N = b_n + y_n + m_n + o_n$, show that the population stays the same size after one more time step, and so by induction the population of trees is a constant N.

Adding the four equations gives

$$\begin{aligned} b_{n+1} + y_{n+1} + m_{n+1} + o_{n+1} &= \alpha b_n + \beta y_n + \gamma m_n + \delta o_n + (1-\alpha)b_n \\ &\qquad + (1-\beta)y_n + (1-\gamma)m_n + (1-\delta)o_n \\ &= b_n + y_n + m_n + o_n \\ &= N. \end{aligned}$$

Therefore, the population of trees stays the same size after one more time step, and so by induction the population of trees is a constant N.

2. Equations (1) – (4) are linear in the dependent variables so that matrices are a way of solving the problem. Define $\underset{\sim}{\boldsymbol{v}}_n = (b_n, y_n, m_n, o_n)^{\mathrm{T}}$ (a column vector) and write down the matrix $\underline{\underline{\boldsymbol{T}}}$ that you would use in the matrix model $\underset{\sim}{\boldsymbol{v}}_{n+1} = \underline{\underline{\boldsymbol{T}}}\,\underset{\sim}{\boldsymbol{v}}_n$ of these equations.

Equations (1) – (4) can be written in matrix form as $\underset{\sim}{\boldsymbol{v}}_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$$\underset{\sim}{\boldsymbol{v}}_{n+1} = \begin{bmatrix} b_{n+1} \\ y_{n+1} \\ m_{n+1} \\ o_{n+1} \end{bmatrix} = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ 1-\alpha & 0 & 0 & 0 \\ 0 & 1-\beta & 0 & 0 \\ 0 & 0 & 1-\gamma & 1-\delta \end{bmatrix} \begin{bmatrix} b_n \\ y_n \\ m_n \\ o_n \end{bmatrix},$$

so that

$$\underline{\underline{\boldsymbol{T}}} = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ 1-\alpha & 0 & 0 & 0 \\ 0 & 1-\beta & 0 & 0 \\ 0 & 0 & 1-\gamma & 1-\delta \end{bmatrix}.$$

3. Take $\alpha = 0.2$, $\beta = 0.5$, $\gamma = 0.3$, $\delta = 0.2$ and $N = 1000$. Write down the matrix $\underline{\underline{\boldsymbol{T}}}$ with these values.

$$\underline{\underline{\boldsymbol{T}}} = \begin{bmatrix} 0.2 & 0.5 & 0.3 & 0.2 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.7 & 0.8 \end{bmatrix}.$$

4. Four initial conditions are needed in order to fully solve this matrix equation. Assume all baby trees initially.

Run the matrix model in Problems 2 and 3 through 10 cycles (150 years). Do the individual populations appear to be stabilising?

Run the model $\underset{\sim}{\boldsymbol{v}}_{n+1} = \underset{=}{\boldsymbol{T}}\underset{\sim}{\boldsymbol{v}}_n$ through 10 cycles with $\underset{\sim}{\boldsymbol{v}}_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, that is all baby trees initially: with $\underset{=}{\boldsymbol{T}}$ in Mat A and $\underset{\sim}{\boldsymbol{v}}_0$ in Mat B, execute the command Mat A Mat B $\rightarrow$ Mat B, then press $\boxed{\text{EXE}}$ nine times. Alternatively, use the POP program.

$$\underset{\sim}{\boldsymbol{v}}_{10} = \begin{bmatrix} 278.5 \\ 223.2 \\ 112.1 \\ 386.2 \end{bmatrix}.$$

The number in each population appears to be stabilising: b_n at around 279; y_n at around 223; m_n around 112; and o_n around 386.

5. Use the EIGENV5 program to find the eigenvalues of $\underset{=}{\boldsymbol{T}}$.

The eigenvalues are -0.358, -0.2, 0,558 and 1.

6. Find the eigenvector of $\underset{=}{\boldsymbol{T}}$ corresponding to $\lambda = 1$ (*explain why we use this value*) and relate this to the behaviour of the model that you found in Problem 4.

An eigenvector of 1 means the total population does not change, the case here. The corresponding eigenvector from EIGENV5, normalised to a maximum component of 1, is

$$\begin{bmatrix} 0.7142 \\ 0.5714 \\ 0.2857 \\ 1 \end{bmatrix},$$

giving a total population of 2.5714.

To relate this to the problem here, we multiply the eigenvector by 1000/2.5714 to give a total population of 1000. The new eigenvector, giving the four populations after a large (actually infinite) number of cycles, is (values rounded to integers)

$$\begin{bmatrix} 278 \\ 222 \\ 111 \\ 389 \end{bmatrix},$$

in good agreement with our results after 10 cycles in Problem 4.

20 Financial Mathematics 2

20.1 Introduction

The TVM notes (Section 20.2) are based closely on Chapter 11 of *Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS* by Barry Kissane. This book is a real bible on everything a graphics calculator can do and how to do it. Still very relevant but sadly now hard to find.[28]

The notes here concentrate on calculations which involve compound interest, such as savings with or without regular payments, loans, discounting, effective rate of interest, annuities, sinking funds, leasing and bonds. Other calculations such as simple interest, percentage calculations, days and dates, cost price/selling price/margins and cash-flow analysis, you will find in the above reference and in *Financial Calculation (TVM) Examples* (see References).

The good news is that all the compound-interest calculations are essentially the same: you specify the values of all the variables except one, then find the one you don't know. This is much simplified by using the TVM module of the calculator.

Some calculations should be done by hand first, so that students understand what they are are calculating. However, hand calculations very quickly become tedious once regular payments are involved. To be able eventually to do some useful financial modelling, such as comparing loans, etc, the TVM module is essential.

There are plenty of TVM exercises and activities with solutions in Sections 20.3 and 20.4.

References

Financial Calculation (TVM) Examples[29]
From the calculator manual
A number of examples of financial calculations. Lots of keystrokes, but not much explanation.

Mathematical Interactions: Financial Mathematics[2]
Barry Kissane and Anthony Harridine
Shriro Australia, 2000. ISBN 1 876543 60 4.
Available at *casioeducation.com.au/wp-content/uploads/2020/07/MIfinmath.pdf* (18/4/2022)
An introduction to financial maths, ending with simple compound-interest calculations.

Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS[1]
Barry Kissane
The Mathematical Association of Western Australia, 2003, ISBN 1 876583 24 X.
A must-have for teachers of Years 10 − 12 using Casio calculators.

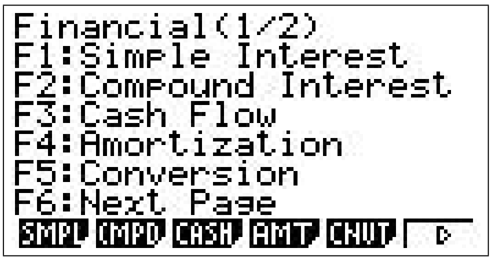
Quick Guide to Financial Keystrokes on a Casio cfx-9850G PLUS[2]
Barry Kissane.
A one-page summary of the Casio commands.

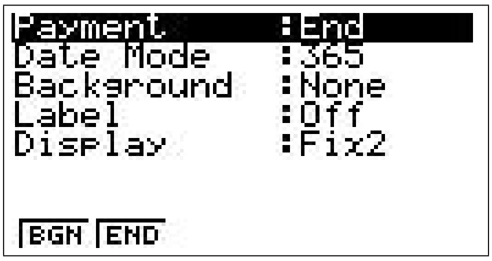
[28] I have a copy.
[29] on the CMA website *canberramaths.org.au*

20.2 Using the TVM Solver

20.2.1 Setting up

Enter TVM mode by pressing MENU C (below left — press F6 to see the remaining commands).





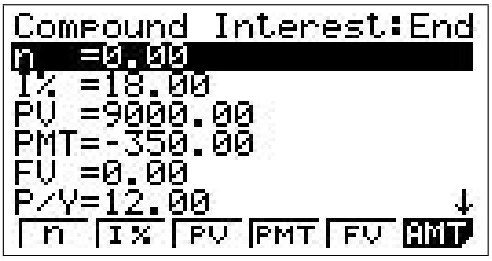
Then select SET UP (SHIFT MENU) and set up your calculator as shown above right. Fix 2 sets two decimal places, which is fine for dollar answers, but you may need to increase the number of decimal places if you are calculating interest rates.

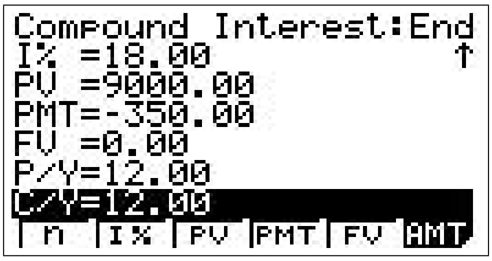
Press EXIT when you have finished with SET UP.

20.2.2 Compound interest

Most interest calculations involve compound interest, as people expect to pay interest only on the actual amount of money still owing on a loan or to earn interest on the actual amount of money in the account (not just on what was originally deposited).

Press F2 in TVM mode to select *Compound Interest* and display screens like those below. Note that you have to scroll down to see all the entries (below right).





The screens show that a number of variables are possibly involved in calculations involving compound interest. To use the TVM module, you need to specify the value of each of the variables except one; the calculator works out the missing one for you. Here is what the variables mean.

n the number of payment periods involved. If there are no regular payments, this variable refers instead to the number of years involved.

I% the annual rate of interest, given as a percentage.

PV the present value of the loan or investment. In the case of a loan, the amount borrowed. In the case of an investment, the original deposit.

PMT the regular payment — either repaying a loan, depositing money for investment or a regular payment from a superannuation or investment account.

FV the future value of the loan or investment. For a loan, this is usually zero, since the loan must be all paid off.

P/Y the number of payments made per year. Set to 1 if there are no payments made.

C/Y the number of compounding periods per year (in case it is different from P/Y).

For each variable, enter the necessary value, followed by EXE.

The calculator will set P/Y and C/Y to be the same as soon as you enter P/Y, since these two are usually the same. If they are not the same, you must enter P/Y first, then C/Y.

Note that AMT against F6 stands for Amortization, not for Amount.

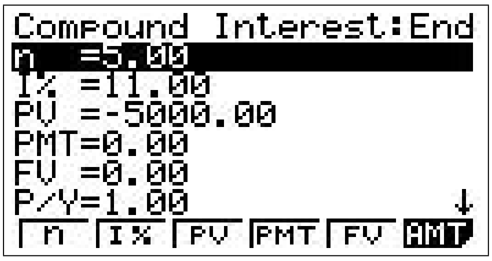
The least complicated case for compound interest involves depositing some money into an account and leaving it there some time to accrue interest.

Example 1

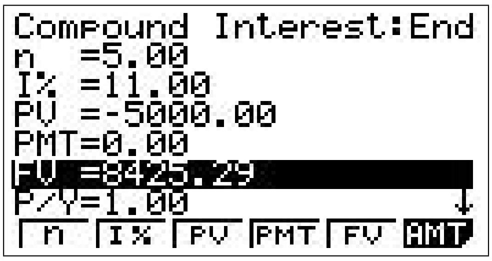
An inheritance of \$5000 has been placed into an investment account paying 11% interest per annum, compounded annually. How much is in the account after five years?

As shown in the figure below, n, the number of years, has been set to 5, I% to 11, PMT to 0, since there are no regular payments, P/Y and C/Y to 1. Note that 1 is the minimum value permitted by the calculator for P/Y, even though there are no payments.

PV, the present value, is -5000, negative because this amount has been paid to the bank.



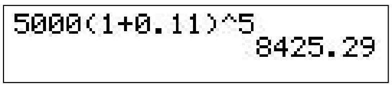
To find the future value of this investment (the amount after five years), press F5 (FV), to obtain the screen on the left below. Press F1 (REPT) to return to the TVM screen, below right, now displaying all the values.



These screens show that, at the end of five years, the investment will have grown to \$8425.29, positive because it will be paid to you.

It is important to realise that compound-interest results can also be obtained without using the TVM module. The variables here are related by the standard formula.[30]

$$\mathrm{FV} = \mathrm{PV}\left(1 + \frac{\mathrm{I\%}}{100}\right)^{\mathrm{n}}$$



The screenshot shows the result of this calculation in RUN mode, the same answer as as was found using the TVM module.

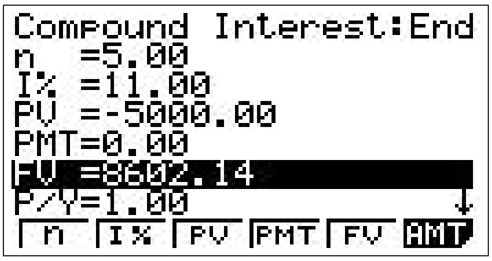
Other TVM calculations can also be done directly, but it is easier (and sometimes a lot easier) to let the calculator to do them for you. However, you should know what the calculator is doing.

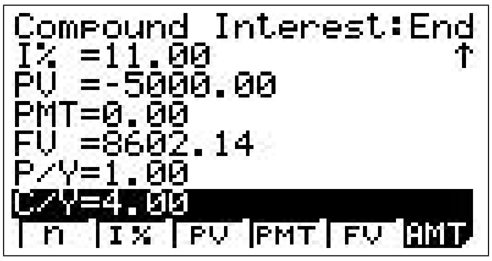
[30] If this formula is not familiar, you need to go back to *Financial Mathematics 1* in Volume 1: write down how you calculate the amount in the account after 1 year, 2 years, etc, that is derive the formula.

Example 2

How much is in the account in Example 1 if the interest is now compounded every three months?

Here we just change C/Y to 4, leaving the other variables the same, and again evaluate FV. The screens below show the result after pressing F1 (REPT).





Although the future value has increased to \$8602.14, the change to quarterly compounding does not have a dramatic effect, earning only an extra \$176.85.

20.2.3 Discounting

Compound-interest calculations are often used to find the future value of an investment with a certain rate of interest. However, sometimes we are interested in the opposite problem: finding the present value of a sum of money offered some time in the future.

One good reason for doing this is to compare various kinds of investments. If all are expressed in present values, then a direct comparison can be made. We could also compare investments by calculating their future values, but the choice of present values allows for today's money values to be used directly.

The process of finding the present value of a guaranteed future sum of money is sometimes called *discounting*, and the interest rate involved is sometimes called the *discount rate* in that case.

The relationship above between PV and FV can be rearranged to show how PV can be calculated:

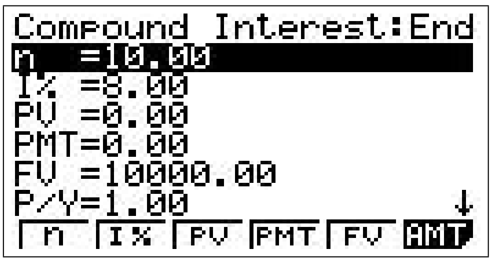
$$\mathrm{PV} = \frac{\mathrm{FV}}{\left(1 + \frac{\mathrm{I\%}}{100}\right)^{\mathrm{n}}}.$$

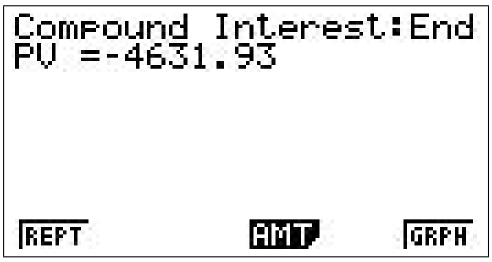
In the TVM module, discounting can be performed directly by entering the appropriate values for FV, I% and n, then calculating PV.

Example 3

What is the present value of \$10,000 in 10 years' time, assuming a discount rate of 8% per annum?

The screens below show the TVM set up (C/Y is 1) and the result.





The answer is \$4631.93, shown as negative because you would have to pay out this amount to have the \$10,000 (FV positive) paid to you in 10 years' time.

You can use the calculator to check this result by calculating that an investment of \$4631.93 at 8% per annum for 10 years will yield \$10,000.

20.2.4 Effective rate of interest

As seen above, if interest is compounded more often than annually, the amount of interest involved increases, since there is 'interest on the interest'. So, to compare interest rates, you need to consider how often the interest is compounded. Most interest rates are given as a rate per annum, sometimes called the *nominal rate* (annual compounding). The *effective rate* of interest takes into account the fact that the interest is compounded more than once a year. The effective rate of interest is useful for comparing different financial situations.

Essentially we first calculate the total amount of interest earned on an amount of say \$1000 (the amount does not matter) for one year under whatever compounding scheme is used. We then calculate the annual interest rate that would have generated this amount if there were no compounding: this is the effective rate.

Example 4

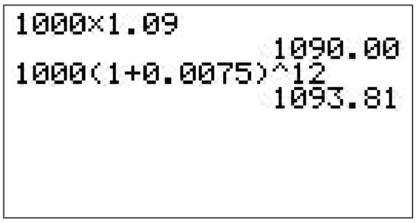
Consider an investment of \$1000 for a year at 9% per annum. After one year, if there is no compounding (or if the compounding period is one year), the total interest is 9% of \$1000 or \$90, and the future value \$1090.

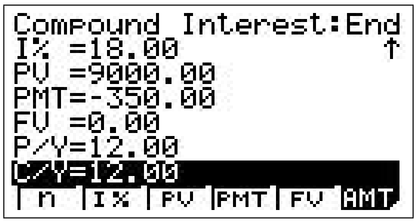
If the interest were compounded monthly instead of annually, the interest rate would be 0.75% per month (9% ÷ 12), and the future value can be determined from the usual formula (the interest rate in the hand calculations here must be the monthly interest rate):

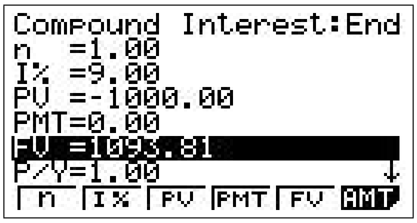
$$\mathrm{FV} \;=\; \mathrm{PV}\left(1+\frac{\mathrm{I\%}}{1200}\right)^{12\mathrm{n}} \;=\; 1000\big(1+0.0075\big)^{12} \;=\; \$1093.81.$$

The screens below show these calculations both in RUN mode and using the TVM module (after F1 (REPT) has been pressed).

Note: for monthly compounding for a year, $n=1$, P/Y = 1 (no regular payments) but C/Y = 12.

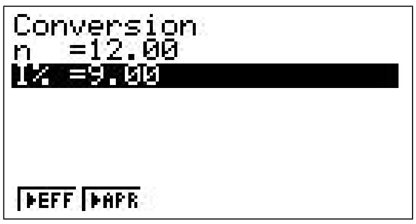


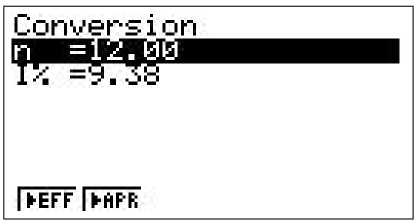


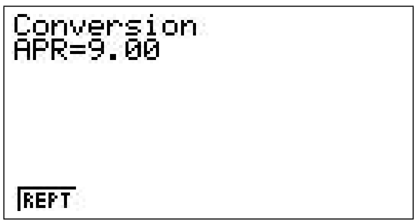


The investment in this case has earned interest of \$93.81. This is the amount of interest that would be earned with a nominal rate of interest of 9.381% (compounded annually). This is the effective rate of interest.

Once you understand the concept of effective rate of interest, you can use the TVM mode to calculate it directly using the Conversion module. For the present case, press F5 (Conversion) and enter the values for n and I%, as shown in the screen below left.







Pressing F1 (EFF) gives the effective rate of interest, 9.38% (rounded to 2 decimal places), as shown in the middle screen above after pressing F1 (REPT).

If you now press $\boxed{\text{F2}}$ (APR), the calculator will give the nominal rate corresponding to an effective rate of 9.38%, the reverse calculation.

Note that the calculator uses APR, the Annual Percentage Rate, for the nominal rate.

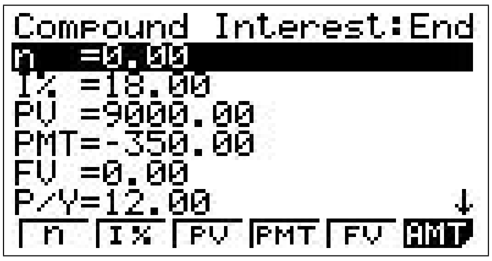
20.2.5 Analysing a loan

Most loans involve compound interest, as well as regular payments, so that the calculator is useful for analysing them. In particular, we can explore what happens if the variables are changed. In practice, this means we can ask questions like: *What if I pay fortnighly rather than monthly? What if the interest rate changes? What if pay a bit more each payment?*, and so on. Here we explore some of those possibilities in an example.

Example 5

I borrow $9000 to buy a car. The interest rate is 18% per annum, compounded monthly, and I plan to make monthly payments of $350. How long will it take to pay off the loan?

The screen below shows the variables entered into the TVM Compound Interest module.



Here PV is positive because the bank has paid the money to me, PMT is negative because I pay this to the bank and FV is zero because I will pay off the loan. P/Y and C/Y are both 12 because the payment and compounding periods are monthly. Press $\boxed{\text{F1}}$ to find n, the length of the loan, to give the result below.

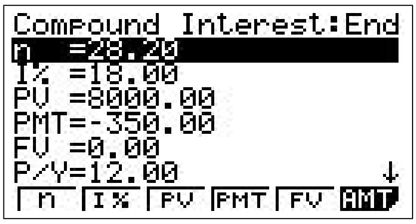
The value for n, 32.73, doesn't make sense because we can't have partial payment periods. The interpretation of n is that 32 payments of $350 are needed, and that the last payment will be less than this. The loan will certainly be paid off within 33 months or 2 years 9 months.

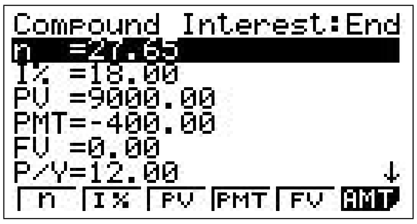
Details (and graphs) of the payment-by-payment values — the balance, how much of each payment is interest and how much goes to paying off the principal, the total interest paid and total paid off the principal to that payment — can be found using the Amortization module by pressing $\boxed{\text{F6}}$ (AMT). We won't do this here — the interested reader is referred to Barry Kissane's book *Mathematics with a Graphics Calculator: Casio cfx-9850G PLUS* and to *Financial Calculation (TVM) Examples* (see References).

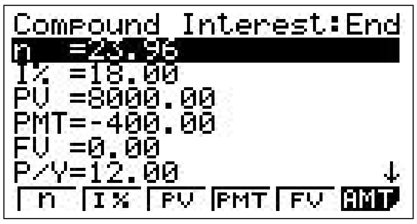
In calculating the total interest paid, we can use the Amortization module or we can just do the calculation: *total interest* = *number of payments* × *payment amount* − *loan amount.*

Here we have total interest $= 32.73 \times 350 - 9000 = \$2455.50$.

Let's now look at three alternatives to the original loan, shown in the three screens below. In each case, we have calculated n, then pressed F1 (RPT). C/Y is 12 in all cases.





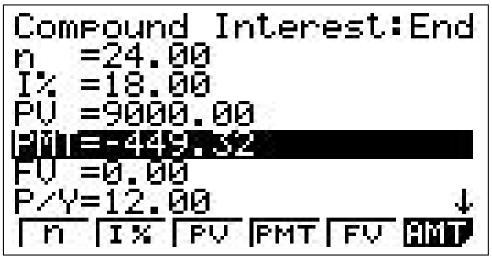


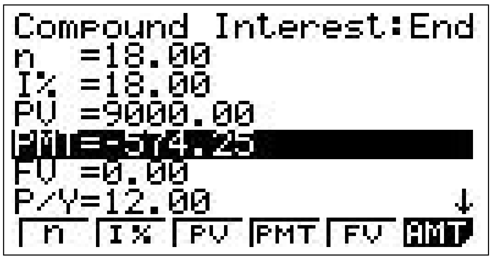
The first screen shows that if only \$8000 is borrowed, perhaps after paying a deposit of \$1000, the loan is paid off in a little over 28 months.

The second screen shows that increasing the payment from \$350 to \$400 each month on the original loan of \$9000 will reduce the length of the loan to a little less than 28 months.

The third screen shows that making both these changes will reduce the length of the loan to less than 2 years. The total interest paid here, $23.96 \times 400 - 8000 = \$1584$, is a saving of \$871 on the interest paid on the original loan.

Alternatively, we can ask what the payment needs to be made to pay off the loan in say 2 years or 18 months. The two screens below show these cases.





The loan can be paid off in 2 years with monthly payments of \$449.32, or in 18 months with monthly payments of \$574.25.

The total interest paid is, respectively, about \$1784 and \$1337.

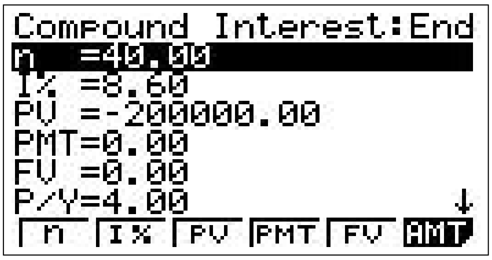
20.2.6 Annuities

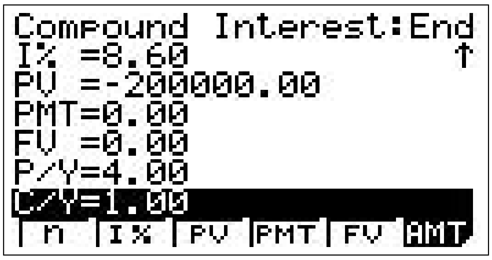
An annuity involves the payment of a regular sum of money, with a fixed rate of interest, over a period of time. Although you can regard payments on a loan as a kind of annuity, the term usually refers to other financial arrangements, such as a superannuation scheme (which has regular payouts) or a savings scheme (which has regular deposits. All these essentially involve compound interest, and so can be investigated efficiently using the financial functions of the calculator.

Example 6

A family has won a prize of \$200,000 in a lottery and decided to use it as an annuity to provide them with a regular amount of money every three months over the next ten years. They invest the money in an account paying 8.6% interest per annum, compounded annually. How much money will they get every three months?

Here there are four payments per year for ten years, giving 40 payment periods altogether, so that $\mathrm{n} = 40$. The present value PV is $-200{,}000$ (they are paying it to the bank), and the final value FV is zero, since all the money (and interest) will be used. There are four payments per year, so that $\mathrm{P/Y} = 4$ and $\mathrm{C/Y} = 1$, since the interest is compounded annually. The only missing variable is the payment PMT. The screens below show the details of entering these data.





The payment is found by pressing F4 (PMT) to get:

So this annuity provides the family with $7419.22 every 3 months for 10 years.

Financial Note: The use of 8.6% per annum as the interest rate assumes that there are no taxes and charges involved. In many situations, the interest earned would be subject to tax, and there will also be bank fees and charges. What you calculate might therefore be quite different to what you receive! We ignore these taxes and charges in our calculations.

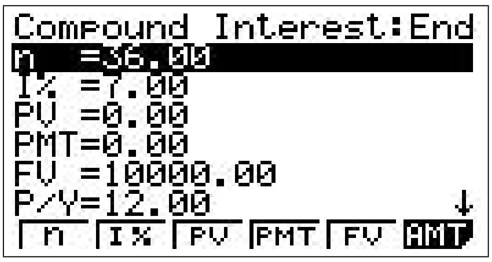
20.2.7 Sinking funds

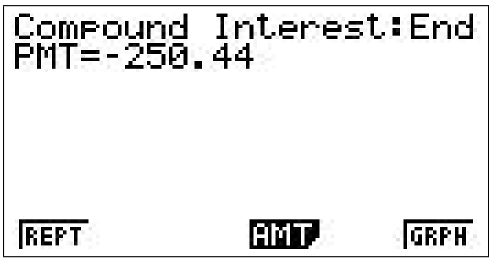
A sinking fund is a further example of an annuity, as it involves regular payments over a certain period, with a fixed rate of compound interest. The idea of a sinking fund is that money is invested in instalments into a fund for some particular purpose, and accumulates over time.

Example 7

You estimate you will need $10,000 for a long overseas holiday in 3 years' time. The rate of interest available in an investment account is 7%, compounded monthly. How much should you save each month to reach the goal?

In this case, $\text{PV} = 0$, $\text{FV} = 10{,}000$ and $\text{n} = 3 \times 12 = 36$, as the screen below left shows.





The required payment is $250.44 per month.

20.2.8 To lease or to buy?

TVM is a powerful idea for comparing alternative financial arrangements over different times. One example of this involves deciding whether it's financially better to purchase something now or buy it over a period of time.

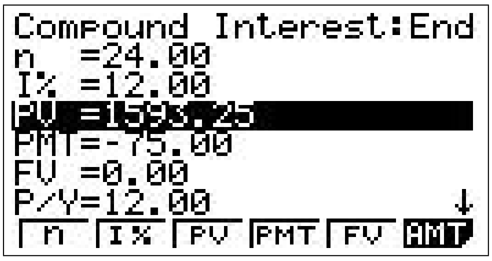
Example 8

A computer store advertises a computer system for sale for \$1999 or offers an alternative involving leasing. The store will lease the same computer system to you for \$75 a month over two years, with an interest rate of 12% per annum (compounded monthly). At the end of the two-year lease, you will be given an option to purchase the system for \$400. Should you buy the computer system now or lease it now and buy it in 2 years' time?

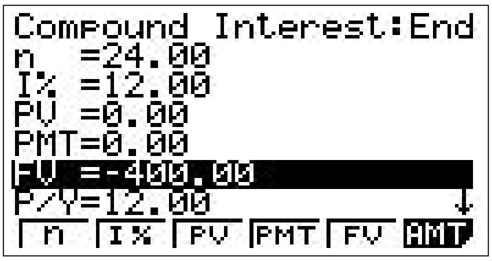
A quick comparison suggests that there is not much difference between these two choices, as the total payments for the lease-and-purchase option are $24 \times \$75 + \$400 = \$2200$, a bit more expensive than the cash purchase.

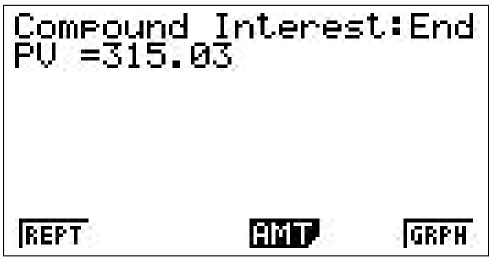
To explore these options more carefully, we need to have a way of comparing their monetary value at the same time. One way of doing this is to find the present value of the leasing option, and compare it with the present value of the cash purchase.

The leasing option comprises two parts: the monthly payments and the residual value of \$400. The screen below shows the 24 monthly payments of \$75 each. To determine the present value of these payments, press $\boxed{\text{F3}}$ (PV) and then $\boxed{\text{F1}}$ (REPT).



To determine the present value of the residual payment of \$400, enter the values as shown below left and evaluate PV. The resulting value is shown below right.

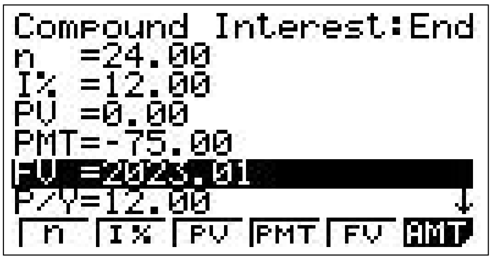


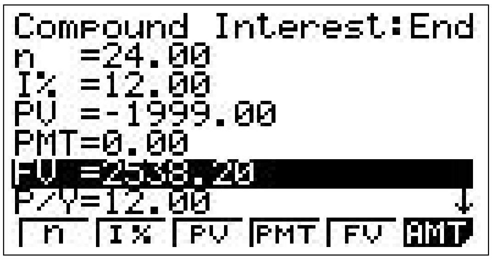


So the present value of the leasing option is \$1593.25 + \$315.03 = \$1908.28.

The cash purchase has a present value of \$1999, almost \$100 more than this, so it is a financially better deal to take the lease-and-purchase option in this case.

We could have also compared these two options using their respective future values (in two years' time). In the case of the lease, the screen below left shows that the monthly payments have FV = 2023.01. Added to the \$400 purchase price in 2 years' time (which of course has FV = 400), this gives a total future value of \$2423.01.





The cash-purchase option has a future value of \$2538.20, as shown above right.

Again, the analysis shows that leasing is the better option in this case.

20.2.9 Bonds

A bond is an agreement by a debtor to pay a certain amount of money (the *face value*) at a certain future date (the *maturation date*) and also to pay another sum of money periodically, the sum being a fixed percentage of the face value of the bond. Bonds are issued by public authorities, governments and companies as a way of raising money to finance their activities. In many cases, a bond has attached to it small dated coupons that can be used to obtain the periodical payment, so that the periodical payments are usually called *coupon payments*, and the interest rate of the coupon payments is called a *coupon rate*. Unlike private bonds issued by a company seeking to raise capital, government bonds may have some associated tax implications (which are not considered here).

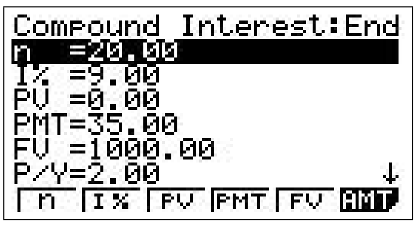
After they have been issued, bonds can be bought and sold by investors. Naturally enough then, the main interest in bonds is in determining their present value, deciding how much they should fetch when traded and thus deciding whether or not they are a good investment. Government bonds are usually regarded as more secure investments than private bonds, as there is less risk of default (i.e. there is less risk that the issuing body will not be able to redeem (pay for) the bonds at the maturation date or make the coupon payments). You can use the TVM module to analyse bonds.

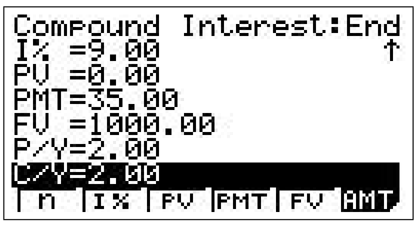
Example 9

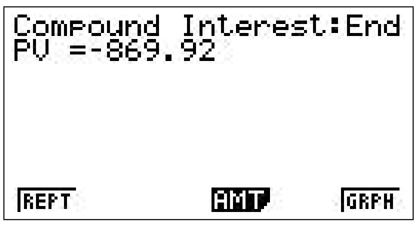
A bond is issued on 1 July 2006 by an electricity authority in order to finance a new power plant. The bond has a face value of \$1000, a maturation date of 1 July 2016 (that is, it matures in ten years) and a coupon rate of 7%, compounded and paid semi-annually.

You want a rate of return of 9% on your money. Do you buy the bond when it is issued?

In this case, the payments are fixed at $7.5\% \div 2 = 3.5\%$ of \$1000 = \$35 every six months, and the future value is fixed at the face value of \$1000. The required rate of return is I% = 9, and both P/Y and C/Y are set to 2 for semi-annual payment and compounding. There are 20 compounding periods, 2 each year for 10 years. The left-hand and middle screens below show how the data are entered into the TVM module.







To obtain the present value of the bond, press $\boxed{\text{F3}}$ (PV) (right-hand screen above).

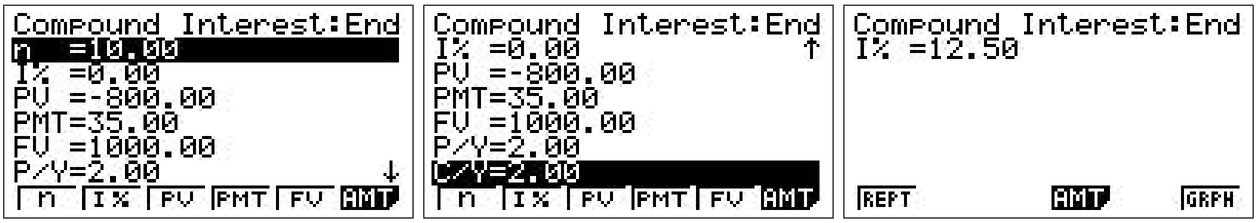
So the bond will return 9% on your money if you pay \$869.92 for it when it is issued. You will get a better rate of return than 9% if you pay less than this.

You can use the same module to calculate the Yield to Maturity (YTM) of a bond. The YTM is the equivalent interest rate you will earn if you retain the bond until it matures and you redeem it, as well as collecting the coupon payments up till then.

Example 10

An investor pays $800 for an electricity-authority bond as described in the previous example, 5 years before it matures (and just after a coupon payment was made). What is the YTM?

The left-hand and middle screens below show the relevant data entered into the calculator.



To calculate the YTM, press $\boxed{\text{F2}}$ (I%) to calculate the nominal interest rate (right-hand screen above).

So, in this case, the investment will yield $12\frac{1}{2}$% interest per annum, compounded semi-annually (assuming of course that the electricity authority is able to redeem the bond when it matures).

20.2.10 A cautionary note

The methods outlined here are generally less complicated than those used in practice, and do not include taxes, fees and other charges. You should seek proper financial advice before making any financial decisions, rather than relying on the calculations outlined here.

20.3 TVM exercises with solutions

These exercises and activities are taken from Barry Kissane's book, keeping the original numbering. Those questions marked with an asterisk cover material not in these notes, but the solutions are given here. See Barry Kissane's book for this material.

Note that pressing $\boxed{F1}$ *REPT, where this is a menu option, takes you back to the previous screen for more calculations.*

Note also that n *in the TVM calculations stands for different things in different modules.*

The main purpose of these exercises is to help you develop your calculator skills.

All results are rounded to two decimal places.

***1.** Find the simple interest on a 90-day loan of \$1600 at 5.6% per annum.

TVM $\boxed{\textbf{F1}}$ **Simple Interest**

n = 90 number of *days*
I% = 5.6 annual percentage interest rate
PV = 1600 amount paid to you by the bank (a loan), hence positive

$\boxed{\text{F1}}$: SI = −22.09 negative because you have to pay this to the bank

The total interest is \$22.09.

***2.** How many days are there between January 12 2003 and June 3 2003?

TVM $\boxed{\textbf{F6}}$ $\boxed{\textbf{F2}}$ **Days Calculation**

d1 = 1.122003 January 12 2003
d2 = 6.032003 June 3 2003

$\boxed{\text{F1}}$: PRD = 142

There are 142 days between January 12 2003 and June 3 2003.

***3.** A store buys a skateboard for \$45. What selling price will give a 30% margin?

TVM $\boxed{\textbf{F6}}$ $\boxed{\textbf{F1}}$ **Cost/Sel/Margin**

Cst = 45
Sel = ? doesn't matter what is here — this is what we have to find
Mrg = 30

$\boxed{\text{F2}}$: SEL = 64.29

The selling price should be \$64.29.

Note that a 30% margin means the return is 30% of the selling price. A 30% profit means a return of 30% of the cost price.

4. **(a)** How much compound interest will be earned by an investment of \$1600 for ten years at $9\frac{3}{4}\%$ per annum?

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = 10 with no regular payments, n is the number of *years*
I% = 9.75 annual percentage interest rate
PV = −1600 negative because paid to bank
PMT = 0 no regular payments
FV = ? doesn't matter what is here — this is what we have to find
P/Y = 1 payments per year — must be set to 1 if no payments
C/Y = 1 compounding periods per year (assumed to be equal to P/Y)

$\boxed{\text{F5}}$: FV = 4056.63 positive because paid to you

The final amount is \$4056.63.

Interest = \$4056.63 − \$1600 = \$2456.63.

(b) How much more interest will be earned if the interest were compounded quarterly?

Press $\boxed{\text{F1}}$ REPT and change C/Y to 4 (compounding periods per year).

$\boxed{\text{F5}}$: FV = 4192.56.

The final amount is \$4192.56.

Interest = \$4192.56 − \$1600 = \$2592.56.

Extra interest = \$2592.56 − \$2456.63 = \$135.93.

You could also just subtract one final amount from the other, as the difference is the extra interest.

5. For how long should you leave \$10,000 in the bank at 6% per annum interest, compounded monthly, until it grows to \$12,000?

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = ? doesn't matter what is here — this is what we have to find
I% = 6 annual percentage interest rate
PV = −10000 negative because paid to bank
PMT = 0 no regular payments
FV = 12000 the amount we want to receive
P/Y = 1 payments per year — must be set to 1 if no payments
C/Y = 12 compounding periods per year

$\boxed{\text{F1}}$: n = 3.046 because there are no regular payments, n is in years

You would need to leave the money in the bank 3.046 years or 3 years 17 days.

You might want to check that the bank will in fact give you the interest for the 17 days if you withdraw the money then. You may have to leave the money in the bank until the next monthly interest credit. By then you will have \$12,026.64.

6. A credit card company charges 16% per annum interest, compounded monthly. What is the effective rate of interest?

Here you can use the *Conversion* module of TVM (Method A). However, this is a bit of a black box. More intuitively, we can do the calculations on an amount of say \$1000 (Method B).

Method A

TVM $\boxed{\textbf{F5}}$ **Conversion**

n = 12 here n is the *number of compounding periods in a year*
I% = 16 annual percentage interest rate

$\boxed{\text{F1}}$: EFF = 17.23

The effective interest rate is 17.23% (to two decimal places).

Method B

Here say we owe the credit card company an amount of \$1000 over a year (the amount does not matter — you always get the same answer).

If the interest is compounded once a year, the amount of interest to be paid is $0.16 \times 1000 =\$160$.

Now we calculate how much interest is to be paid *if the interest is compounded monthly.*

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = 1 number of *years*, because no regular payments
I% = 16 annual percentage interest rate
PV = 1000 positive because it is a 'loan' paid to us
PMT = 0 no regular payments
FV = ? doesn't matter what is here — this is what we have to find
P/Y = 1 payments per year — must be set to 1 if no payments
C/Y = 12 compounding periods per year

$\boxed{\text{F5}}$: FV = 1172.27

The total amount of interest paid this time is $\$1172.27 - \$1000 = \$172.27$.
If this were the result of compounding once a year, the rate of interest would have to have been $172.27/1000 = 0.17227$ or $17.227\% \approx 17.23\%$. This is the effective rate of interest.

7. **(a)** To finance extensions to her house, Susan borrowed \$7000 at 11% per annum. What monthly payments are needed to repay the loan in three years?

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = $12 \times 3 = 36$ with regular payments, n is the total number of *payments*
I% = 11 annual percentage interest rate
PV = 7000 positive because paid to Susan
PMT = ? doesn't matter what is here — this is what we have to find
FV = 0 loan paid off
P/Y = 12 payments per year — monthly payments here
C/Y = 12 compounding periods per year — assumed equal to P/Y

$\boxed{\text{F4}}$: PMT = −229.17 negative because paid to the bank

The monthly payment is \$229.17.

(b) How much interest will she pay for this loan?

Total interest is total money paid minus the loan amount (principal), here
$36 \times 229.17 - 7000 = \$1250.12$.

Alternatively, press $\boxed{\text{F4}}$ AMT (Amortisation Table) after calculating the monthly payment, set PM1 = 1 (first payment), PM2 = 36 (last payment) and press $\boxed{\text{F4}}$ ΣINT (total interest for the specified payments) to get \$1250.16.
The small difference between the two amounts is due to rounding the monthly payment to two decimal places in the manual calculation.

8. Julian's lump-sum superannuation amounts to \$350,000. He invests this sum to receive a regular payment each month for the next twenty years. Assuming the interest rate stays at 6.8% per annum, find the size of each payment.

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = $12 \times 20 = 240$ total number of *payments*
I% = 6.8 annual percentage interest rate
PV = −350000 negative because paid to bank
PMT = ? doesn't matter what is here — this is what we have to find
FV = 0 capital and interest all used up after 20 years
P/Y = 12 payments per year
C/Y = 12 compounding periods per year — assumed equal to P/Y

$\boxed{\text{F4}}$: PMT = 2671.69

The monthly payment to Julian is \$2671.69.

9. A school establishes a sinking fund to save $90,000 over the next two years for a new computer laboratory. How much should be deposited into the fund each month if the interest rate stays at $7\frac{1}{2}\%$ per annum?

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = 12 × 2 = 24 total number of *payments*
I% = 7.5 annual percentage interest rate
PV = 0 no capital
PMT = ? doesn't matter what is here — this is what we have to find
FV = 90000 final amount required
P/Y = 12 payments per year
C/Y = 12 compounding periods per year — assumed equal to P/Y

$\boxed{\text{F4}}$: PMT = −3487.46

The monthly payment is $3487.46.

10. What is worth more? $1200 today or $100 per month for a year? Assume the interest rate stays constant at 7.1% per annum.

We have to find the present value of $100 per month for a year at 7.1% per annum.

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = 12 total number of *payments*
I% = 7.1 annual percentage interest rate
PV = ? doesn't matter what is here — this is what we have to find
PMT = −100 monthly payment to bank
FV = 0 you need to put 0 here
P/Y = 12 payments per year
C/Y = 12 compounding periods per year — assumed equal to P/Y

$\boxed{\text{F3}}$: PV = 1155.10

The present value of $100 per month for a year at 7.1% per annum is $1155.10. This is less than $1200, so $1200 today is worth more than $100 per month for a year at 7.1% per annum.

PTO for the rest of the answer

We could have found the future value of both amounts, although this is more work. Proceed as follows.

Press $\boxed{\text{F1}}$ REPT, set PV = 0 and press $\boxed{\text{F5}}$ FV to find the future value of $100 per month for a year at 7.1% per annum to be $1239.83.

Convert $1200 today to its future value:

n = 1 total number of *years* (no regular payments)
I% = 7.1
PV = −1200 present value
PMT = 0 no regular payments
FV = ? doesn't matter what is here — this is what we have to find
P/Y = 1 payments per year — must be set to 1 if no payments
C/Y = 12 compounding periods per year

$\boxed{\text{F5}}$: FV = 1288.03, so that the future value of the $1200 is $1288.03

We reach the same conclusion as above: take the $1200!

20.4 TVM activities with solutions

The activities are to help you use your calculator to learn mathematics.
All results rounded to two decimal places.

***1. (a)** Find the number of days between 12 January 2003 and 12 January 2004.

TVM $\boxed{\textbf{F6}}$ $\boxed{\textbf{F2}}$ **Days Calculation**

d1 = 1.122003 12 January 2003
d2 = 1.122004 12 January 2004

$\boxed{\text{F1}}$: PRD = 365

There are 365 days between 12 January 2003 and 12 January 2003.

(b) Find the number of days between 12 January 2004 and 12 January 2005.

TVM $\boxed{\textbf{F6}}$ $\boxed{\textbf{F2}}$ **Days Calculation**

d1 = 1.122004 12 January 2004
d2 = 1.122005 12 January 2005

$\boxed{\text{F1}}$: PRD = 366

There are 366 days between 12 January 2003 and 12 January 2003.

(c) Explain why the answers to Questions 1a and 1b are different.

2004 was a leap year.

2. Banks sometimes advertise that interest is compounded daily on customers' investments, in order to attract more business. But how much difference does more-frequent compounding make? To find out, consider an investment of \$100 in an account earning compound interest at a rate of 8% per annum. Compare the interest obtained by compounding: **(a)** annually; **(b)** quarterly; **(c)** monthly; **(d)** daily.

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = 1 number of *years* (no regular payments)
I% = 8 annual percentage interest rate
PV = −100 the amount invested — negative because paid to the bank
PMT = 0 no regular payments
FV = ? doesn't matter what is here — we calculate this
P/Y = 1 payments per year — must be set to 1 if no payments
C/Y = see table compounding periods per year

Set the value of C/Y and calculate FV by pressing $\boxed{\text{F5}}$.

Compounding	C/Y	FV (\$)	Interest (\$)
annually	1	108.00	8.00
quarterly	4	108.24	8.24
monthly	12	108.30	8.30
daily	365	108.33	8.33

You can also do these calculations with the Conversion module.

3. The effective rate of interest associated with more-frequent compounding rapidly approaches $e^r - 1$, where r is the annual rate of interest and e is the base of the exponential function ($e = 2.718281828\ldots$). Investigate this claim by taking some values for r and examining the effect of more-frequent compounding (such as weekly, daily, hourly, etc).

See Exercise 6 above for details on how to calculate effective rate of interest. Here we use various values for r = I%/100, and work out the effective rates of interest for different compounding periods. The results are shown in the table below. In the table, n is the number of compounding periods in a year. Interest rates are rounded to 3 decimal places.

The continuously compounded interest rate is $100\big(e^r - 1\big)$.

		Effective interest rate % when compounded ...					
I%	r	**yearly** ($N=1$)	**monthly** ($N=12$)	**weekly** ($N=52$)	**daily** ($N=365$)	**hourly** ($N=8760$)	**continuously** ($N\to\infty$)
5	0.05	5	5.116	5.125	5.127	5.127	5.127
7.5	0.075	7.5	7.763	7.783	7.788	7.789	7.789
10	0.1	10	10.471	10.506	10.516	10.517	10.517
20	0.2	20	21.939	22.093	22.134	22.140	22.140

Most of the change in the effective interest rate is that between annual compounding (n = 1) and monthly compounding (n = 12). The change between monthly and continuous compounding $100(e^r - 1)$ is relatively small, but larger for higher interest rates.

***4.** Jill purchased a new car for $11,999 by obtaining a five-year loan from a finance company. The interest on the loan was at a flat rate of 14% per year. Compare the amounts of interest paid on this loan each year with the amounts paid for a loan of the same amount at a rate of 14% reducible compound interest per annum, with monthly payments and compounding.

The flat rate means she pays 14% of $11,999, or $1679.86, each year for five years.

For the reducing rate: **TVM** $\boxed{\textbf{F2}}$ **Compound Interest**

n = $12 \times 5 = 60$ number of *payments*
I% = 14 annual percentage interest rate
PV = 11999 the loan amount
PMT = ? we need to find this first
FV = 0 loan paid off
P/Y = 12 payments per year
C/Y = 12 compounding periods per year

$\boxed{\text{F4}}$: PMT = −279.20

The monthly repayments are $279.20.

The simplest way to find the amount of interest paid each year is to press $\boxed{\text{F4}}$ (AMT) (Amortisation Table). For the amount of interest paid in the first year, set PM1 = 1, PM2 = 12 and press $\boxed{\text{F4}}$ (ΣINT) (total interest).

For the second year, press $\boxed{\text{F1}}$ (REPT) to go back, set PM1 = 13, PM2 = 24 and press $\boxed{\text{F4}}$ (ΣINT) again. Repeat for each year. You should get the amounts shown in the table below.

Year	**Flat-rate Interest ($)**	**Reducing Interest ($)**
1	1679.86	1568.39
2	1679.86	1302.27
3	1679.86	996.40
4	1679.86	644.86
5	1679.86	240.82
Total	8399.30	4752.74

As a check, you should compare the total reducing interest payable with the value found from the amortisation table with PM1 = 1 and PM2 = 60.

5. A couple are hoping to buy their first home. They expect to obtain a 25-year loan, and their monthly payments cannot exceed $920 if their living expenses are taken into account. If the interest rate is assumed to be constant at 7% per annum, how much money will they be able to borrow?

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = 12 × 25 = 300 number of *payments*
I% = 7 annual percentage interest rate
PV = ? doesn't matter what is here — we calculate this
PMT = −920 monthly payments to bank
FV = 0 loan paid off
P/Y = 12 payments per year
C/Y = 12 compounding periods per year — assumed equal to P/Y

$\boxed{\text{F3}}$: PV = 130,167.95

The couple can only borrow $130,167. This is why first-home buyers are struggling at present.

6. A student purchased a new laptop multimedia computer system with software for \$4500 by obtaining a personal loan at 13.4% per annum interest, compounded monthly.

(a) How much will each payment be if the loan is spread over 3 years?

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = 12 × 3 = 36 number of *payments*
I% = 13.4 annual percentage interest rate
PV = 4500 amount of loan
PMT = ? doesn't matter what is here — we calculate this
FV = 0 loan paid off
P/Y = 12 payments per year
C/Y = 12 compounding periods per year

$\boxed{\text{F4}}$: PMT = −152.49

The monthly repayment is \$152.49.

(b) How much interest will be paid over the 3 years?

Total interest = total payments − amount of loan = 36 × 152.49 − 4500 = \$989.64.

***(c)** If she arranged for a 3-year loan, as in Parts (a) and (b), but then decided to pay the entire loan balance at the end of 2 years, how much would she save?

To find out the total cost of this scheme, we need to find the balance owing after 2 years and the total payments in the first 2 years.

In the Amortisation Table ($\boxed{\text{F6}}$), set PM1 = 1 and PM2 = 24 (2 years). Press $\boxed{\text{F1}}$ (BAL) to obtain the balance after 2 years (\$1703.72).

Next press $\boxed{\text{F1}}$ (REPT) and $\boxed{\text{F4}}$ (ΣINT) to find the total interest paid in 2 years (\$863.51). Then press $\boxed{\text{F1}}$ (REPT) and $\boxed{\text{F5}}$ (ΣPRN) to find the total principal paid in 2 years (\$2796.28). The sum of these, \$3659.79, is the total payments made in 2 years. Alternatively calculate 24 × 152.49 = \$3659.76 (roundoff error).

The total payment here is then the total payment for 2 years plus the balance owing after 2 years, that is \$3659.79 + \$1703.72 = \$5363.51.

The total payment for the full 3e-year loan is principal + interest, giving from (b), \$4500 + \$989.68 = \$5489.68.

The total saving for the scheme of Part (c) is then \$5489.68 − \$5363.51 = \$126.17.

7. Jeremy decided to invest money in a superannuation account. He deposited \$400 per month into an account earning 9% interest per annum, compounded monthly.

(a) How much money will be in the account after 25 years?

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n $= 12 \times 25 = 300$ number of *payments*
I% $= 9$ annual percentage interest rate
PV $= 0$ nothing in the account to start with
PMT $= -400$ monthly payment to the bank
FV = ? doesn't matter what is here — we calculate this
P/Y $= 12$ payments per year
C/Y $= 12$ compounding periods per year

$\boxed{\text{F5}}$: FV $= 448448.77$

The amount in the account after 25 years is \$448,448.77.

(b) At the end of that time, Jeremy arranged to use the accumulated funds in the account to receive a monthly payment for the next 25 years. If the interest rate remains the same, find the size of the payment each month.

n $= 12 \times 25 = 300$ number of *payments*
I% $= 9$ annual percentage interest rate
PV $= -448448.77$ accumulated funds
PMT = ? doesn't matter what is here — we calculate this
FV $= 0$ all funds used up after 25 years
P/Y $= 12$ payments per year
C/Y $= 12$ compounding periods per year

$\boxed{\text{F4}}$: PMT $= 3763.37$

The amount payed to Jeremy each month is \$3763.37.

Seems like a good deal! Pay in \$400 per month for 25 years, then receive \$3763.37 per month for the next 25 years. What's the catch?

8. Carole and José have taken out a 25-year home loan of \$120,000 to buy an apartment at an interest rate of $7\frac{1}{2}\%$ per annum, compounded monthly.

(a) Check that their monthly payment for this loan is \$886.79.

TVM $\boxed{\text{F2}}$ **Compound Interest**

n = 12 × 25 = 300 number of *monthly payments*
I% = 7.5 annual percentage interest rate
PV = 120000 amount of loan
PMT = ? doesn't matter what is here — we calculate this
FV = 0 loan paid off
P/Y = 12 payments per year
C/Y = 12 compounding periods per year

$\boxed{\text{F4}}$: PMT = −886.79

The monthly repayment is \$886.79.

(b) What would be the effect on the length of their loan if they were to pay an extra \$50 per month?

The new payment is \$886.79 + \$50 = \$936.79.
Set PMT = −936.79.

$\boxed{\text{F1}}$: n = 258.8 (months)

The length of the loan would be reduced to 259 months or 21 years 7 months (the last payment would be less than the full monthly payment).

(c) What would be the effect on the length of their loan of a drop in interest rates to 7% (assuming they pay \$886.79 per month)?

I% = 7
PMT = −886.79

$\boxed{\text{F1}}$: n = 267.8 (months)

The length of the loan would be reduced to 268 months or 22 years 4 months (the last payment would be less than the full monthly payment).

(d) Carole suggests paying \$443.40 every two weeks instead of \$886.79 per month, even though the interest rate remained at $7\frac{1}{2}\%$ per annum interest, compounded monthly. How would this plan affect the length of the loan?

n = ?
I% = 7.5
PV = 120000
PMT = −443.40 fortnightly payment
FV = 0
P/Y = 26 fortnightly payments per year
C/Y = 12 compounding periods per year

$\boxed{\text{F1}}$: n = 525.5 (fortnights)

The length of the loan would be reduced to 526 fortnights or 20 years 12 weeks (the last payment would be less than the full fortnightly payment).

(e) If they were to borrow only \$115,000, yet still pay \$886.79 per month, with interest arrangements still the same, for how long will the loan last?

n = ?
I% = 7.5

PV = 115000 new loan amount
PMT = −886.79 monthly payment
FV = 0
P/Y = 12 monthly payments per year
C/Y = 12

$\boxed{\text{F1}}$: n = 267.0 (months)

The length of the loan would be reduced to 267 months or 22 years 3 months.

(f) Calculate how much time Carole and José can save off the original repayment time of 25 years if all of the above strategies are adopted: an interest rate of 7% is obtained, compounded monthly; payments of $468.40 are made every two weeks (an extra $25 per fortnight); and they only borrow $115,000 instead of $120,000.

How much interest would they save?

n = ?
I% = 7
PV = 115000
PMT = −468.40
FV = 0
P/Y = 26
C/Y = 12

$\boxed{\text{F1}}$: n = 401.8 (fortnights)

The length of the loan would be reduced from 25 years to 402 fortnights or 15 years 24 weeks.

They would pay a total of $73,2203.12 in interest, compared to $146,037 for the original loan, a saving of about $72,830.

9. A *perpetuity* is an annuity that lasts forever.

Find the present value of a perpetual annual payment of \$1000, assuming an interest rate of 6% per annum. (This is the amount you would need to invest today to provide enough funds for an annual payment of \$1000.)

In TVM mode, explore this situation by choosing a large number for the number of compounding periods n, and set FV to be zero.

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = 1000 number of compounding periods (years)
I% = 6 annual percentage interest rate
PV = ? doesn't matter what is here — we calculate this
PMT = 1000 positive because the bank pays this out to the beneficiary
FV = 0 nothing left at the end (of time)
P/Y = 1 payments per year
C/Y = 12 compounding periods per year — assumed to be 12

$\boxed{\text{F3}}$: PV = −16213.29

You would need to invest an amount of about \$16,213.

Changing n to 10,000 years also gives PV = −16213.29, so we can say the amount invested is enough to make the payment 'perpetual'.

Here, the amount of interest on the principal of \$16,213.29 is just enough to cover the annual \$1000 payment. The principal actually remains the same from year to year.

We can check that this is the case using the TVM Solver.

n = 1 one payment a year
I% = 6 annual percentage interest rate
PV = −16213.29 the principal in the bank
PMT = ? doesn't matter what is here — we calculate this
FV = 16213.29 principal remains the same at the end of the year
P/Y = 1 payments per year
C/Y = 12 compounding periods per year

$\boxed{\text{F4}}$: PMT = 1000, as we predicted.

10. You may have heard of the *Rule of 72*, which provides a rough approximation to the amount of time it will take for an investment to double when interest is compounded annually at a rate of I%. The Rule claims

$$\text{doubling time} \approx \frac{72}{\text{I\%}}.$$

So, according to the *Rule of 72*, an investment at a rate of 6%, compounded annually, will take about 72/6 = 12 years to double in value.

Investigate this Rule using the TVM mode of the calculator. Over what range of values for I% does it provide a useful approximation?

TVM $\boxed{\textbf{F2}}$ **Compound Interest**

n = ? doesn't matter what is here — we calculate this
I% = see table annual percentage interest rate
PV = −100 choose any amount
PMT = 0 no regular payments
FV = 200 double the PV
P/Y = 1 payments per year — must be set to 1 if no payments
C/Y = 1 compounding periods per year

Set values of I% and calculate n, the time to double the original amount, by pressing $\boxed{\text{F1}}$ each time. The results are shown in the table below.

I%	**n** (yrs)	**72/I%**	**% error**
1	69.7	72.0	3.3
2.5	28.1	28.8	2.5
5	14.2	14.4	1.4
7.5	9.6	9.6	0.0
10	7.3	7.2	−1.4
20	3.8	3.6	−5.3

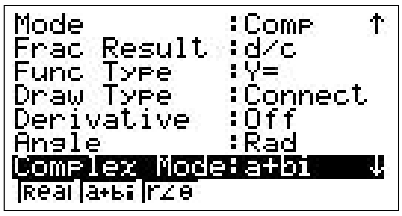
The approximation is good except for I% > 20, but particularly good around I% = 7.5.

21 Complex Numbers

21.1 Setting complex mode

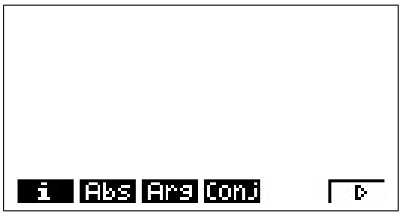
To work with complex numbers, press SET UP (SHIFT MENU) and scroll down to Complex Mode. Select a+bi, the Cartesian form of complex numbers. This determines how the answers are displayed; they may be input in either Cartesian or polar form.[31]

Note $r\angle\theta$, the polar form (Section 21.3). This notation is common in Engineering whereas, in Mathematics, we tend to use the exponential form $re^{i\theta}$ or $r\text{cis}(\theta)$.



While you are on this screen, select Rad, radians, for the angles.

Complex calculations are all done on the RUN screen (MENU 1). To access the complex operations, first press OPTN to give the screen below left. Then press F3 (CPLX) to give the screen below right.



We shall use $z_1 = 1+2i$ and $z_2 = 3-i$ in our examples. i is F1 in the CPLX menu.

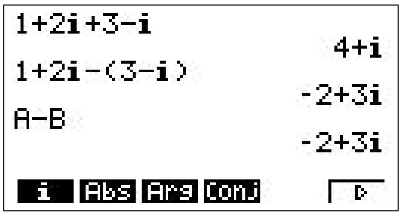
Complex numbers can be stored in the same way as ordinary numbers. Store z_1 in memory A: $1+2i$ $\rightarrow$ A; z_2 in memory B: $3-i$ $\rightarrow$ B.

[31]In *Real* mode, complex calculations will stil be carried out if you include an i in the calculation.

21.2 Basic operations

21.2.1 Addition and subtraction

Just as you would expect.

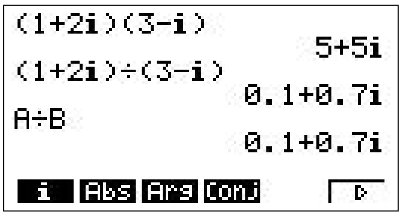


$1+2i+3-i = 4+i$ or $A+B = 4+i$

$1+2i-(3-i) = -2+3i$ or $A-B = -2+3i$

21.2.2 Multiplication and division

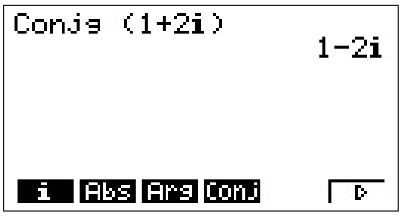
Again as you would expect. Implied multiplication works too.



$(1+2i)(3-i) = 5+5i$ or $AB = 5+5i$

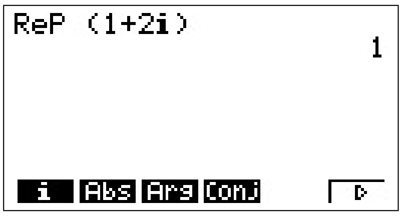
$(1+2i) \div (3-i) = 0.1+0.7i$ or
$A \div B = 0.1+0.7i$

21.2.3 Conjugation



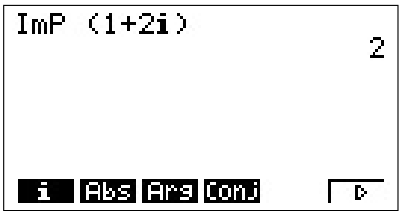
Finding the complex conjugate:
$\bar{z}_1 = \text{Conjg}\,(1+2i) = 1-2i$ *Conjg* is F4;
brackets are necessary unless the number is pure real or pure imaginary.

21.2.4 Real part



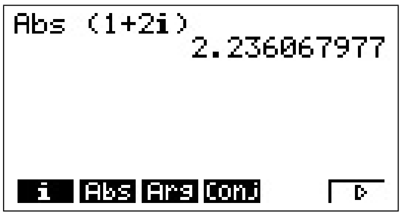
$\text{Re}(z_1) = \text{ReP}\,(1+2i) = 1$ *ReP* is F6 F1;
again brackets are necessary.

21.2.5 Imaginary part



$\text{Im}(z_1) = \text{ImP}\,(1+2i) = 2$ *ImP* is F6 F2;
again brackets are necessary.

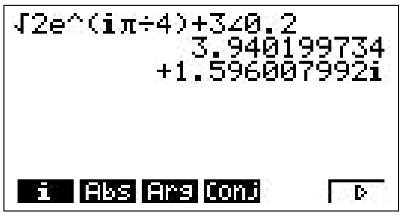
21.2.6 Modulus



Sometimes called length or absolute value.
$|z_1| = \text{Abs}\,(1+2i) = \sqrt{1^2+2^2} = \sqrt{5} \approx 2.236$.
Abs is $\boxed{\text{F2}}$; include brackets.

21.3 Polar form

The polar form of a complex number can be entered into the 9860 in either exponential form $re^{i\theta}$ or in the form $r\angle\theta$. The angle symbol $\angle$ is on the $\boxed{\text{X},\theta,\text{T}}$ key.
You can do all the usual operations with this form, even mixing the two forms. The result is given in Cartesian or polar form, depending on the setting in Complex Mode in $\boxed{\text{SETUP}}$.



21.3.1 Modulus and angle

The CPLX operation *Abs*, used in Section 21.2.6, finds the modulus r, while *Arg* $\left(\boxed{\text{F3}}\right)$ finds the angle θ, of a complex number given in Cartesian form. The angle is given in radians or degrees, depending on the angle setting in $\boxed{\text{SETUP}}$.

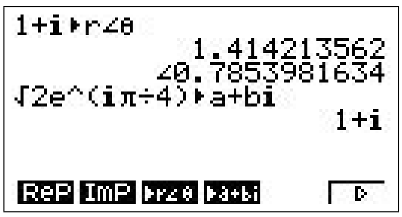
$\text{Arg}\,(1+2i) \approx 63.435^\circ \approx 1.107$ radians.

21.3.2 Conversion between forms

To convert complex numbers from Cartesian or rectangular form to polar form and vice versa, use the conversions in the $\boxed{\text{CPLX}}$ menu — press $\boxed{\text{F6}}$ to see these.

In *Radian* mode, converting $1+i$ to polar form:

$1+i$ $\boxed{\text{F3}}$ $\boxed{\text{EXE}}$ gives $r \approx 1.414213562$ ($\sqrt{2}$) and $\theta \approx 0.7853981634$ ($\pi/4$).



Converting $\sqrt{2}e^{i\pi/4}$ to Cartesian or rectangular form:

$\sqrt{2}e^\wedge(i\pi \div 4)$ $\boxed{\text{F4}}$ $\boxed{\text{EXE}}$ gives the Cartesian form $1+i$.

21.4 Powers and roots

21.4.1 Powers

As for real numbers. Non-integer and negative powers work too.

$(1+2i)^7 = 8 - 8i$.

$(1+2i)^{-1} = 0.2 - 0.4i$.

21.4.2 Roots

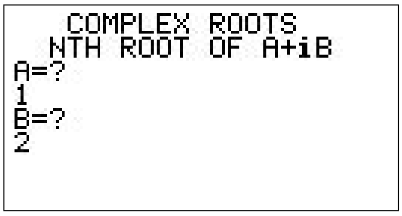
Only the square-root key $\boxed{\sqrt{\ }}$ works, and it only gives you one root of a complex number. To find all the roots,[32] use the program CMPXROOT.

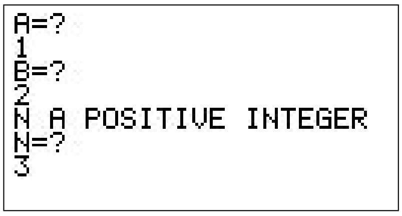
The CMPXROOT program

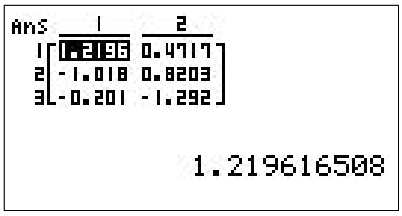
Calculates the Cartesian form $x+iy$ of each of the N Nth roots of the complex number $A+iB$.

Use: Run the program. Input values for A, B and N (positive integer) when prompted. The program displays the roots in a matrix, with the real parts x in the first column and the imaginary parts y in the second column.

Example: The cube roots (N = 3) of $1+2i$ are $1.220+0.472i$, $-1.018+0.820i$ and $-0.201-1.292i$, all rounded to three decimal places.







[32]A complex number has n nth roots (n a positive integer).

21.5 Exercises

$$z_1 = 3 + 4i \qquad z_2 = 2 + 3i \qquad z_3 = \sqrt{2}\,\text{cis}\,(\pi/4) \qquad z_4 = \text{cis}\,(\pi/2)$$

Find

1. $z_1 + z_2$
2. $2z_1 + 3z_2$
3. $z_1 - z_2$
4. $4z_1 - 2z_2$
5. $z_1 z_2$
6. z_1/z_2
7. $\bar{z_1}$
8. $z_1\bar{z_1}$
9. $|z_1|^2$
10. $\text{Re}(z_1)$
11. $\text{Im}(z_2)$
12. z_1^2
13. z_1^4
14. $\sqrt{z_1}$
15. $z_3 z_4$ in polar form
16. z_3/z_4 in polar form
17. $\sqrt{z_4}$ in polar and Cartesian form
18. z_1 in polar form
19. z_3 in Cartesian form
20. z_4 in Cartesian form

21.6 Answers to exercises

1. $5 + 7i$
2. $12 + 17i$
3. $1 + i$
4. $8 + 10i$
5. $-6 + 17i$
6. $18/13 - i/13 \approx 1.3846 - 0.0769i$
7. $3 - 4i$
8. 25
9. 25
10. 3
11. 3
12. $-7 + 24i$
13. $-527 - 336i$
14. $2 + i$ and $-2 - i$
15. $\sqrt{2}\angle 3\pi/4 \approx 1.4142\angle 2.3562$
16. $\sqrt{2}\angle -\pi/4 \approx 1.4142\angle -0.7854$
17. $1\angle \pi/4$ and $1\angle 5\pi/4$ or $\pm(1+i)/\sqrt{2}$
18. $5\angle 0.927295218$
19. $1 + i$
20. i